

DIFFERENCES OF COMPOSITION OPERATORS BETWEEN WEIGHTED BANACH SPACES OF HOLOMORPHIC FUNCTIONS

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Abstract

We consider differences of composition operators between given weighted Banach spaces H_v^∞ or H_v^0 of analytic functions with weighted sup-norms and give estimates for the distance of these differences to the space of compact operators. We also study boundedness and compactness of the operators. Some examples illustrate our results.

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1. Introduction

Let v and w be strictly positive bounded continuous functions (*weights*) on the open unit disk D in the complex plane. In this paper we are interested in operators defined on Banach spaces of analytic functions of the following form:

$$H_v^\infty := \left\{ f \in H(D); \|f\|_v = \sup_{z \in D} v(z)|f(z)| < \infty \right\},$$
$$H_v^0 := \left\{ f \in H(D); \lim_{|z| \rightarrow 1^-} v(z)|f(z)| = 0 \right\},$$

endowed with the norm $\|\cdot\|_v$. Here $H(D)$ denotes the space of all analytic functions equipped with the compact open topology. These spaces appear in the study of growth conditions of analytic functions and have been studied in various articles, see, for example, [1, 2, 16–18, 22, 23].

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Let $\phi, \psi : D \rightarrow D$ be analytic mappings. These maps induce through composition linear composition operators $C_\phi(f) = f \circ \phi$ and $C_\psi(f) = f \circ \psi$ between spaces of holomorphic functions of the type defined above. We will consider differences of composition operators $(C_\phi - C_\psi)(f) = f \circ \phi - f \circ \psi$ acting on these spaces of holomorphic functions.

Composition operators have been studied on various spaces of analytic functions. We refer the reader to the excellent monographs [7] and [21], and the article [15]. The case of operators defined on weighted Banach spaces of the type defined above was treated, for example, in [4, 5] and [6]. Differences of composition operators have been investigated more recently; see [10, 13, 14, 19] and [20]. In this article we are mainly interested in finding an expression for the essential norm $\|C_\phi - C_\psi\|_e$, that is, the distance of $C_\phi - C_\psi$ to the space of compact operators, when $C_\phi - C_\psi$ is a bounded operator from H_v^∞ into H_w^∞ ; compare with [10] and [12] for the case of H^∞ . It is known that if $\|\phi\|_\infty < 1$, then C_ϕ is a compact operator from H_v^∞ into H_w^∞ . Therefore, we are interested in the case $\max\{\|\phi\|_\infty, \|\psi\|_\infty\} = 1$. In our investigation we also study boundedness and compactness of $C_\phi - C_\psi$. It turns out that we obtain similar conditions to those obtained in [5] and [4], at least when the weight v is radial and satisfies certain natural conditions; see the details below.

2. Notation and definitions

We refer the reader to [7, 9, 11] and [21] for notation on composition operators and spaces of analytic functions on the unit disc. The closed unit ball of H_v^∞ , respectively H_v^0 , is denoted by B_v^∞ , respectively B_v^0 . The formulation of many results on weighted spaces of analytic functions and on operators between them requires the so-called *associated weights* (see [3]). For a weight v the associated weight \tilde{v} is defined as

$$\tilde{v}(z) := (\sup\{|f(z)| : f \in H_v^\infty, \|f\|_v \leq 1\})^{-1} = 1/\|\delta_z\|_{H_v^\infty}, \quad z \in D,$$

where δ_z denotes the point evaluation of z . The associated weights are also continuous and $\tilde{v} \geq v > 0$ (see [3]). Furthermore, for each $z \in D$ there exists $f_z \in H_v^\infty$, $\|f_z\|_v \leq 1$, such that $|f_z(z)| = 1/\tilde{v}(z)$. A weight is said to be *essential* if there is a constant $C > 0$ with

$$v(z) \leq \tilde{v}(z) \leq Cv(z) \quad \text{for every } z \in D.$$

For examples of essential weights and conditions when weights are essential see [3, 5] and [4]. Especially interesting are *radial weights* v , that is, weights which satisfy $v(z) = v(|z|)$ for every $z \in D$. Every radial weight which is nonincreasing with respect to $|z|$ and such that $\lim_{|z| \rightarrow 1} v(z) = 0$ is called a *typical weight*. If the weight v is typical, then the unit ball B_v^∞ coincides with the closure of B_v^0 for the compact open topology. *Throughout this article every radial weight is assumed to be nonincreasing.*

