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THE ASTEROSEISMIC CALIBRATION OF SOLAR-TYPE STARS

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INTRODUCTION

The addition of seismic parameters to stellar-model calibrations substantially increases the constraints one can place on the properties of stars. We present some preliminary calculations to assess the accuracy with which certain stellar parameters can be inferred. For simplicity we use just two of the three most basic seismic parameters characterizing the low-degree p modes that might be measured from intensity variations by instruments such as photometers planned for the ESA spacecraft PRISMA. We ascertain the accuracy of a calibration of an isolated star and of a cluster of N solar-type stars.

THE SEISMIC CONSTRAINTS

We employ two seismic parameters Δ and d_0, defined in the manner of Gough and Novotny (1990) and obtained from fits of Tassoul's (1980) asymptotic formula to the frequency set. These parameters are essentially representative values of $\Delta_{nl} = \nu_{n,l} - \nu_{n-1,l}$ and $d_{n0} = \nu_{n,0} - \nu_{n-1,2}$, where $\nu_{n,l}$ is the cyclic frequency of the mode of order n and degree l. The frequency separation Δ depends on the gross structure of the star, principally on M/R^3, which is proportional to the mean density. The separation d_0 depends strongly on the variation of sound speed in the core, and is thus sensitive to the amount of helium that has been produced by the nuclear reactions; d_0 is therefore sensitive to the age of the star. We do not use Φ, a representative value of $\nu_{n,l}/\Delta - \left(n + \frac{1}{2}l\right)$, because it measures conditions near the stellar surface and is not well modelled by theory.

As Christensen-Dalsgaard (1986) has pointed out, the mass M and age t of a late-type main-sequence star could be determined from Δ and d_0, if only the composition and mixing-length parameter α were known and if the stellar evolution theory were presumed to be correct. But if the composition and α are not known, additional astronomical data must be used. We ask how well the stellar parameters can then be determined.

Stellar models are typically defined by M, α, t and the initial hydrogen and heavy-element abundances X and Z; we denote these parameters by α_i ($i = 1, 2, ..., 5$) $= (\ln M, \ln\alpha, \ln t, \ln X, \ln Z)$. They predict $a_k = (\ln R,\ \ln L,\ \Delta,\ d_0)$, where R and L are the radius and luminosity of the star. The problem in hand is to determine from these models how sensitively the inferred properties of stars depend on errors in the observations.

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ISOLATED STARS

As an example, consider a star for which $\beta_i = (\ln T_e,\ \ln L,\ Z/X,\ \Delta,\ d_0)$, where T_e is effective temperature, have been measured. In this and in the subsequent section, we consider the seismic parameters Δ and d_0 to have been determined from modes with $0 \le l \le 2$ and $14 \le n + \frac{1}{2}l \le 28$. The sensitivities of the inferred parameters $b_k = (\ln t,\ \ln M,\ \ln Y,\ \ln\alpha)$, where $Y = 1 - X - Z$, are obtained simply by transforming the partial derivatives $\partial a_k/\partial\alpha_i$ of the theory to the corresponding derivatives $\partial b_k/\partial\beta_i$ in the space of observations.

Derivatives $\partial a_k/\partial\alpha_i$ for a $1M_\odot$ stellar model of age $t = 4.6\times10^9$y in which the mixing length was taken to be α pressure scale heights are listed in Table I. They were computed by linear regression against sequences of five models with varying α_i and fixed $\alpha_j (j \neq i)$. Their transforms $\partial b_k/\partial\beta_i$, multiplied by perhaps somewhat optimistic observational errors $\delta\beta_i$, are listed in Table II. Thus the entries in Table II represent the corresponding contributions to the errors in the inferred values of b_k. Of course, each row scales linearly with the presumed error $\delta\beta_i$ in β_i. The total standard errors, for the values of $\delta\beta_i$ (assumed uncorrelated) listed in the second column, are recorded in the bottom row. It is interesting to observe in Table II that age is most susceptible to errors in d_0, as might have been expected. The accuracy of the other stellar properties depends mainly on the errors in L.

