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GAUSSIAN AND NON-GAUSSIAN DISTRIBUTION-VALUED ORNSTEIN-UHLENBECK PROCESSES

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1. **Introduction.** Generalized (distribution-valued) Ornstein-Uhlenbeck processes, which by definition are solutions of generalized Langevin equations, arise in many investigations on fluctuation limits of particle systems (eg. Bojdecki and Gorostiza [1], Dawson, Fleischmann and Gorostiza [5], Fernández [7], Gorostiza [8,9], Holley and Stroock [10], Itô [12], Kallianpur and Pérez-Abreu [16], Kallianpur and Wolpert [14], Kotelenez [17], Martin-Löf [19], Mitoma [22], Uchiyama [25]). The state space for such a process is the strong dual Φ' of a nuclear space Φ. A generalized Langevin equation for a Φ'-valued process $X \equiv \{X_t\}$ is a stochastic evolution equation of the form

$$dX_t = A_t^* X_t\, dt + dZ_t, \tag{1.1}$$

where $\{A_t\}$ is a family of linear operators on Φ and $Z \equiv \{Z_t\}$ is a Φ'-valued semimartingale (in some sense) with independent increments. Equations of the type (1.1) where Z does not have independent increments also arise in applications (eg. [9,14,20]) but here we are interested precisely in the case when Z has independent increments (we restrict the term generalized Langevin equation to this case in accordance with the classical Langevin equation).

Existence and uniqueness of solutions of some classes of stochastic evolution equations of the type (1.1) have been studied by various authors (eg. [16]). In these studies the equation is assumed to be given. On the other hand, in the analysis of limits of particle systems one usually obtains the finite-dimensional distributions of the limit process X, and from this information one wishes to determine if X is a generalized Ornstein-Uhlenbeck process and to find the Langevin equation it satisfies. This is the question addressed in the present paper.

Most often the processes X encountered as fluctuation limits of particle systems are Gaussian. In this case the question above has been treated in [1,2], where a convenient criterion is given in terms of the covariance functional of X. Recently Dawson *et al.* [5] have obtained a fluctuation limit process which is a stable (non-Gaussian) generalized Ornstein-Uhlenbeck process; the Langevin equation is derived in [6]. This result arises in a natural way and may be considered typical for a large class of particle systems (having a branching mechanism with infinite variance). Clearly in this situation the covariance

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criterion of [1,2] is not applicable and it becomes necessary to seek a criterion with a wider range of applicability. In this paper we give such a criterion, which should prove useful for the analysis of particle systems with fluctuation limits which may be non-Gaussian.

Our main results are a general criterion to determine if a given Φ'-valued process X is a generalized Ornstein-Uhlenbeck process (not necessarily Gaussian) and to find the Langevin equation (Theorem 2.2), and a special case of the criterion with a more explicit result, which however covers a large class of applications (Theorem 2.5). The criterion is given in terms of the conditional characteristic functional of X (which in principle can be derived from the finite-dimensional distributions). In order to obtain a reciprocal result we also prove a theorem on existence and uniqueness of solutions of equation (1.1); as usual, the unique solution is expressed as a mild solution, or variation of constants formula (Theorem 2.11). Existence and uniqueness results of this type are given in [16], assuming Z is a square-integrable martingale. We restrict ourselves to the case when Φ is a nuclear Fréchet space, but we think our results can be extended to other classes of nuclear spaces.

The relationship between the distributions of the processes X and Z in (1.1) contained in Theorem 2.5 is relevant in connection with the *fluctuation-dissipation relations* of statistical mechanics.

In Section 2 we state and prove our results. Section 3 contains a general result for the Gaussian case and the example of a stable process of [5,6].

We shall assume that the reader is familiar with the basics of nuclear spaces as presented in [23] or [24].

2. **Results.** Let Φ be a nuclear Fréchet space with a (shift-invariant) metric ρ, and let Φ' be the strong dual of Φ. It is well-known that Φ' is nuclear as well [24]. The canonical bilinear form on (Φ', Φ) will be denoted by $\langle \cdot, \cdot \rangle$.

A positive number T is fixed throughout.

In what follows we will consider families $\{A_t : 0 \leq t \leq T\}$ and $\{U_{s,t} : 0 \leq s \leq t \leq T\}$ of continuous linear operators from Φ into Φ such that

(a) the function $t \longmapsto A_t\varphi$ is continuous for each $\varphi \in \Phi$,

(b) the function $(s,t) \longmapsto U_{s,t}\varphi$ is continuous for each $\varphi \in \Phi$,

(c) $U_{t,t} = I$ and $U_{r,t} = U_{r,s}U_{s,t}$ for $0 \leq r \leq s \leq t \leq T$,

(d) $U_{s,t}\varphi = \varphi + \int_s^t U_{s,r}A_r\varphi \, dr$ for $0 \leq s \leq t \leq T$, $\varphi \in \Phi$.

DEFINITION 2.1. Let $\{Z_t : 0 \leq t \leq T\}$ be a Φ'-valued right-continuous with left limits (abbr. cadlag) process, $Z_0 = 0$. A Φ'-valued process $\{X_t : 0 \leq t \leq T\}$ is said to be a solution of the equation

$$dX_t = A_t^* X_t \, dt + dZ_t \tag{2.1}$$

if

$$\langle X_t, \varphi \rangle = \langle X_0, \varphi \rangle + \int_0^t \langle X_s, A_s\varphi \rangle \, ds + \langle Z_t, \varphi \rangle \text{ a.s.} \tag{2.2}$$

for each $\varphi \in \Phi$, $0 \leq t \leq T$.

THEOREM 2.2. *Let $A_t, U_{s,t}$ satisfy (a), (b), (c), (d) and assume that $X \equiv \{X_t : 0 \leq t \leq T\}$ is a Φ'-valued cadlag process such that for each $\varphi \in \Phi$, $0 \leq s \leq t \leq T$,*

$$E[e^{i\langle X_t,\varphi\rangle} \mid \langle X_r, \psi\rangle, r \leq s, \psi \in \Phi] = e^{i\langle X_s, U_{s,t}\varphi\rangle} H(s,t;\varphi) \tag{2.3}$$

for some deterministic complex-valued $H(s,t;\varphi)$. Then X is a Markov process and it satisfies (2.1), where Z is a process with independent increments.