In order to handle differences of composition operators we need the so-called *pseudohyperbolic metric*. Recall that, for any $z \in D$, φ_z is the Möbius transformation of D which interchanges the origin and z , namely,

$$\varphi_z(w) = \frac{z - w}{1 - \bar{z}w}, \quad w \in D.$$

The pseudohyperbolic distance $\rho(z, w)$ for $z, w \in D$ is defined by $\rho(z, w) = |\varphi_z(w)|$. We refer the reader to [9] for more details. According to [8] we define $\rho_v(z, p) := \sup\{|f(z)|\tilde{v}(z); f \in B_v^\infty, f(p) = 0\}$. Note that, for any $z, p \in D$,

$$\rho(z, p) \leq \rho_v(z, p).$$

Indeed, let $f(p) = 0, f \in H^\infty, \|f\|_\infty \leq 1$. For each $z \in D$ there exists $g_z \in H_v^\infty, \|g_z\|_v \leq 1$, such that $|g_z(z)|\tilde{v}(z) = 1$. Hence $|f(z)| = |f(z)g_z(z)|\tilde{v}(z) \leq \rho_v(z, p)$.

In the case when v is a radial weight such that the condition (which is due to Lusky [17])

$$\inf_k \frac{v(1 - 2^{-k-1})}{v(1 - 2^{-k})} > 0 \tag{L1}$$

holds, then it is proved in [8] that ρ is equivalent to ρ_v . Several conditions equivalent to (L1) can be seen in [8]. In particular it is equivalent to a condition considered in [23].

An operator $T \in L(E, F)$ from the Banach space E to the Banach space F is said to be *compact* if it maps the closed unit ball of E onto a relatively compact set in F . We recall that operators $T : E \rightarrow F$ which take weakly null sequences in E to norm null sequences in F are said to be *completely continuous*. The essential norm of a continuous linear operator T is defined by $\|T\|_e := \inf\{\|T - K\| : K \text{ is compact}\}$. Since $\|T\|_e = 0$ if and only if T is compact, the estimates on $\|T\|_e$ lead to conditions for T to be compact.

3. Results

We start with an auxiliary result.

LEMMA 1. *Let v be a radial weight satisfying condition (L1). There exists a constant $C_v > \phi$ (depending only on the weight v) such that, for all $f \in H_v^\infty$,*

$$|f(z) - f(p)| \leq C_v \|f\|_v \max \left\{ \frac{\rho(z, p)}{v(z)}, \frac{\rho(z, p)}{v(p)} \right\}$$

for all $z, p \in D$.

PROOF. By Lemma 1(a) in [8], there exist an $0 < s < 1$ and a constant $0 < C < \infty$ such that $v(z)/v(p) \leq C$ for all $z, p \in D$ with $\rho(z, p) \leq s$. Hence it follows by Lemma 14 in [8] that

$$|f(z) - f(p)|v(z) \leq \frac{4C}{s} \|f\|_v \rho(z, p),$$

for all $z, p \in D$ with $\rho(z, p) \leq s/2$. If $\rho(z, p) > s/2$, then

$$|f(z) - f(p)| \min\{v(z), v(p)\} \leq 2\|f\|_v \leq \frac{4\|f\|_v}{s} \rho(z, p).$$

Therefore, we conclude that

$$|f(z) - f(p)| \min\{v(z), v(p)\} \leq C_v \|f\|_v \rho(z, p),$$

for all $z, p \in D$, from which the assertion follows. \square

Now we characterize bounded operators $C_\phi - C_\psi$. Recall that not every composition operator C_ϕ is bounded on H_v^∞ ; see [5].

