A note on rectifiable curves

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Let f be a continuous complex-valued function of a real parameter whose real and imaginary parts are of bounded variation in the range (a, b) of the parameter, so that the range of f is a rectifiable plane curve. The main results connecting the arc-length s with the parametrization are as follows:

THEOREM 1 (Tonelli). *For any rectifiable curve,*

$$\int_{\alpha}^{\beta} | f'(t) | \, dt \leqslant s(\beta) - s(\alpha),$$

equality holding for all α, β $(a \leqslant \alpha < \beta \leqslant b)$ *if and only if* f *is absolutely continuous in* (a, b).

THEOREM 2 (Pollard). *If* f' *exists in some neighbourhood of* t_0 *and if* $| f' |$ *is upper semi-continuous at* t_0, *then* s *is differentiable at* t_0 *and* $s'(t_0) = | f'(t_0) |$.

The purpose of this note is to show how these results may be deduced from the following simple lemma:

If f *is absolutely continuous in* (α, β), *then*

$$s(\beta) - s(\alpha) \leqslant \int_{\alpha}^{\beta} | f'(t) | \, dt. \tag{1}$$

To prove this, let $\alpha = t_0 < t_1 < \ldots . < t_n = \beta$ be any sub-division of (α, β). Then

$$\begin{aligned} \sum_{i=1}^{n} | f(t_i) - f(t_{i-1}) | &= \sum_{i=1}^{n} \left| \int_{t_{i-1}}^{t_i} f'(t) dt \right| \\ &\leqslant \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} | f'(t) | \, dt \\ &= \int_{\alpha}^{\beta} | f'(t) | \, dt. \end{aligned}$$

The arc-length being by definition the upper bound of the lengths of

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inscribed polygons, the result follows by taking the upper bound of the left hand member with respect to all subdivisions of (α, β) into a finite set of subintervals.

We use also the fact that, for any rectifiable curve,

$$| f(\beta) - f(\alpha) | \leqslant s(\beta) - s(\alpha) \tag{2}$$

(since the chord does not exceed the arc-length).

To deduce Theorem 1, we note that f' and s' exist p.p. and are integrable, because we are dealing with functions of bounded variation. It follows from (2) that $| f'(t) | \leqslant s'(t)$ p.p., whence

$$\int_{\alpha}^{\beta} | f'(t) | \, dt \leqslant \int_{\alpha}^{\beta} s'(t) dt \leqslant s(\beta) - s(\alpha),$$

since s is non-decreasing. If f is absolutely continuous in (a, b), then equality holds because of (1). Conversely, if equality holds then s is an integral, hence absolutely continuous and (2) implies that f also is absolutely continuous.

To prove Theorem 2 we observe that there is a neighbourhood of t_0 in which f' is bounded and hence f absolutely continuous. If t is any point of this neighbourhood other than t_0, we deduce from (1) and (2) that

$$\left| \frac{f(t) - f(t_0)}{t - t_0} \right| \leqslant \frac{s(t) - s(t_0)}{t - t_0} \leqslant \frac{1}{t - t_0} \int_{t_0}^{t} | f'(u) | \, du \leqslant \sup_{|u - t_0| < |t - t_0|} | f'(u) | \,.$$

In the limit we have equality between the extreme members and hence equality throughout: $s'(t_0)$ exists and has the value $| f'(t_0) |$.

The argument just given would establish the conclusion of Theorem 2 on weaker assumptions than those made (but more cumbersome). Since the argument uses absolute continuity of f in some neighbourhood of t_0, it will not, however, produce necessary and sufficient conditions.

The proofs given are not restricted to plane curves, but apply equally well to higher dimensions. It is necessary only to regard f as taking its values in n-dimensional space, to interpret requirements such as having real and imaginary parts of bounded variation as applying to each coordinate separately, and to replace the absolute value of a complex number by the Euclidean distance of a point from the origin (i.e. by the norm or length of the vector). It would even be possible to carry out the argument in certain spaces of infinite dimension.

The theorem (due to Lebesgue) that a rectifiable curve has a tangent p.p. is an easy corollary of the above. Assuming the curve parametrized with respect to its arc-length, equation (2) is a Lipschitz condition and hence equality holds in Tonelli's theorem; this gives $|f'(s)| = 1$ p.p., as required. It should be noted that a rectifiable curve can always be parametrized with respect to its arc-length (and in particular it is not necessary for s to be a strictly increasing function of t). One need only define $\phi(u) = f(t)$ for any t such that $s(t) = u$: this is justified by (2), which shows that if $s(\alpha) = s(\beta)$, then $f(\alpha) = f(\beta)$. It is now easy to verify that the arc-length is the same for both parametrizations, and that for the second the arc-length is u.

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