PROOF. The Markov property of X follows directly from (2.3). The Banach-Steinhaus theorem holds for Φ since it is barrelled [24], hence for each $\varphi \in \Phi$ the process $\{\langle X_t, A_t\varphi\rangle : 0 \leq t \leq T\}$ is cadlag and we can define the process Z by the formula

$$\langle Z_t, \varphi\rangle = \langle X_t, \varphi\rangle - \langle X_0, \varphi\rangle - \int_0^t \langle X_r, A_r\varphi\rangle \, dr. \tag{2.4}$$

The regularization theorem [11] implies that $Z_t \in \Phi'$, $0 \leq t \leq T$, and by [21] Z has a cadlag version. We will prove that Z has independent increments.

For any $s \in [0,T]$ we denote by $\mathcal{G}_s$ the σ-algebra generated by the random variables $\langle X_r, \psi\rangle$ for $r \leq s$, $\psi \in \Phi$.

It clearly suffices to prove that $E[\exp\{i(\langle Z_t,\varphi\rangle - \langle Z_s,\varphi\rangle)\} \mid \mathcal{G}_s]$ is deterministic for each $\varphi \in \Phi$, $0 \leq s \leq t \leq T$. Fix φ, s, t and let $s = r_0^n < r_1^n < \cdots < r_{m_n}^n = t$, $n = 1, 2, \ldots$, be an arbitrary normal sequence of partitions of $[s,t]$. By (2.4) we have

$$\langle Z_t, \varphi\rangle - \langle Z_s, \varphi\rangle = \langle X_t, \varphi\rangle - \langle X_s, \varphi\rangle - \int_s^t \langle X_r, A_r\varphi\rangle \, dr = \lim_{n\to\infty} B_n,$$

where

$$B_n = \langle X_t, \varphi\rangle - \langle X_s, \varphi\rangle - \sum_{k=1}^{m_n} \langle X_{r_k^n}, A_{r_k^n}\varphi\rangle (r_k^n - r_{k-1}^n),$$

and of course,

$$\lim_{n\to\infty} E[e^{iB_n} \mid \mathcal{G}_s] = E[\exp\{i(\langle Z_t,\varphi\rangle - \langle Z_s,\varphi\rangle)\} \mid \mathcal{G}_s] \text{ a.s.} \tag{2.5}$$

We transform $E[e^{iB_n} \mid \mathcal{G}_s]$ as follows (writing r_k instead of r_k^n, Δr_k instead of $r_k^n - r_{k-1}^n$ and m instead of m_n):

$$\begin{aligned}
&E[e^{iB_n} \mid \mathcal{G}_s] \\
&\quad = E\Big[E[\exp\{i\langle X_{r_m}, \varphi - A_{r_m}\varphi\Delta r_m\rangle\} \mid \mathcal{G}_{r_{m-1}}] \\
&\qquad \cdot \exp\Big\{i\Big(-\sum_{k=1}^{m-1}\langle X_{r_k}, A_{r_k}\varphi\rangle\Delta r_k - \langle X_{r_0}, \varphi\rangle\Big)\Big\}\Big|\mathcal{G}_s\Big] \\
&\qquad \text{(by (2.3))} \\
&\quad = E\Big[\exp\Big\{i\Big(\langle X_{r_{m-1}}, U_{r_{m-1},r_m}\varphi - U_{r_{m-1},r_m}A_{r_m}\varphi\Delta r_m\rangle \\
&\qquad - \sum_{k=1}^{m-1}\langle X_{r_k}, A_{r_k}\varphi\rangle\Delta r_k - \langle X_{r_0}, \varphi\rangle\Big)\Big\}\Big|\mathcal{G}_s\Big] H(r_{m-1}, r_m; \varphi - A_{r_m}\varphi\Delta r_m)
\end{aligned}$$

(iterating the same procedure, by (2.3) and (c))

$$= \exp\Big\{ i\langle X_s, U_{s,t}\varphi - \sum_{k=1}^{m} U_{s,r_k} A_{r_k}\varphi \Delta r_k - \varphi\rangle \Big\}$$
$$\cdot \prod_{k=1}^{m} H\Big(r_{k-1}, r_k; U_{r_k,t} - \sum_{j=k}^{m} U_{r_k,r_j} A_{r_j}\varphi \Delta r_j \Big).$$

Once again by the Banach-Steinhaus theorem, the function $r \longmapsto U_{s,r}A_r\varphi$ is continuous, hence

$$\lim_{n\to\infty} \langle X_s, U_{s,t}\varphi - \varphi - \sum_{k=1}^{m_n} U_{s,r_k^n} A_{r_k^n}\varphi (r_k^n - r_{k-1}^n)\rangle$$
$$= \Big\langle X_s, U_{s,t}\varphi - \varphi - \int_s^t U_{s,r}A_r\varphi\, dr \Big\rangle = 0$$

by assumption (d), and the proof is complete by virtue of (2.5).

COROLLARY 2.3. *Under the assumptions of Theorem 2.2, the process Z is unique in law and the distributions of its increments are given by the formula*

$$E\big[\exp\{ i(\langle Z_t, \varphi\rangle - \langle Z_s, \varphi\rangle)\}\big]$$
$$= \lim_{n\to\infty} \prod_{k=1}^{m_n} H\Big(r_{k-1}^n, r_k^n; U_{r_k^n,t}\varphi - \sum_{j=k}^{m_n} U_{r_k^n, r_j^n} A_{r_j^n}\varphi (r_j^n - r_{j-1}^n)\Big) \tag{2.6}$$

for each $\varphi \in \Phi$, $0 \le s < t \le T$, where $s = r_0^n < r_1^n < \cdots < r_{m_n}^n = t$, $n = 1, 2, \ldots$, is an arbitrary normal sequence of partitions of $[s, t]$.

COROLLARY 2.4. *Under the assumptions of Theorem 2.2, if the function $(s, t, \varphi) \longmapsto H(s, t, \varphi)$ is real-valued, then the distribution of the increments of Z are symmetric.*

Theorem 2.2 is fairly general and formula (2.6) expresses the distribution of Z by means of the distribution of X; however, being so general, formula (2.6) certainly is not (and probably cannot be) completely explicit. The following theorem yields a sufficient condition which permits to obtain the distributions of the increments of Z in an explicit form. In Section 3 it will be seen that this condition turns out to be general enough to cover important cases that appear in applications.