PROPOSITION 2. *Let v and w be weights. If $C_\phi - C_\psi : H_v^\infty \rightarrow H_w^\infty$ is bounded, then*

$$\max \left\{ \sup_{z \in D} \frac{w(z)}{\tilde{v}(\phi(z))} \rho(\phi(z), \psi(z)), \sup_{z \in D} \frac{w(z)}{\tilde{v}(\psi(z))} \rho(\phi(z), \psi(z)) \right\} < \infty.$$

If v also is radial and satisfies condition (LI), then

$$\max \left\{ \sup_{z \in D} \frac{w(z)}{\tilde{v}(\phi(z))} \rho(\phi(z), \psi(z)), \sup_{z \in D} \frac{w(z)}{\tilde{v}(\psi(z))} \rho(\phi(z), \psi(z)) \right\} < \infty,$$

implies the boundedness of $C_\phi - C_\psi : H_v^\infty \rightarrow H_w^\infty$.

PROOF. Assume that $C_\phi - C_\psi : H_v^\infty \rightarrow H_w^\infty$ is bounded. Hence

$$\begin{aligned} \sup_{z \in D} \frac{w(z)}{\tilde{v}(\phi(z))} \rho(\phi(z), \psi(z)) &\leq \sup_{z \in D} \frac{w(z)}{\tilde{v}(\phi(z))} \rho_v(\phi(z), \psi(z)) \\ &\leq \sup_{z \in D} \frac{w(z)}{\tilde{v}(\phi(z))} \tilde{v}(\phi(z)) \sup\{|f(\phi(z)) - f(\psi(z))|; f \in B_v^\infty\} \\ &= \|C_\phi - C_\psi\| < \infty. \end{aligned}$$

Similarly,

$$\sup_{z \in D} \frac{w(z)}{\tilde{v}(\psi(z))} \rho(\phi(z), \psi(z)) < \infty.$$

For the converse implication we first notice that v is essential by Proposition 2(b) in [8]. Now we apply Lemma 1, so

$$\begin{aligned} \|C_\phi - C_\psi\| &= \sup_{z \in D} w(z) \sup\{|f(\phi(z)) - f(\psi(z))|; f \in B_v^\infty\} \\ &\leq \sup_{z \in D} w(z) C_v \max \left\{ \frac{\rho(\phi(z), \psi(z))}{v(\phi(z))}, \frac{\rho(\phi(z), \psi(z))}{v(\psi(z))} \right\} < \infty, \end{aligned}$$

and $C_\phi - C_\psi : H_v^\infty \rightarrow H_w^\infty$ is bounded. \square

Since $C_\phi - C_\psi : H(D) \rightarrow H(D)$ is continuous, we immediately get the following result.

PROPOSITION 3. *Let v be a weight such that $\overline{B_v^0} = B_v^\infty$. If $C_\phi - C_\psi : H_v^0 \rightarrow H_w^0$ is bounded, then $C_\phi - C_\psi : H_v^\infty \rightarrow H_w^\infty$ is bounded.*

EXAMPLE 4. We give an example of nonbounded composition operators such that their difference is bounded.

Choose $w(z) = 1$ and $v(z) = 1 - |z| = \tilde{v}(z)$ which are radial weights on D . Obviously, v satisfies condition (L1). Moreover, select, $\phi(z) = (z + 1)/2$ and $\psi(z) = (z + 1)/2 + t(z - 1)^3$, $z \in D$, such that t is real and $|t|$ so small that ψ maps D into D . By [5, Proposition 2.1], $C_\phi : H_v^\infty \rightarrow H_w^\infty$ is not bounded, because for $z = r \in \mathbb{R}$ we have $w(r)/\tilde{v}(\phi(r)) = 2/(1 - r)$ which tends to ∞ if $r \rightarrow 1$. The fact that $C_\psi : H_v^\infty \rightarrow H_w^\infty$ is not bounded follows in an analogous way: for $z = r \in \mathbb{R}$ we obtain

$$\frac{w(r)}{\tilde{v}(\psi(r))} = \frac{1}{1 - ((r + 1)/2 - t(r - 1)^3)} \rightarrow \infty \quad \text{if } r \rightarrow 1.$$

By [19, Example 1] we know that $\rho(\phi(z), \psi(z)) \leq (|t|/\delta)|z - 1|$, where $\delta > \phi$ is a constant. This yields

$$\begin{aligned} \sup_{z \in D} \frac{w(z)}{\tilde{v}(\phi(z))} \rho(\phi(z), \psi(z)) &\leq \sup_{z \in D} \left(1 - \left| \frac{z + 1}{2} \right| \right)^{-1} \frac{|t|}{\delta} |z - 1| < \infty \quad \text{and} \\ \sup_{z \in D} \frac{w(z)}{\tilde{v}(\psi(z))} \rho(\phi(z), \psi(z)) &\leq \sup_{z \in D} \left(1 - \left| \frac{z + 1}{2} + t(z - 1)^3 \right| \right)^{-1} \frac{|t|}{\delta} |z - 1| < \infty. \end{aligned}$$

Hence, $C_\phi - C_\psi : H_v^\infty \rightarrow H_w^\infty$ is bounded.

