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Proof of a conformal mapping relationship

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A proof is given of a result deduced from numerical work on the conformal mapping of the region exterior to a quadrilateral possessing an axis of symmetry. It is shown that the result follows from known relationships between hypergeometric functions.

1. Introduction and preliminary work

In a recent paper [2], Hughes mentioned a relationship he had found in the course of work on a conformal mapping problem. He wished to map the region exterior to a quadrilateral ABCD , with an axis of symmetry AC , on to the complex plane and to make use of the symmetry he took points

$$F_1(-\infty),\ A_1(-1),\ B_1(0),\ C_1(r),\ G_1(+\infty)$$

on the real axis in the z-plane and mapped them into

$$F(-\infty),\ A\left(-d_1\cos\alpha\pi\right),\ B(hi),\ C\left(d_2\cos\beta\pi\right),\ G(+\infty)$$

in the w-plane by a Schwarz-Christoffel transformation

$$w = hi + \int_0^z \left\{u^{\alpha+\beta}(u+1)^{-\alpha}(u-r)^{-\beta}\right\}du \ . \tag{1.1}$$

In the triangle ABC , d_1 and d_2 are the lengths of the sides AB and BC , while $\alpha\pi$ and $\beta\pi$ are the (interior) angles at A and C . The height of the triangle is $h = d_1\sin\alpha\pi = d_2\sin\beta\pi$. In the mapping, the positive constant r is initially unknown and has to be chosen so that A

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and C lie on the real axis in the w-plane. Hughes found from numerical work that the appropriate solution for r is

$$r = \alpha/\beta \ , \tag{1.2}$$

but was unable to verify this analytically.

From equation (1.1),

$$d_1 \exp(i\alpha\pi) = \int_{-1}^{0} \{u^{\alpha+\beta}(u+1)^{-\alpha}(u-r)^{-\beta}\} du \ ,$$

$$d_2 \exp(-i\beta\pi) = \int_{0}^{r} \{u^{\alpha+\beta}(u+1)^{-\alpha}(u-r)^{-\beta}\} du \ ,$$

and hence

$$d_1 = \int_0^1 \{x^{\alpha+\beta}(1-x)^{-\alpha}(r+x)^{-\beta}\} dx \ , \tag{1.3}$$

$$d_2 = \int_0^r \{x^{\alpha+\beta}(1+x)^{-\alpha}(r-x)^{-\beta}\} dx \ . \tag{1.4}$$

These equations correspond to equations (1.3) and (1.4) in Hughes's paper, which unfortunately both contain minor slips. (His equation (1.3) requires a minus sign on the right-hand side and the integrand in his equation (1.4) should have $x^{\alpha+\beta}$ in place of $r^{\alpha+\beta}$.)

The substitution $x = rw/(r+1-w)$ in (1.3) gives

$$d_1 = r^{\alpha+1} s^{\alpha+\beta+1} \int_0^1 (1-w)^{-\alpha} w^{\alpha+\beta} (1-sw)^{-2} dw \ , \tag{1.5}$$

where $s = 1/(r+1)$. Since s lies between 0 and 1 , $(1-sw)^{-2}$ can be expanded in the integrand and this leads to

$$d_1 = s^{\beta}(1-s)^{\alpha+1}\{\Gamma(\alpha+\beta+1)\Gamma(1-\alpha)/\Gamma(\beta+2)\}F(2,\ \alpha+\beta+1;\ \beta+2;\ s) \ . \tag{1.6}$$

Essentially, this is using Euler's formula for the hypergeometric function $F(a, b; c; z)$ [1, p. 59, equation (10)]. Hughes [2, p. 103] gives an equivalent expression for d_1 in terms of a hypergeometric function but in the sequel it turns out to be an advantage to have one of the parameters an integer.