THEOREM 2.5. *Let A_t, $U_{s,t}$ satisfy (a), (b), (c), (d) and assume that $X = \{X_t : 0 \le t \le T\}$ is a Φ'-valued cadlag process such that if $\varphi \in \Phi$, $0 \le s \le t \le T$, then*

$$E[e^{i\langle X_t, \varphi\rangle} \mid \langle X_r, \psi\rangle, r \le s, \psi \in \Phi] = \exp\Big\{ i\langle X_s, U_{s,t}\varphi\rangle + \int_s^t h_r(U_{r,t}\varphi)\, dr \Big\}, \tag{2.7}$$

where h_r is a complex-valued function on Φ for $0 \le r \le T$, such that for each $\varphi \in \Phi$,

$$\sup_{0\le r\le T} |h_r(\varphi) - h_r(\psi)| \longrightarrow 0 \text{ as } \psi \longrightarrow \varphi, \tag{2.8}$$

and the function $r \longmapsto h_r(\varphi)$ is Borel-measurable. Then X satisfies (2.1), where Z is a process with independent increments such that

$$E[\exp\{ i(\langle Z_t, \varphi\rangle - \langle Z_s, \varphi\rangle)\}] = \exp\Big\{ \int_s^t h_r(\varphi)\, dr \Big\} \tag{2.9}$$

for each $\varphi \in \Phi$, $0 \le s \le t \le T$.

The proof of this theorem will be preceded by several lemmas. The first two of them are straightforward consequences of the assumptions of the theorem and of the fact that Φ is barrelled, so their proofs are omitted.

LEMMA 2.6. *(i) The function* $(s, t, \varphi) \mapsto U_{s,t}\varphi$ *is continuous.*

(ii) The function $(s, t, \varphi) \mapsto U_{s,t}A_t\varphi$ *is continuous.*

(iii) For each compact set $K \subset \Phi$ *the set* $\{ U_{s,t}\varphi : 0 \le s \le t \le T, \varphi \in K\}$ *is compact in* Φ.

(iv) For each $\varphi \in \Phi$ *the set* $\{ U_{s,t}A_t\varphi : 0 \le s \le t \le T\}$ *is compact in* Φ.

LEMMA 2.7. *For any compact set* $K \subset \Phi$ *and for each* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that if* $\varphi \in K$, $\psi \in \Phi$ *and* $\rho(\psi, \varphi) \le \delta$, *then*

$$\sup_{t \le T} |h_t(\varphi) - h_t(\psi)| \le \varepsilon.$$

Now, let us fix $\varphi \in \Phi$, $0 \le s < t \le T$ and $s = r_0^n < r_1^n < \cdots < r_{m_n}^n = t$, $n = 1, 2, \ldots$, a normal sequence of partitions of $[s, t]$.

LEMMA 2.8. *There exists a compact set* $K \subset \Phi$ *such that*

$$\left\{ U_{r_k^n,t}\varphi - \sum_{j=k}^{m_n} U_{r_k^n,r_j^n}A_{r_j^n}\varphi(r_j^n - r_{j-1}^n) : k = 1, \ldots, m_n,\ n = 1, 2, \ldots \right\} \subset K.$$

PROOF. By Lemma 2.6 (iv), the set

$$K_0 = \{ U_{r_k^n,r_j^n}A_{r_j^n}\varphi : k = 1, \ldots, m_n,\ n = 1, 2, \ldots\}$$

is relatively compact and

$$\left\{ \sum_{j=k}^{m_n} U_{r_k^n,r_j^n}A_{r_j^n}\varphi(r_j^n - r_{j-1}^n) : k = 1, \ldots, m_n,\ n = 1, 2, \ldots \right\} \subset (t-s)^{-1}\operatorname{co}(K_0, 0),$$

where $\operatorname{co}(K_0, 0)$ denotes the convex hull of $K_0 \cup \{0\}$. It is well-known that the set $K_1 = (t-s)^{-1}\operatorname{co}(K_0, 0)$ is relatively compact [24], and by Lemma 2.6(iii) the set $K_2 = \{ U_{r_k^n,t}\varphi : k = 1, \ldots, m_n,\ n = 1, 2, \ldots\}$ is also relatively compact. To complete the proof of the lemma it suffices to observe that the set $K_2 - K_1 = \{ \psi_2 - \psi_1 : \psi_1 \in K_1, \psi_2 \in K_2\}$ is relatively compact.

LEMMA 2.9. *For each compact set* $K \subset \Phi$,

$$\max_{1 \le k \le m_n} \sup_{r_{k-1}^n \le u \le r_k^n} \sup_{\psi \in K} |h_u(U_{u,r_k^n}\psi) - h_u(\psi)| \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

PROOF. For fixed K and $\varepsilon > 0$, let δ be given by Lemma 2.7. By Lemma 2.6(i) the function $(u, r, \psi) \mapsto U_{u,r}\psi$ is uniformly continuous on compact sets, hence

$$\max_{1 \le k \le m_n} \sup_{r_{k-1}^n \le u \le r_k^n} \sup_{\psi \in K} \rho(U_{u,r_k^n}\psi, \psi) < \delta$$

for n sufficiently large, since $\max_k(r_k^n - r_{k-1}^n) \longrightarrow 0$ as $n \longrightarrow \infty$. Now it is enough to apply Lemma 2.7.

LEMMA 2.10.

$$\max_{1\leq k\leq m_n}\sup_{0\leq u\leq T}\left|h_u\Big(U_{r_k^n,t}\varphi-\sum_{j=k+1}^{m_n}U_{r_k^n,r_j^n}A_{r_j^n}\varphi(r_j^n-r_{j-1}^n)\Big)-h_u(\varphi)\right|\longrightarrow 0$$

as $n\longrightarrow\infty$.