EXAMPLE 5. We give a nontrivial example of a nonbounded difference of composition operators.

Choose $\phi(z) = (z + 1)/2$, $\psi(z) = (z + 1)/2 + t(z - 1)^3$, where t is real and $|t|$ is so small that ψ maps D into D . Now select $w(z) = v(z) = e^{-(1/1 - |z|)} = \tilde{v}(z)$, which are radial weights not satisfying (L1). By [5, Proposition 2.1], $C_\phi : H_v^\infty \rightarrow H_w^\infty$ is not bounded since for $z = r \in \mathbb{R}$ we have

$$\begin{aligned} \frac{w(r)}{\tilde{v}(\phi(r))} &= e^{-(1/(1-r)) + (1/(1 - ((r+1)/2)))} \\ &= e^{-(1/(1-r)) + (2/(1-r))} = e^{(1/(1-r))} \rightarrow \infty \quad \text{if } r \rightarrow 1. \end{aligned}$$

Analogously $C_\psi : H_v^\infty \rightarrow H_w^\infty$ is not bounded since for $z = r \in \mathbb{R}$ we have

$$\begin{aligned} \frac{w(r)}{\tilde{v}(\psi(r))} &= e^{-(1/(1-r)) + (1/(1 - ((r+1)/2 - t(r-1)^3)))} \\ &= e^{-(1/(1-r)) + (2/(1-r - 2t(r-1)^3))} \rightarrow \infty \quad \text{if } r \rightarrow 1. \end{aligned}$$

By [19, Example 1] we know that $\rho(\phi(z), \psi(z)) \leq (|t|/\delta)|z - 1|$, where $\delta > 0$ is a constant. Since $|\phi(z)| \rightarrow 1$ or $|\psi(z)| \rightarrow 1$ is equivalent to $z \rightarrow 1$ and $\lim_{z \rightarrow 1} (|t|/\delta)|z - 1| = 0$, we get

$$\lim_{|\phi(z)| \rightarrow 1} \rho(\phi(z), \psi(z)) = \lim_{|\psi(z)| \rightarrow 1} \rho(\phi(z), \psi(z)) = 0.$$

Now, for $z = r \in \mathbb{R}$, we have

$$\begin{aligned} & \frac{w(r)}{\tilde{v}(\phi(r))} \rho(\phi(r), \psi(r)) \\ &= e^{(1/1-r)} \left| t(r-1)^3 \left(1 - \left(\frac{r+1}{2} \right) \left(\frac{r+1}{2} + t(r-1)^3 \right) \right)^{-1} \right| \\ &= e^{(1/1-r)} |t| \left| (r-1)^3 \left(1 - \left(\frac{(r+1)^2}{4} \right) - \left(\frac{r+1}{2} t(r-1)^3 \right) \right)^{-1} \right|, \end{aligned}$$

and $(w(r)/\tilde{v}(\phi(r)))\rho(\phi(r), \psi(r)) \rightarrow \infty$ for $r \rightarrow 1$. Hence $C_\phi - C_\psi : H_v^\infty \rightarrow H_w^\infty$ is not bounded.

The proof of our next result exploits a method presented in [4].