TABLE I Sensitivity of the properties a_k of a stellar model to the control parameters α_i. The unit of Δ and d_0 is μHz.

$\partial a_k/\partial\alpha_i$

α_i \ a_k	$\ln R$	$\ln L$	Δ	d_0
$\ln M$	2.00	6.09	−320	−27.9
$\ln\alpha$	−0.21	0.05	39.8	0.97
$\ln t$	0.16	0.40	−31.0	−6.75
$\ln X$	−1.62	−6.25	308	31.3
$\ln Z$	−0.16	−0.79	30.8	2.79

It is a straightforward matter to transform to a five-dimensional space spanned by a different set of β_i; for example, one might replace T_e by surface gravity g. Alternatively, one might suppose that the dimension of the space of observations is greater than five, say, by adding g to the data, so that the calibration problem is formally overdetermined. One can then use this property to reduce the formal uncertainty in the inferences. Of course, one could obtain more information about the star by analysing the individual frequencies $\nu_{n,l}$ rather than only the gross parameters Δ and d_0. Such an analysis is presented by Gough and Kosovichev (these proceedings).

CALIBRATION OF A SET OF SIMILAR STARS IN A CLUSTER

The members of a star cluster are normally assumed to have the same chemical composition and the same age. These constraints can reduce the uncertainty in

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TABLE II Errors δb_k for an isolated star induced by the errors $\delta\beta_i$ listed in column 2. The last row is the total error in b_k, assuming that $\delta\beta_i$ are independent.

		$(\partial b_k/\partial\beta_i)\delta\beta_i$			
β_i	$\delta\beta_i$	$\delta t/t$	$\delta M/M$	$\delta Y/Y$	$\delta\alpha/\alpha$
$\ln T_e$	0.005	−0.017	−0.031	0.082	−0.024
$\ln L$	0.1	0.005	0.153	−0.353	0.209
Z/X	0.003	−0.020	−0.001	0.040	0.012
Δ	0.2 μHz	0.003	0.003	−0.009	0.008
d_0	0.5 μHz	−0.112	−0.001	0.023	−0.033
All		0.12	0.16	0.37	0.21

the determination of Y and t dramatically. We denote the control parameters of stellar evolution theory for each star s by $\alpha_i^s = (\ln M^s, \ln\alpha^s, \ln t, \ln X, \ln Z)$ and the predicted properties by $a_k^s = (\ln R^s, \ln L^s, \Delta^s, d_0^s)$. Note that since the value of the mixing length is an assumption of the theory, the parameter α^s must be permitted to be different for each star. We consider an example in which the values of $\beta_i^s = (\ln T_e^s, \ln L^s, \Delta^s, d_0^s, Z/X)$ have been measured with standard error σ_i in a set of N solar-type stars of a cluster. We assume that the stars are sufficiently similar that their properties can be obtained by linearization about our stellar model of the previous section, using the partial derivatives in Table I. Moreover, for simplicity, we assume σ_i to be the same for all the stars. The calibration is an overdetermined problem (if $N \geq 2$), and we carry it out by minimizing amongst $b_k^s = (\ln M^s, \ln\alpha^s, \ln t, \ln Y, \ln Z)$ the deviation

$$\mathcal{X}^2 = \sum_s \sum_i \sigma_i^{-2} \left(\beta_i^s - \beta_{\mathrm{obs},i}^s\right)^2 ,$$