PROOF. Denote

$$\psi_{n,k}=U_{r_k^n,t}\varphi-\sum_{j=k+1}^{m_n}U_{r_k^n,r_j^n}A_{r_j^n}\varphi(r_j^n-r_{j-1}^n).$$

Fix $\varepsilon>0$, and let δ be given by Lemma 2.7 for $K=\{\varphi\}$. It suffices to prove that $\rho(\psi_{n,k},\varphi)\leq\delta$ for $k=1,\ldots,m_n$, for n sufficiently large. The metric ρ being invariant under shifts, it is enough to show that

$$\psi_{n,k}-\varphi\in B(\delta) \tag{2.10}$$

for $k=1,\ldots,m_n$, for n large, where $B(\delta)=\{\psi:\rho(0,\psi)\leq\delta\}$. We have

$$\varphi=U_{r_k^n,t}\varphi-\int_{r_k^n}^t U_{r_k^n,v}A_v\varphi\,dv$$

(assumption(d)), hence

$$\begin{aligned}\psi_{n,k}-\varphi&=\int_{r_k^n}^t U_{r_k^n,v}A_v\varphi\,dv-\sum_{j=k+1}^{m_n}U_{r_k^n,r_j^n}A_{r_j^n}\varphi(r_j^n-r_{j-1}^n)\\&=\int_{r_k^n}^t\Big(\tilde\psi_{n,k}(v)-\tilde{\tilde\psi}_{n,k}(v)\Big)\,dv,\end{aligned}$$

where

$$\begin{aligned}\tilde\psi_{n,k}(v)&=U_{r_k^n,v}A_v\varphi=\sum_{j=k+1}^{m_n}U_{r_k^n,v}A_v\varphi I_{[r_{j-1}^n,r_j^n[}(v),\\ \tilde{\tilde\psi}_{n,k}(v)&=\sum_{j=k+1}^{m_n}U_{r_k^n,r_j^n}A_{r_j^n}\varphi I_{[r_{j-1}^n,r_j^n[}(v).\end{aligned}$$

By Lemma 2.6(ii) the function $(r,v)\longmapsto U_{r,v}A_v\varphi$ is uniformly continuous on $\{(r,v):0\leq r\leq v\leq T\}$; hence there exists n_0 such that if $n\geq n_0$ then $U_{r_k^n,v}A_v\varphi-U_{r_k^n,r_j^n}A_{r_j^n}\varphi\in(t-s)^{-1}B(\delta)$ for $k=1,\ldots,m_n$, $v\in[r_{j-1}^n,r_j^n[$, $j=k+1,\ldots,m_n$. Therefore if $n\geq n_0$, then $\tilde\psi_{n,k}(v)-\tilde{\tilde\psi}_{n,k}(v)\in(t-s)^{-1}B(\delta)$ for $k=1,\ldots,m_n$, $v\in[r_k^n,t[$. But $(t-s)^{-1}B(\delta)$ is closed and convex (the metric ρ can be chosen that way), hence

$$(t-r_k^n)^{-1}\int_{r_k^n}^t\Big(\tilde\psi_{n,k}(v)-\tilde{\tilde\psi}_{n,k}(v)\Big)\,dv\in(t-s)^{-1}B(\delta)$$

for $k=1,\ldots,m_n$, $n\geq n_0$. (2.10) follows from this and the lemma is proved.

PROOF OF THEOREM 2.5. By virtue of Theorem 2.2 and Corollary 2.3 it suffices to prove that if $\varphi\in\Phi$, $0\leq s<t\leq T$ and $s=r_0^n<r_1^n<\cdots<r_{m_n}^n=t$, $n=1,2,\ldots$, is a normal sequence of partitions of $[s,t]$, then

$$\int_s^t h_r(\varphi)\,dr=\lim_{n\to\infty}\sum_{k=1}^{m_n}\int_{r_{k-1}^n}^{r_k^n}h_u\Big(U_{u,r_k^n}\Big(U_{r_k^n,t}\varphi-\sum_{j=k}^{m_n}U_{r_k^n,r_j^n}A_{r_j^n}\varphi(r_j^n-r_{j-1}^n)\Big)\Big)\,du.$$

The sum under the lim can be written as $a_n + b_n + c_n + d_n$, where (we omit the index n)

$$a_n = \sum_{k=1}^{m} \int_{r_{k-1}}^{r_k} \Big[h_u U_{u,r_k}\Big(\Big(U_{r_k,t}\varphi - \sum_{j=k}^{m} U_{r_k,r_j} A_{r_j}\varphi(r_j - r_{j-1})\Big)\Big)$$
$$-h_u\Big(U_{r_k,t}\varphi - \sum_{j=k}^{m} U_{r_k,r_j}A_{r_j}\varphi(r_j - r_{j-1})\Big)\Big]\, du,$$
$$b_n = \sum_{k=1}^{m} \int_{r_{k-1}}^{r_k} \Big[h_u\Big(U_{r_k,t}\varphi - \sum_{j=k}^{m} U_{r_k,r_j}A_{r_j}\varphi(r_j - r_{j-1})\Big)$$
$$-h_u\Big(U_{r_k,t}\varphi - \sum_{j=k+1}^{m} U_{r_k,r_j}A_{r_j}\varphi(r_j - r_{j-1})\Big)\Big]\, du,$$
$$c_n = \sum_{k=1}^{m} \int_{r_{k-1}}^{r_k} \Big[h_u\Big(U_{r_k,t}\varphi - \sum_{j=k+1}^{m} U_{r_k,r_j}A_{r_j}\varphi(r_j - r_{j-1})\Big) - h_u(\varphi)\Big]\, du,$$
$$d_n = \sum_{k=1}^{m} \int_{r_{k-1}}^{r_k} h_u(\varphi)\, du = \int_s^t h_u(\varphi)\, du.$$

The theorem will be proved if we show that $a_n \longrightarrow 0$, $b_n \longrightarrow 0$, $c_n \longrightarrow 0$ as $n \longrightarrow \infty$.

Fix $\varepsilon > 0$, and let K be the compact subset of Φ given by Lemma 2.8. We have by Lemma 2.9,

$$|a_n| \leq \sum_{k=1}^{m_n} \int_{r_{k-1}^n}^{r_k^n} \Big| h_u\Big(U_{u,r_k^n}(\ldots)\Big) - h_u(\ldots)\Big|\, du \leq \sum_{k=1}^{m_n} \int_{r_{k-1}^n}^{r_k^n} \varepsilon\, du = (t-s)\varepsilon$$

for n sufficiently large, so $a_n \longrightarrow 0$. Next, let δ be given by Lemma 2.7. We have

$$\rho\Big(U_{r_k^n,t}\varphi - \sum_{j=k}^{m_n} \ldots, U_{r_k^n,t}\varphi - \sum_{j=k+1}^{m_n} \ldots\Big) = \rho\Big(A_{r_k^n}\varphi(r_k^n - r_{k-1}^n), 0\Big) < \delta$$

for $k = 1, \ldots, m_n$ and n sufficiently large, since the set $\{A_{r_k^n}\varphi : k = 1, \ldots, m_n, n = 1, 2, \ldots\}$ is bounded in Φ. Then Lemma 2.7 implies that

$$|b_n| \leq \sum_{k=1}^{m_n} \int_{r_{k-1}^n}^{r_k^n} \varepsilon\, du = (t-s)\varepsilon$$

for n sufficiently large, so $b_n \longrightarrow 0$. Finally, Lemma 2.10 implies that $|c_n| \leq (t-s)\varepsilon$ for n sufficiently large, hence $c_n \longrightarrow 0$ and the proof is complete.