THEOREM 6. *Let v and w be radial weights such that v is typical and satisfies condition (LI). There is a constant $C_v > 0$ such that, if $\phi, \psi : D \rightarrow D$ are analytic maps such that $\max\{\|\phi\|_\infty, \|\psi\|_\infty\} = 1$ and $C_\phi - C_\psi : H_v^\infty \rightarrow H_w^\infty$ is bounded, then*

$$\begin{aligned} & \max \left\{ \limsup_{|\phi(z)| \rightarrow 1} \frac{w(z)}{\tilde{v}(\phi(z))} \rho(\phi(z), \psi(z)), \limsup_{|\psi(z)| \rightarrow 1} \frac{w(z)}{\tilde{v}(\psi(z))} \rho(\phi(z), \psi(z)) \right\} \\ & \leq \|C_\phi - C_\psi\|_e \\ & \leq C_v \max \left\{ \limsup_{|\phi(z)| \rightarrow 1} \frac{w(z)}{\tilde{v}(\phi(z))} \rho(\phi(z), \psi(z)), \limsup_{|\psi(z)| \rightarrow 1} \frac{w(z)}{\tilde{v}(\psi(z))} \rho(\phi(z), \psi(z)) \right\}. \end{aligned}$$

PROOF. We first prove the lower estimate of the essential norm by contradiction. Assume we can find $b > c > d > 0$, a compact operator $K : H_v^\infty \rightarrow H_w^\infty$ and a sequence $(z_n) \in D$ with $|\phi(z_n)| \rightarrow 1$ such that

$$\frac{w(z_n)}{\tilde{v}(\phi(z_n))} \rho(\phi(z_n), \psi(z_n)) \geq b > c > d > \|C_\phi - C_\psi - K\| \quad \text{for all } n.$$

Now we select an increasing sequence $(\alpha(n))_n$ of natural numbers going to infinity such that $|\phi(z_n)|^{\alpha(n)} \geq c/b$ for all n . Since v is typical, it follows that for every n we can find $f_n \in B_v^0$ such that $|f_n(\phi(z_n))| \geq (1/\tilde{v}(\phi(z_n))) (d/c)$.

Set $h_n(z) := z^{\alpha(n)} \varphi_{\psi(z_n)}(z) f_n(z)$. Thus, $h_n \in H_v^0$ with $\|h_n\|_v \leq 1$. Moreover, (h_n) converges to zero in the compact open topology, and consequently $h_n \rightarrow 0$ weakly

in H_v^0 ; see, for example, [25]. Since the operator K is compact, $\lim_{n \rightarrow \infty} \|Kh_n\|_w = 0$. Thus, for each n ,

$$c > \|C_\phi - C_\psi - K\| \geq \|(C_\phi - C_\psi)h_n\|_w - \|Kh_n\|_w,$$

and we conclude that

$$\begin{aligned} d > \|C_\phi - C_\psi - K\| &\geq \limsup_n \|(C_\phi - C_\psi)h_n\|_w = \limsup_n \|h_n \circ \phi - h_n \circ \psi\|_w \\ &\geq \limsup_n w(z_n) |h_n(\phi(z_n)) - h_n(\psi(z_n))| \\ &= \limsup_n w(z_n) |\phi(z_n)|^{\alpha(n)} |\varphi_{\psi(z_n)}(\phi(z_n)) f_n(\phi(z_n))| \\ &\geq \frac{d}{c} \limsup_n \frac{w(z_n)}{\tilde{v}(\phi(z_n))} \rho(\psi(z_n), \phi(z_n)) |\phi(z_n)|^{\alpha(n)} \geq c \frac{d}{c}, \end{aligned}$$

which is a contradiction.

We now prove the upper estimate. Take the sequence of linear operators $C_k : H(D) \rightarrow H(D)$, $k \in \mathbb{N}$, defined by $C_k f(z) = f((k/k + 1)z)$, which are continuous for the compact open topology and $C_k f \rightarrow f$ uniformly on every compact subset of D . Moreover, the operators $C_k : H_v^\infty \rightarrow H_v^\infty$ are well defined and compact with $\|C_k\| \leq 1$.