where $\beta_i^s\,(b_k^s)$ are the result of theory, obtained by transforming $a_k^s\,(\alpha_i^s)$; and $\beta_{\mathrm{obs},i}^s$ are measured values. We do not assume stellar evolution theory to be perfect, so the minimum value of $\mathcal{X}^2$ would not necessarily be zero in the absence of observational errors. But in the presence of random errors $\delta\beta_i^s$ in $\beta_{\mathrm{obs},i}^s$, errors δb_k^s in b_k^s arise. Let the contribution from $\delta\beta_i^s$ to the standard deviation of δb_k^s be e_{ki}^s. Then we can define the error sensitivity to be $\partial e_{ki}^s/\partial\sigma_i$. Error sensitivities for a group of 10 stars are listed in Table III. In this example, we have assumed the errors $\delta\beta_i^s$ in the intrinsic quantities β_i ($i = 1,2,3,4$) to be uncorrelated. The quantity Z/X, on the other hand, is assumed to be a cluster average, and its error is common to every star. In reality, there is also a common contribution to the error in $\ln L^s$, for example, which arises from an error in the measurement of the distance to the cluster, so that the errors $\delta\beta_2^s$ are not wholly uncorrelated. However, we have ignored that complication here, which results in the error sensitivities being independent of σ_i. The bottom row in Table III provides the total uncertainty in the calibration. Those entries were computed assuming that the errors for different i are uncorrelated.

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TABLE III Error sensitivities $\partial e^s_{ki}/\partial\sigma_i$ of the calibration of a cluster of 10 stars to observational errors in $\beta^s_{\mathrm{obs},i}$ with standard deviation σ_i, the unit of σ_3 and σ_4 being μHz. The measurement errors for each star were assumed to be uncorrelated for $i = 1, 2, 3, 4$, and the error in Z/X ($i = 5$) was taken to be common to all the stars. The bottom row is the total uncertainty in b^s_k obtained by adopting the values of σ_i listed in column 2. For clarity, we have adopted the notation: $\frac{\partial e(\ln M^s)}{\partial\sigma_i}$ for $\partial e^s_{1i}/\partial\sigma_i$, etc.

β^s_i	σ_i	$\frac{\partial e(\ln M^s)}{\partial\sigma_i}$	$\frac{\partial e(\ln \alpha^s)}{\partial\sigma_i}$	$\frac{\partial e(\ln t)}{\partial\sigma_i}$	$\frac{\partial e(\ln Y)}{\partial\sigma_i}$	$\frac{\partial e(\ln Z)}{\partial\sigma_i}$
$\ln T^s_e$	0.005	2.06	5.12	1.05	5.20	1.78
$\ln L^s$	0.1	0.49	0.66	0.02	1.12	0.38
Δ^s	0.2μHz	0.005	0.015	0.005	0.014	0.005
d^s_0	0.5μHz	0.001	0.022	0.071	0.014	0.005
Z/X	0.005	0.02	0.41	6.73	13.5	31.9
All		0.05	0.07	0.05	0.13	0.16

The redundancy amongst the measurements that has been introduced by assuming that age and composition are the same for all the stars has caused a degree of cancellation amongst the errors. Comparison of Tables II and III reveals that the uncertainties in all the calibrated quantities have been reduced, notwithstanding the increased value of the uncertainty in Z/X that has been adopted. Larger values of N lead to a further reduction, but the errors in the intrinsic variables $\ln M^s$ and $\ln \alpha^s$ soon stabilize. The errors in the global quantities $\ln t$, $\ln Y$ and $\ln Z$ continue to decline, approximately in proportion to $N^{-1/2}$. Indeed, if it is assumed that σ_i ($i = 1, 2, 3, 4$) are independent of N, but that σ_5 is proportional to $N^{-1/2}$, as one might expect from a cluster average, then the standard erors e^s_{ki} in the global quantities b_k ($k = 3, 4, 5$) eventually decline to zero strictly in proportion to $N^{-1/2}$.

It is straightforward to include other information in the analysis, such as g^s, as it is, also, to take correlated errors into account. In particular, in view of the high sensitivity of the inferred values of b^s_k to L^s, such a calibration might lead to a determination of the distance to the cluster that is more accurate than the methods currently in use. We shall report on our analysis in more detail elsewhere.

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