We will show now that in most cases condition (2.3) is necessary for a process in Φ' to satisfy (2.1), with Z having independent increments. To this end we shall need an existence and uniqueness result permitting to write down the solution of (2.1) in an explicit form.

We start by recalling briefly Ustunel's definition of the stochastic integral with respect to a Φ'-valued semimartingale [26, 27]. It should be observed, however, that in the Gaussian case (see Section 3) a more direct approach, developed in [3,4], can be used.

We will say that a set $V \subset \Phi'$ is an H-neighbourhood of 0 if it is convex, symmetric and closed, and its Minkowsky functional is a continuous separable Hilbertian seminorm in

Φ'. By $\Phi'(V)$ we will denote the corresponding Hilbert space, and the canonical mapping $\Phi' \longrightarrow \Phi'(V)$ will be denoted by $k(V)$. If K is a bounded subset of Φ then there exists B such that $K \subset B$ and B is the polar of an H-neighbourhood of 0 in Φ'. The polar B° of B is an H-neighbourhood of 0 in Φ' and the (Hilbert) spaces $\Phi[B]$, $\Phi'(B^\circ)$ are in duality, where $\Phi[B]$ denotes the completion of span(B) with respect to the Minkowsky functional of B.

Let $f\colon \Phi \longrightarrow K$ be a measurable function and $Z \equiv \{Z_t : 0 \leq t \leq T\}$ be a process in Φ' such that for each $\varphi \in \Phi$ the process $\{\langle Z_t, \varphi\rangle : 0 \leq t \leq T\}$ is a semimartingale (with respect to some filtration which is assumed to be given in advance). Then for any H-neighbourhood V of 0, the process $k(V)Z$ is a $\Phi'(V)$-valued semimartingale and the stochastic integral $\int_0^t \langle dZ_s, f(s)\rangle$ can be defined as $\int_0^t \langle dk(B^\circ)Z_s, f(s)\rangle$, the last integral being the usual stochastic integral in Hilbert space. This integral does not depend on a particular choice of B.

We are now ready to state our next result.

THEOREM 2.11. *(i) Let $A_t, U_{s,t}$ satisfy (a), (b), (c), (d) and let $\{\mathcal{F}_t : 0 \leq t \leq T\}$ be a filtration. Assume that $Z \equiv \{Z_t : 0 \leq t \leq T\}$ is a Φ'-valued cadlag process, $Z_0 = 0$, such that for each $\varphi \in \Phi$ the process $\{\langle Z_t, \varphi\rangle : 0 \leq t \leq T\}$ is a semimartingale. Then equation (2.1) has a solution of the form*

$$X_t = U^*_{0,t}X_0 + \int_0^t U^*_{s,t}\, dZ_s, \tag{2.11}$$

or, more precisely,

$$\langle X_t, \varphi\rangle = \langle X_0, U_{0,t}\varphi\rangle + \int_0^t \langle dZ_s, U_{s,t}\varphi\rangle \tag{2.12}$$

for $\varphi \in \Phi$, $0 \leq t \leq T$.

(ii) If $A_t, U_{s,t}$ satisfy additionally the "backward equation"

$$U_{s,t}\varphi = \varphi + \int_s^t A_r U_{r,t}\varphi\, dr \tag{2.13}$$

for $\varphi \in \Phi$, $0 \leq s \leq t \leq T$, then the solution given by (2.11) is unique up to modification.

PROOF. Given $\varphi \in \Phi$, we define

$$K = \{U_{s,t}\varphi : 0 \leq s \leq t \leq T\} \cup \{U_{s,t}A_t\varphi : 0 \leq s \leq t \leq T\},$$

which is a bounded subset of Φ by Lemma 2.6. We take some B as described above, and we have the well-defined integrals

$$\int_0^t \langle dZ_s, U_{s,t}\varphi\rangle = \int_0^t \langle dk(B^\circ)Z_s, U_{s,t}\varphi\rangle$$

and

$$\int_0^t \langle dZ_s, U_{s,t}A_t\varphi\rangle = \int_0^t \langle dk(B^\circ)Z_s, U_{s,t}A_t\varphi\rangle.$$

It is easy to see that the regularization theorem [11] can be applied to obtain that $\varphi \mapsto \int_0^t \langle dZ_s, U_{s,t}\varphi \rangle$ is (has a version which is) a Φ'-valued random variable, hence X defined by (2.12) is a Φ'-valued process.

The function $(r,s) \mapsto U_{r,s}A_s\varphi$ is measurable and bounded in $\Phi[B]$, hence, by a stochastic Fubini theorem in Hilbert space [18] the integral $\int_0^s \langle dk(B^\circ)Z_r, U_{r,s}A_s\varphi \rangle$ has a measurable version (as a function of (s,ω)) and

$$\int_0^t \int_0^s \langle dk(B^\circ)Z_r, U_{r,s}A_s\varphi \rangle \, ds = \int_0^t \langle dk(B^\circ)Z_r, \int_r^t U_{r,s}A_s\varphi \, ds \rangle$$
$$= \int_0^t \langle dZ_r, U_{r,t}\varphi \rangle - \langle Z_t, \varphi \rangle \text{ a.s.}$$

by assumption (d). This, together with (2.12) and assumption (d) once more yield

$$\int_0^t \langle X_s, A_s\varphi \rangle \, ds = \int_0^t \langle X_0, U_{0,s}A_s\varphi \rangle \, ds + \int_0^t \int_0^s \langle dk(B^\circ)Z_r, U_{r,s}A_s\varphi \rangle \, ds$$
$$= \langle X_0, U_{0,t}\varphi \rangle - \langle X_0, \varphi \rangle + \int_0^t \langle dZ_r, U_{r,t}\varphi \rangle - \langle Z_t, \varphi \rangle$$
$$= \langle X_t, \varphi \rangle - \langle X_0, \varphi \rangle - \langle Z_t, \varphi \rangle \text{ a.s.}$$

Thus, the first part of the theorem is proved (see (2.2)).