For fixed $k \in \mathbb{N}$, we have

$$\|C_\phi - C_\psi\|_e \leq \|C_\phi - C_\psi - (C_\phi - C_\psi)C_k\| = \|(C_\phi - C_\psi)(\text{Id} - C_k)\|.$$

Let $f \in H_v^\infty$ with $\|f\|_v \leq 1$ and fix an arbitrary $r \in (0, 1)$. Set $g_k := (\text{Id} - C_k)f$. Then $g_k \in H_v^\infty$ and $\|g_k\|_v \leq 2$. We set $X := \{z \in \mathbb{C} \mid |\phi(z)| \leq r\}$ and $Y := \{z \in \mathbb{C} \mid |\psi(z)| \leq r\}$. Hence

$$\begin{aligned} \|(C_\phi - C_\psi)g_k\|_w &\leq \sup_{z \in X \cap Y} |g_k(\phi(z)) - g_k(\psi(z))|w(z) \\ &\quad + \sup_{z \in \mathbb{C} \setminus (X \cap Y)} |g_k(\phi(z)) - g_k(\psi(z))|w(z) \\ &\leq \sup_{z \in X} |g_k(\phi(z))|w(z) + \sup_{z \in Y} |g_k(\psi(z))|w(z) \\ &\quad + \sup_{z \in \mathbb{C} \setminus (X \cap Y)} |g_k(\phi(z)) - g_k(\psi(z))|w(z). \end{aligned}$$

The sequence of operators $(\text{Id} - C_k)_k$ satisfies $\lim_k (\text{Id} - C_k)g = 0$ for each g in $H(D)$, and the space $H(D)$ is a Fréchet space. By the Banach–Steinhaus theorem, $(\text{Id} - C_k)_k$ converges to zero uniformly on the compact subsets of $H(D)$. Since the closed unit ball of H_v^∞ is a compact subset of $H(D)$, we obtain that

$$\lim_k \sup_{\|f\|_v \leq 1} \sup_{|\xi| \leq r} |((\text{Id} - C_k)f)(\xi)| = 0.$$

By Lemma 1,

$$|f(\phi(z)) - f(\psi(z))|w(z) \leq C_v \max \left\{ \frac{w(z)\rho(\phi(z), \psi(z))}{v(\phi(z))}, \frac{w(z)\rho(\phi(z), \psi(z))}{v(\psi(z))} \right\},$$

for all $z \in D$ and $f \in H_v^\infty$, $\|f\|_v \leq 1$. Since v is nonincreasing we conclude from this that

$$\begin{aligned} & \lim_k \|(C_\phi - C_\psi)(\text{Id} - C_k)\| \\ & \leq 2C_v \max \left\{ \sup_{z \in \mathbb{C} \setminus X} \frac{w(z)\rho(\phi(z), \psi(z))}{v(\phi(z))}, \sup_{z \in \mathbb{C} \setminus Y} \frac{w(z)\rho(\phi(z), \psi(z))}{v(\psi(z))} \right\}. \end{aligned}$$

Consequently,

$$\begin{aligned} & \|C_\phi - C_\psi\|_e \\ & \leq 2C_v \max \left\{ \lim_{r \rightarrow 1} \sup_{|\phi(z)| > r} \frac{w(z)\rho(\phi(z), \psi(z))}{v(\phi(z))}, \lim_{r \rightarrow 1} \sup_{|\psi(z)| > r} \frac{w(z)\rho(\phi(z), \psi(z))}{v(\psi(z))} \right\}. \end{aligned}$$

Since every radial weight with condition (L1) is essential (see [8, Proposition 2]), we are done. □

COROLLARY 7. *Let v and w be radial weights such that v is typical and satisfies condition (L1). Then $C_\phi - C_\psi : H_v^\infty \rightarrow H_w^\infty$ is compact if and only if*

$$\limsup_{|\phi(z)| \rightarrow 1} \frac{w(z)}{\tilde{v}(\phi(z))} \rho(\phi(z), \psi(z)) = \limsup_{|\psi(z)| \rightarrow 1} \frac{w(z)}{\tilde{v}(\psi(z))} \rho(\phi(z), \psi(z)) = 0.$$

PROOF. If $C_\phi - C_\psi$ is compact, then the conditions are satisfied by Theorem 6. Conversely, Theorem 6 implies the compactness of $C_\phi - C_\psi$ as soon as we know that $C_\phi - C_\psi$ is bounded. But by assumption we can choose $r < 1$ such that

$$\max \left\{ \sup_{|\phi(z)| > r} \frac{w(z)}{\tilde{v}(\phi(z))} \rho(\phi(z), \psi(z)), \sup_{|\psi(z)| > r} \frac{w(z)}{\tilde{v}(\psi(z))} \rho(\phi(z), \psi(z)) \right\} \leq 1.$$