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In the same way, the substitution $x = rw/(r+1-rw)$ in (1.4) gives

$$(1.7)\quad d_2 = r^{\alpha+1}s^{\alpha+\beta+1}\int_0^1 (1-w)^{-\beta}w^{\alpha+\beta}\{1-(1-s)w\}^{-2}dw$$

$$= s^{\beta}(1-s)^{\alpha+1}\{\Gamma(\alpha+\beta+1)\Gamma(1-\beta)/\Gamma(\alpha+2)\}F(2,\ \alpha+\beta+1;\ \alpha+2;\ 1-s)\ .$$

2. Solution for r

The equation to determine r , or s , is that

$$(2.1)\qquad d_1\sin\alpha\pi = d_2\sin\beta\pi$$

and from equations (1.6) and (1.7) this can be written

$$(2.2)\qquad u_2 = u_1\{\Gamma(1-\alpha)\Gamma(2+\alpha)\sin\alpha\pi\}/\{\Gamma(1-\beta)\Gamma(\beta+2)\sin\beta\pi\}$$

$$= u_1\{\alpha(1+\alpha)/\beta(1+\beta)\}\ ,$$

where

$$(2.3)\qquad u_1 = F(2,\ \alpha+\beta+1;\ \beta+2;\ s)\ ,\quad u_2 = F(2,\ \alpha+\beta+1;\ \alpha+2;\ 1-s)\ ;$$

the usual gamma function properties

$$\Gamma(\alpha+2) = (\alpha+1)\alpha\Gamma(\alpha)\ ,\quad \Gamma(\alpha)\Gamma(1-\alpha)\sin\alpha\pi = \pi\ ,$$

have been used to obtain the second line of equation (2.2). In general, if

$$(2.4)\qquad u_1 = F(a,\ b;\ c;\ z)\ ,\quad u_2 = F(a,\ b;\ a+b+1-c;\ 1-z)\ ,$$

then u_1 and u_2 are solutions of the hypergeometric equation

$$(2.5)\qquad z(1-z)u'' + \{c-(a+b+1)z\}u' - abu = 0$$

and when $a = 2$, the equation also has a solution of finite form

$$(2.6)\qquad u_5 = z^{1-c}(1-z)^{c-b-2}[1-\{(b-1)/(c-2)\}z]\ ;$$

[1, Section 2.2]. In this case, u_1 and u_5 are linearly independent solutions of equation (2.5) and u_2 can be written as a linear combination of u_1 and u_5 in the form

$$(2.7)\qquad u_2 = \frac{\Gamma(b+3-c)\Gamma(1-c)}{\Gamma(3-c)\Gamma(b+1-c)}u_1 + \frac{\Gamma(b+3-c)\Gamma(c-1)}{\Gamma(b)}u_5$$

[1, p. 107, equation (35)]. In the problem we are concerned with $a = 2$, $b = \alpha + \beta + 1$, $c = \beta + 2$ and hence

$$(2.8) \qquad \frac{\Gamma(b+3-c)\Gamma(1-c)}{\Gamma(3-c)\Gamma(b+1-c)} = \frac{\Gamma(\alpha+2)\Gamma(-1-\beta)}{\Gamma(1-\beta)\Gamma(\alpha)} = \frac{(\alpha+1)\alpha}{(\beta+1)\beta} .$$

Thus equation (2.7) becomes

$$(2.9) \qquad u_2 = u_1\{\alpha(1+\alpha)/\beta(1+\beta)\} + u_5\{\Gamma(\alpha+2)\Gamma(\beta+1)/\Gamma(\alpha+\beta+1)\} .$$

Comparing this with equation (2.2) gives $u_5 = 0$, that is,

$$(2.10) \qquad 0 = s^{-1-\beta}(1-s)^{-1-\alpha}[1-\{(\alpha+\beta)/\beta\}s] ,$$

and hence $s = \beta/(\alpha+\beta)$ and $r = \alpha/\beta$.

In view of the protean character of the hypergeometric function [3, Chapter 23, introductory lines], the main problem in obtaining the expression for r was to select appropriate forms and relationships between them. In doing this, essential guidance came from the clear way in which the properties of the hypergeometric function are explained and tabulated in [1].

Added in proof, 20 December 1973. Dr O.F. Hughes has independently obtained a proof of the relationship $r = \alpha/\beta$, using similar methods.

References

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