The second part follows immediately from the deterministic argument given in [15,16], so the proof is complete.

REMARK. Instead of the stochastic Fubini theorem, the integration by parts formula could be used, analogously as in [28], where a very special case is considered.

The following result is a reciprocal of Theorem 2.2.

COROLLARY 2.12. *Let $A_t, U_{s,t}$ satisfy (a), (b), (c), (d) and (2.13). Assume that X is a solution of (2.1) where Z is a process with independent increments, cadlag, $Z_0 = 0$, such that for each $\varphi \in \Phi$ the function $t \mapsto Ee^{i\langle Z_t, \varphi \rangle}$ has finite variation in $[0,T]$, and X_0 is independent of Z. Then (2.3) is satisfied for some deterministic H.*

PROOF. It suffices to observe that for any $\varphi \in \Phi$ the process $\{\langle Z_t, \varphi \rangle : 0 \le t \le T\}$ is a semimartingale with respect to the filtration generated by Z (eg. [13], Ch. II, Theorem 4.14), and then to apply Theorem 2.11, since (2.12) together with (c) imply

$$\langle X_t, \varphi \rangle = \langle X_s, U_{s,t}\varphi \rangle + \int_s^t \langle dZ_r, U_{r,t}\varphi \rangle \, dr \text{ a.s.}$$

for $\varphi \in \Phi$, $0 \le s \le t \le T$, and $\int_s^t \langle dZ_s, U_{r,t}\varphi \rangle$ is clearly independent of the σ-algebra generated by $\{X_r, \psi : 0 \le r \le s, \psi \in \Phi\}$.

COROLLARY 2.13. *If X satisfies the hypotheses of Theorem 2.5 and the backward equation (2.13) holds, then X is the unique solution of equation (2.1) and it is given in the form (2.11).*

REMARK. In the Gaussian case [1,2] we have referred to the relationship between the covariance functionals of the processes X and Z in equation (1.1) (formula (3.2) below) as a "fluctuation-dissipation relation," abstracting this concept from statistical mechanics. In this case the distributions of X and Z are completely determined by their respective covariance functionals. In the setting of Theorem 2.5 the distributions of X and Z are determined by formulas (2.7) and (2.9), and these expressions also contain the relationship between the two distributions. Therefore, abstracting further, we may call the pair (2.7)–(2.9) a "generalized fluctuation-dissipation relation."

3. **Special cases.** In this section we show some examples of applications of Theorem 2.2 and Theorem 2.5. Firstly we will consider the Gaussian case. This case was discussed in [1,2] and we now present it in the new framework.

THEOREM 3.1. *Let $A_t, U_{s,t}$ satisfy (a), (b), (c), (d). Assume that $X \equiv \{X_t : 0 \leq t \leq T\}$ is a continuous, centered Gaussian Φ'-valued process whose covariance functional*

$$K_X(s,\varphi;t,\psi) = E(\langle X_s,\varphi\rangle\langle X_t,\psi\rangle), \quad \varphi,\psi \in \Phi, s,t \in [0,T],$$

is such that

$$K_X(s,\varphi;t,\psi) = K_X(s,\varphi;s,U_{s,t}\psi) \tag{3.1}$$

for $0 \leq s \leq t \leq T, \varphi,\psi \in \Phi$. Then X is a Markov process and satisfies equation (2.1), where the process Z is centered Gaussian, continuous, with independent increments, and its convariance functional is given by

$$\begin{aligned} K_Z(s,\varphi,t,\psi) &= K_X(s\wedge t,\varphi;s\wedge t,\psi) - K_X(0,\varphi;0,\psi) \\ &\quad - \int_0^{s\wedge t} [K_X(r,A_r\varphi;r,\psi) + K_X(r,\varphi;r,A_r\psi)]\,dr. \end{aligned} \tag{3.2}$$

PROOF. Fix $0 \leq s < t < T$ and $\varphi \in \Phi$. Let $\mathcal{G}_s$ denote the σ-algebra generated by $\{\langle X_r,\psi\rangle : r \leq s, \psi \in \Phi\}$. The conditional distribution of $\langle X_t,\varphi\rangle$ under the condition of $\mathcal{G}_s$ is Gaussian with mean $E[\langle X_t,\varphi\rangle \mid \mathcal{G}_s]$ and variance

$$E\langle X_t,\varphi\rangle^2 - E(E[\langle X_t,\varphi\rangle \mid \mathcal{G}_s])^2.$$

On the other hand, (3.1) and (c) imply that

$$E[\langle X_t,\varphi\rangle \mid \mathcal{G}_s] = \langle X_s, U_{s,t}\varphi\rangle,$$

hence we obtain

$$E[e^{i\langle X_t,\varphi\rangle} \mid \mathcal{G}_s] = e^{i\langle X_s,U_{s,t}\varphi\rangle} \exp\Big\{-\frac{1}{2}[K_X(t,\varphi;t,\varphi) - K_X(s,U_{s,t}\varphi;s,U_{s,t}\varphi)]\Big\}.$$

Thus we have obtained (2.3) with

$$H(s,t;\varphi) = \exp\Big\{-\frac{1}{2}[K_X(t,\varphi;t,\varphi) - K_X(s,U_{s,t}\varphi;s,U_{s,t}\varphi)]\Big\}, \tag{3.3}$$

so we can apply Theorem 2.2. It is clear that the process Z given by that theorem is centered Gaussian and continuous. The covariance functional of Z can now be obtained from (3.3) with the help of Corollary 2.3.