Hence the boundedness follows from

$$\begin{aligned} & \max \left\{ \sup_{z \in D} \frac{w(z)}{\tilde{v}(\phi(z))} \rho(\phi(z), \psi(z)), \sup_{z \in D} \frac{w(z)}{\tilde{v}(\psi(z))} \rho(\phi(z), \psi(z)) \right\} \\ & \leq \max \left\{ 1, \sup_{z \in D} \frac{w(z)}{\tilde{v}(r)} \right\}. \end{aligned} \quad \square$$

Corollary 7 and the proof of the lower estimate in Theorem 6 permit us to obtain the following consequence.

COROLLARY 8. *Let v and w be radial weights such that v is typical and satisfies condition (LI). Then $C_\phi - C_\psi : H_v^\infty \rightarrow H_w^\infty$ is completely continuous if and only if $C_\phi - C_\psi$ is compact.*

THEOREM 9. *Let v and w be typical weights such that v satisfies condition (LI). There is a constant $C_v > 0$ such that, if $\phi, \psi : D \rightarrow D$ are analytic maps such that $\max\{\|\phi\|_\infty, \|\psi\|_\infty\} = 1$ and $C_\phi - C_\psi : H_v^0 \rightarrow H_w^0$ is bounded, then*

$$\begin{aligned} & \max \left\{ \limsup_{|z| \rightarrow 1} \frac{w(z)}{\tilde{v}(\phi(z))} \rho(\phi(z), \psi(z)), \limsup_{|z| \rightarrow 1} \frac{w(z)}{\tilde{v}(\psi(z))} \rho(\phi(z), \psi(z)) \right\} \\ & \leq \|C_\phi - C_\psi\|_e \\ & \leq C_v \max \left\{ \limsup_{|z| \rightarrow 1} \frac{w(z)}{\tilde{v}(\phi(z))} \rho(\phi(z), \psi(z)), \limsup_{|z| \rightarrow 1} \frac{w(z)}{\tilde{v}(\psi(z))} \rho(\phi(z), \psi(z)) \right\}. \end{aligned}$$

PROOF OF THEOREM 9. The difference with the proof of the lower bound of Theorem 6 is that now we get $b > c > d > 0$, a compact operator $K : H_v^0 \rightarrow H_w^0$ and a sequence $(z_n) \in D$ with $|z_n| \rightarrow 1$ such that

$$\frac{w(z_n)}{\tilde{v}(\phi(z_n))} \rho(\phi(z_n), \psi(z_n)) \geq b > c > d > \|C_\phi - C_\psi - K\| \quad \text{for all } n.$$

We can assume that $\phi(z_n) \rightarrow z_0$ for some z_0 with $|z_0| \leq 1$. If $|z_0| \neq 1$, then $0 = \lim_n w(z_n) \geq b v(z_0) > 0$, which is a contradiction. Therefore, $|\phi(z_n)| \rightarrow 1$ and we can continue as in the proof of Theorem 6. Notice also that in the proof of the upper bound the operators $C_k : H_v^0 \rightarrow H_w^0$ are well defined since v is typical.

EXAMPLE 10. We select $\phi(z) = (z + 1)/2$, $\psi(z) = (z + 1)/2 + t(z - 1)^3$, where the real number t is so small that ψ is a self-map on D . Moreover, we choose $w(z) = 1 - |z|$ and $v(z) = (1 - |z|)^3 = \tilde{v}(z)$.

Now $C_\phi, C_\psi : H_v^\infty \rightarrow H_w^\infty$ are not bounded since for $r \in \mathbb{R}$ we have that $w(r)/\tilde{v}(\phi(r)) = 8/(1 - r)^{-2}$ tends to infinity and

$$\frac{w(r)}{\tilde{v}(\psi(r))} = \frac{1 - r}{(1 - ((r + 1)/2 + t(r - 1)^3))^3} \rightarrow \infty \quad \text{if } r \rightarrow 1.$$

It follows from Proposition 2 that the operator $C_\phi - C_\psi : H_v^\infty \rightarrow H_w^\infty$ is bounded (see Example 4). However, it is not compact, since

$$\begin{aligned} \frac{w(r)}{\tilde{v}(\phi(r))} \rho(\phi(r), \psi(r)) &= \frac{8}{(1 - r)^2} \left| t(r - 1)^3 \left(1 - \frac{r + 1}{2} \left(\frac{r + 1}{2} + t(r - 1)^3 \right) \right)^{-1} \right| \\ &\rightarrow 8|t| \quad \text{if } r \rightarrow 1. \end{aligned}$$

For examples of compact and noncompact differences of composition operators $C_\phi - C_\psi : H^\infty \rightarrow H^\infty$, see [19, Example 1]. The change of the behaviour of the operator $C_\phi - C_\psi$ depending on the weights v and w is emphasized in our last example.

EXAMPLE 11. We consider $\phi(z) = (z + 1)/2$, $\psi(z) = (z - 1)/2$, $z \in D$, which are both analytic self-maps of the unit disk. By definition we obtain $\rho(\phi(z), \psi(z)) = |1 - ((z + 1)/2)((z - 1)/2)|^{-1}$. Hence

$$\lim_{|\phi(z)| \rightarrow 1} \rho(\phi(z), \psi(z)) = \lim_{|\psi(z)| \rightarrow 1} \rho(\phi(z), \psi(z)) = 1.$$

(a) Select $w(z) = 1 - |z| = v(z) = \tilde{v}(z)$. Obviously v is typical and satisfies (L1). By Theorem 6 we get

$$\begin{aligned} \limsup_{|\phi(z)| \rightarrow 1} \frac{w(z)}{\tilde{v}(\phi(z))} \rho(\phi(z), \psi(z)) \\ = \limsup_{|\phi(z)| \rightarrow 1} \frac{1 - |z|}{1 - |((z + 1)/2)|} \left| 1 - \frac{\overline{z + 1} z - 1}{2} \frac{z - 1}{2} \right|^{-1} = 2 \end{aligned}$$

and

$$\begin{aligned} \limsup_{|\psi(z)| \rightarrow 1} \frac{w(z)}{\tilde{v}(\psi(z))} \rho(\phi(z), \psi(z)) \\ = \limsup_{|\psi(z)| \rightarrow 1} \frac{1 - |z|}{1 - |((z - 1)/2)|} \left| 1 - \frac{\overline{z + 1} z - 1}{2} \frac{z - 1}{2} \right|^{-1} = 2. \end{aligned}$$

Hence

$$1 \leq \|C_\phi - C_\psi\|_e \leq 2C_v.$$

We conclude that $C_\phi - C_\psi : H_v^\infty \rightarrow H_w^\infty$ is bounded, but not compact.

(b) Choose $w(z) = 1$ and $v(z) = 1 - |z|$. We get

$$\sup_{z \in D} \frac{w(z)}{\tilde{v}(\phi(z))} \rho(\phi(z), \psi(z)) = \sup_{z \in D} \frac{1}{1 - |((z + 1)/2)|} \left| 1 - \frac{\overline{z + 1} z - 1}{2} \frac{z - 1}{2} \right|^{-1} = \infty.$$

Hence $C_\phi - C_\psi : H_v^\infty \rightarrow H_w^\infty$ is not bounded.

(c) Consider $w(z) = 1 - |z|$, $v(z) = 1$ to obtain

$$\limsup_{|\phi(z)| \rightarrow 1} \frac{w(z)}{\tilde{v}(\phi(z))} \rho(\phi(z), \psi(z)) = \limsup_{|\phi(z)| \rightarrow 1} (1 - |z|) \left| 1 - \frac{\overline{z + 1} z - 1}{2} \frac{z - 1}{2} \right|^{-1} = 0$$

and

$$\limsup_{|\psi(z)| \rightarrow 1} \frac{w(z)}{\tilde{v}(\psi(z))} \rho(\phi(z), \psi(z)) = \limsup_{|\psi(z)| \rightarrow 1} (1 - |z|) \left| 1 - \frac{\overline{z + 1} z - 1}{2} \frac{z - 1}{2} \right|^{-1} = 0.$$

Since the upper estimate in Theorem 6 is valid without the assumption that v is typical, we conclude that $\|C_\phi - C_\psi\|_e = 0$, and the operator is compact.

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