REMARKS. (i) In Theorem 3.1 the covariance condition (3.1) can be replaced by

$$E[\langle X_t, \varphi\rangle \mid \langle X_r, \psi\rangle, r \le s, \psi \in \Phi] = \langle X_s, U_{s,t}\varphi\rangle, \quad \varphi \in \Phi, \quad 0 \le s \le t \le T.$$

(ii) If in addition to the assumptions of Theorem 3.1 we require that the function $s \longmapsto K_X(s, \varphi; s, \varphi)$ be continuously differentiable, then the covariance functional of Z has the form

$$K_Z(s, \varphi; t, \psi) = \int_0^{s\wedge t} \Big[\frac{d}{dr} K_X(r, \varphi, r, \psi) - K_X(r, A_r\varphi; r, \psi) - K_X(r, \varphi; r, A_r\psi)\Big]\, dr \tag{3.4}$$

for φ, $\psi \in \Phi$, $s, t \in [0, T]$. Such a process Z is called in [1,2] a generalized Wiener process. See [4] for a discussion of its properties.

(iii) Suppose that the assumptions of Remark (ii) are satisfied and in addition the backward equation (2.13) holds (this situation occurs most frequently in applications). Then formula (3.4) follows from Theorem 2.5 with the function h_r given by

$$h_r(\psi) = -\frac{1}{2}\Big[\frac{d}{dr} K_X(r, \psi; r, \psi) - 2K_X(r, A_r\psi; r, \psi)\Big].$$

Indeed, it is easily seen that the condition (2.8) is satisfied and (see (3.3)) we have

$$K_X(t, \varphi; t, \varphi) - K_X(s, U_{s,t}\varphi; s, U_{s,t}\varphi) = \int_s^t \frac{d}{dr} K_X(r, U_{r,t}\varphi; r, U_{r,t}\varphi)\, dr,$$

so it suffices to observe that

$$\begin{aligned}
&\lim_{h\to 0} \frac{1}{h}[K_X(r+h, U_{r+h,t}\varphi; r+h, U_{r+h,t}\varphi) - K_X(r, U_{r,t}\varphi; r, U_{r,t}\varphi)] \\
&\quad = \lim_{h\to 0}\Big\{\frac{1}{h}[K_X(r+h, U_{r+h,t}\varphi; r+h, U_{r+h,t}\varphi) - K_X(r, U_{r+h,t}\varphi; r, U_{r+h,t}\varphi)] \\
&\qquad + \frac{1}{h}[K_X(r, U_{r+h,t}\varphi; r, U_{r+h,t}\varphi) - K_X(r, U_{r,t}\varphi; r, U_{r+h,t}\varphi)] \\
&\qquad + \frac{1}{h}[K_X(r, U_{r,t}\varphi; r, U_{r+h,t}\varphi) - K_X(r, U_{r,t}\varphi; r, U_{r,t}\varphi)]\Big\} \\
&\quad = -2h_r(U_{r,t}\varphi),
\end{aligned}$$

by (2.13) and the Banach-Steinhaus theorem.

The next example is a special case of a process obtained in [5] as a fluctuation limit of a branching particle system, where the branching law belongs to the domain of normal attraction of a stable law with exponent $1+\beta, 0 < \beta \le 1$.

Let $\mathcal{S}(R^d)$ denote the space of infinitely differentiable functions on R^d, $d \ge 1$, which are rapidly decreasing at infinity together with all their derivatives. Endowed with its usual topology, $\mathcal{S}(R^d)$ is a nuclear Fréchet space [24]. Let $\{T_t, 0 \le t \le T\}$ denote the semigroup associated with the standard Wiener process in R^d. Then $A_t \equiv \frac{1}{2}\Delta$ and $U_{s,t} \equiv T_{t-s}$ satisfy (a), (b), (c), (d) with $\Phi = \mathcal{S}(R^d)$.

The process X found in [5] is an $\mathcal{S}'(R^d)$-valued cadlag process whose conditional characteristic function is given by

$$\begin{aligned} &E[e^{i\langle X_t,\varphi\rangle} \mid \langle X_r,\psi\rangle, r \le s, \psi \in \mathcal{S}(R^d)] \\ &\quad = \exp\Big\{ i\langle X_s, T_{t-s}\varphi\rangle + \int_s^t \int_{R^d} T_r(-iT_{t-r}\varphi)^{1+\beta}(x)\mu(dx)\,dr \Big\}, \end{aligned}$$

for $\varphi \in \mathcal{S}(R^d)$, $0 \le s \le t \le T$, where μ is a Radon measure on R^d. Hence condition (2.7) of Theorem 2.5 is satisfied with

$$h_r(\varphi) = \int_{R^d} T_r(-i\varphi)^{1+\beta}(x)\mu(dx),$$

which is clearly measurable in r.

We must show that condition (2.8) is fulfilled. Note that

$$(-iu)^{1+\beta} = |u|^{1+\beta}\Big(\cos\frac{\pi}{2}(1+\beta) - i(\operatorname{sgn} u)\sin\frac{\pi}{2}(1+\beta)\Big)$$

for real u. We have

$$\sup_{0\le r\le T} |h_r(\varphi) - h_r(\psi)| \le \sup_{0\le r\le T} \int_{R^d} T_r(|(-i\varphi)^{1+\beta} - (-i\psi)^{1+\beta}|)(x)\mu(dx).$$

Now, $\psi \longrightarrow \varphi$ in $\mathcal{S}(R^d)$ implies that $\psi \longrightarrow \varphi$ pointwise and boundedly, hence

$$|(-i\varphi)^{1+\beta} - (-i\psi)^{1+\beta}| \longrightarrow 0$$

pointwise and boundedly, and therefore $T_r|(-i\varphi)^{1+\beta} - (-i\psi)^{1+\beta}|(x) \longrightarrow 0$ for each $x \in R^d$ and $0 \le r \le T$. On the other hand let $\varphi_p(x) = (1+|x|^2)^{-p}$, $x \in R^d$, with some $p > d/2$. Then for each $x \in R^d$ and $0 \le r \le T$ we have

$$\begin{aligned} T_r|(-i\varphi)^{1+\beta} - (-i\psi)^{1+\beta}|(x) &\le T_r|\varphi|^{1+\beta} + T_r|\psi|^{1+\beta} \\ &\le \Big[\sup_{x\in R^d}\big(|\varphi(x)| / \varphi_p(x)\big)^{1+\beta} + \sup_{x\in R^d}\big(|\psi(x)| / \varphi_p(x)\big)^{1+\beta}\Big] T_r\varphi_p(x). \end{aligned}$$

The term in brackets is uniformly bounded as $\psi \longrightarrow \varphi$, and $\sup_{0\le r\le T} T_r\varphi_p(x) \le C\varphi_p(x)$ for some constant C [5]. Finally, φ_p is integrable with respect to μ by assumption, and therefore the desired result follows by the dominated convergence theorem.

We can now apply Theorem 2.5 to conclude that the process X satisfies the $\mathcal{S}'(R^d)$-valued Langevin equation

$$dX_t = \frac{1}{2}\Delta X_t\,dt + dZ_t, \quad 0 \le t \le T,$$

where Z is a process with independent increments such that

$$Ee^{i(\langle Z_t,\varphi\rangle - \langle Z_s,\varphi\rangle)} = \exp\Big\{\int_s^t \int_{R^d} T_r(-i\varphi)^{1+\beta}(x)\mu(dx)\,dr\Big\}$$

for $\varphi \in \mathcal{S}(R^d)$, $0 \leq s \leq t \leq T$.

REMARKS. (i) Since the linear combinations of the random variables $\langle X_{t_1}, \varphi_1 \rangle, \ldots,$ $\langle X_{t_m}, \varphi_m \rangle$ have stable distributions, X is an example of a stable generalized Ornstein-Uhlenbeck process.

(ii) Except in the Gaussian case ($\beta = 1$), the increments of Z are not symmetric (see Corollary 2.4).

(iii) In [5,6] the semigroup $\{T_t\}$ is that of a spherically symmetric stable process in R^d with exponent α, $0 < \alpha \leq 2$. The case $\alpha = 2$ corresponds to the example given above. For $\alpha < 2$, $A = -(-\Delta)^{\alpha/2}$, which does not map $\mathcal{S}(R^d)$ into itself. In this case the formulation developed in this paper is not applicable since the Langevin equation cannot be interpreted in the form (2.2), and a generalized type of solution is needed [6].

REFERENCES

1. T. Bojdecki, L. G. Gorostiza, *Langevin equations for $\mathcal{S}'$-valued Gaussian process and fluctuation limits of particle systems*, Probab. Th. Rel. Fields **73**(1986), 227–244.

2. ———, *Inhomogeneous infinite dimensional Langevin equations*, Stoch. Anal. Appl. **6**(1988), 1–9.

3. T. Bojdecki, J. Jakubowski, *Ito stochastic integral in the dual of a nuclear space*, J. Multiv. Anal. **31**(1989), 40–58.

4. ———, *Stochastic integration for inhomogeneous Wiener process in the dual of a nuclear space*, J. Multiv. Anal. **34**(1990), 185–210.

5. D. A. Dawson, K. Fleischmann, L. G. Gorostiza, *Stable hydrodynamic limit fluctuations of a critical branching particle system in a random medium*, Ann. Probab. **17**(1989), 1083–1117.

6. D. A. Dawson, L. G. Gorostiza, *Generalized solutions of a class of nuciear space valued stochastic evolution equations*, Appl. Math. Optim. **22**(1990), 241–263.

7. B. Fernández, *Markov properties of the fluctuation limit of a particle system*, J. Math. Anal. App. **149**(1990), 160–179.

8. L. G. Gorostiza, *High density limit theorems for infinite systems of unscaled branching Brownian motions*, Ann. Probab. **11**(1983), 374–392.

9. ———, *The demographic variation process of branching random fields*, J. Multiv. Anal. **25**(1988), 174–200.

10. R. Holley, D. W. Stroock, *Generalized Ornstein-Uhlenbeck processes and infinite particle branching Brownian motions*, Publ. RIMS, Kyoto Univ. **14**(1981), 741–788.

11. K. Itô, *Foundations of stochastic differential equations in infinite dimensional spaces*. SIAM, Philadelphia, 1984.

12. ———, *Distribution valued processes arising from independent Brownian motions*, Math. Z. **182**(1988), 17–33.

13. J. Jacod, A. V. Shiryaev, *Limit theorems for stochastic processes*. Springer-Verlag, Berlin, 1987.

14. G. Kallianpur, R. Wolpert, *Infinite dimensional stochastic differential equation models for spatially distributed neurons*, Appl. Math. Optim. **12**(1984), 125–172.

15. G. Kallianpur, V. Pérez-Abreu, *Stochastic evolution equations with values on the dual of a countably Hilbert nuclear space*. Tech. Rep. **145**, Center for Stochastic Processes, Univ. North Carolina, Chapel Hill, 1986.

16. ———, *Stochastic evolution eqautions driven by nuclear space valued martingales*, Appl. Math. Optim. **17**(1988), 237–272.

17. P. Kotelenez, *Fluctuations near homogeneous states of chemical reactions with diffusion*, Adv. Appl. Probab. **19**(1987), 352–370.

18. J. A. León, *Stochastic Fubini theorem for semimartingales in Hilbert space*, Canadian J. Math., **42**(1990) 890–901.

19. A. Martin-Löf, *Limit theorems for the motion of a Poisson system of independent Markovian particles with high density*, Z. Wahrscheinlichkeitstheor. Verw. Geb. **34**(1976), 205–223.

20. S. Méléard, S. Roelly-Coppoleta, *A generalized equation for a continuous measure branching process*. In Stochastic partial differential equations and applications II. (Da Prato, G., Tubaro, L., ed.)., Springer-Verlag Lecture Notes in Math. **1390**. Berlin, 1989, 179–185.

21. I. Mitoma, *On the sample continuity of $\mathcal{S}'$-processes*, J. Math, Soc. Japan **35**(1983), 629–636.

22. ———, *Generalized Ornstein-Uhlenbeck process having a characteristic operator with polynomial coefficients*, Probab. Th. Rel. Fields **76**(1987), 533–555.

23. H. H. Schaefer, *Topological vector spaces*. Springer-Verlag, Berlin, 1971.

24. F. Treves, *Topological Vector spaces, distributions and kernels*. Academic Press, New York, 1967.

25. K. Uchiyama, *Fluctuations of Markovian systems in Kac's charicature of a Maxwellian gas*, J. Math. Soc. Japan **35**(1983), 27–53.

26. S. Ustunel, *Stochastic integration on nuclear spaces and its applications*, Ann. Inst. Henri Poincaré **XVIII**, 1982, 165–200.

27. ———, *A characterization of semimartingales on nuclear spaces*, Z. Wahrscheinlichkeitstheor. Vebw. Geb. **60**(1982), 21–39. Erratum **63**(1983), 553–554.

28. ———, *Some applications of stochastic integration in infinite dimensions*, Stochastics **7**(1982), 255–288.

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