A SPECTRAL RADIUS PROBLEM CONNECTED WITH WEAK COMPACTNESS

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0. Introduction. The asymptotic behaviour has been determined for several natural geometric or topological quantities related to (degrees of) compactness of bounded linear operators on Banach spaces; see for instance [**24**], [**25**] and [**17**]. This paper complements these results by studying the spectral properties of some quantities related to weak compactness.

Let E and F be Banach spaces. The bounded linear operator $S \in L(E, F)$ is weakly compact, and denoted $S \in W(E, F)$, if the image SB_E of the closed unit ball of E is relatively compact in the weak topology of F. The deviation of $S \in L(E, F)$ from weak compactness is measured both by the geometric quantity

$$\omega(S) = \inf\{\varepsilon > 0 \mid SB_E \subset K + \varepsilon B_F, K \text{ weakly compact in } F\}$$

due to de Blasi and by the quotient norm $\|S\|_w = \text{dist}(S, W(E, F))$.

Suppose that E is a complex Banach space. It is known that ω is a submultiplicative seminorm on $L(E)$ that vanishes on the closed ideal $W(E)$ and that $\omega(S) \leq \|S\|_w$ for all operators S (see [**2**]). Hence the limit $\lim_{n\to\infty} \omega(S^n)^{1/n} = \inf_{n\geq 1} \omega(S^n)^{1/n}$ exists for all $S \in L(E)$. This paper considers the natural problem whether it possesses a concrete spectral interpretation. In particular, does

$$\lim_{n\to\infty} \omega(S^n)^{1/n} = \max\{|\lambda| : \lambda \in \sigma(S + W(E))\} \tag{0.1}$$

hold for all $S \in L(E)$ on non-reflexive Banach spaces E? Here $\sigma(S + W(E))$ denotes the spectrum of the quotient element $S + W(E)$ in the generalized Calkin algebra $L(E)/W(E)$ and the right-hand side is its radius $r_\sigma(S + W(E))$. The Gelfand–Beurling spectral radius formula states that

$$r_\sigma(S + W(E)) = \lim_{n\to\infty} \|S^n\|_w^{1/n} \text{ whenever } S \in L(E). \tag{0.2}$$

This problem is approached with the help of algebraic semigroups related to the tauberian and the cotauberian operators. The equality (0.1) is also verified for operators on several classical non-reflexive spaces having the Dunford–Pettis property by comparing ω and $\| \ \|_w$. These computations complement the results of [**3**]. Finally, an asymptotic formula is proved on separable non-reflexive spaces for the inner radius of a spectral subset related to a subclass of the tauberian operators.

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1. The tauberian spectrum. Let A be a Banach algebra. The spectrum of the element $a \in A$ is denoted by $\sigma(a)$ and its radius $\max\{|\lambda| : \lambda \in \sigma(a)\}$ by $r_\sigma(a)$. If E is a Banach space, then $K(E)$ stands for the closed ideal of $L(E)$ consisting of the compact operators on E.

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It was known to Gohberg et al. in the Calkin algebra case that

$$\lim_{n\to\infty} \gamma(S^n)^{1/n} = r_\sigma(S + K(E)) \tag{1.1}$$

for all $S \in L(E)$ and all complex Banach spaces E (see [**9**] or [**24**, 3.3 and 3.4]). The Hausdorff measure of noncompactness

$$\gamma(S) = \inf\{\varepsilon > 0 \mid SB_E \subset K + \varepsilon B_E, K \text{ a finite set in } E\}$$

of $S \in L(E)$ is the compact counterpart of the seminorm ω. The equality (0.1) is clearly suggested by (1.1).

We recall some algebraic semigroups of operators. Any Banach space E is viewed as canonically embedded into its bidual E''. The operator $S \in L(E, F)$ induces an operator $R(S) \in L(E''/E, F''/F)$ through $R(S)(x'' + E) = S''x'' + F$ for $x'' + E \in E''/E$. Set

$$\tau(E, F) = \{S \in L(E, F) : z'' \in E \text{ whenever } S''z'' \in F \text{ and } z'' \in E''\},$$

$$\text{co}\,\tau(E, F) = \{S \in L(E, F) : S' \in \tau(F', E')\},$$

$$\Phi_i(E, F) = \{S \in L(E, F) : R(S) \text{ is bijective}\}.$$

The tauberian operators τ and the cotauberian operators co τ were introduced by Kalton and Wilansky [**12**] respectively by Tacon [**18**]. Alternatively, $S \in \text{co}\,\tau(E, F)$ if and only if $\overline{\text{Im}\, S'' + F} = F''$ [**18**, p. 65]. Evidently $S \in \tau(E, F)$ if and only if $R(S)$ is injective while $S \in \text{co}\,\tau(E, F)$ if and only if $R(S)$ has dense range in F''/F. Moreover, $S \in L(E)$ is W-invertible, denoted by $S \in \Phi_W(E)$, if there are $T_i \in L(E)$ and weakly compact $V_i \in W(E)$ $(i = 1, 2)$ such that $T_1 S = \text{Id} + V_1$ and $ST_2 = \text{Id} + V_2$. It is immediate that $\Phi_W(E) \subset \Phi_i(E) \subset \tau(E) \cap \text{co}\,\tau(E)$. In addition

$$\{S \in L(E) : \text{Im}\, S \text{ closed, Ker}\, S \text{ and } E/\text{Im}\, S \text{ reflexive}\} + W(E) \subset \Phi_i(E)$$

in view of [**23**, 5.1].

The proof of [**24**, 2.1] implies that (0.1) holds for all $S \in L(E)$ if and only if there is $\delta > 0$ such that $\text{Id} + R \in \Phi_W(E)$ whenever $R \in L(E)$ satisfies $\omega(R) < \delta$. However, this perturbation criterion seems difficult to work with and the following connection between the norm of the R-representation and the measure of weak non-compactness appears more useful.

THEOREM 1.1. *Let E and F be Banach spaces. Then*

$$\|R(S)\| \le \omega(S) \quad \textit{for all } S \in L(E, F).$$

Proof. Suppose that $\lambda > \omega(S)$ and pick a weakly compact subset K of F with

$$SB_E \subset K + \lambda B_F.$$

The w^*-density of B_E in $B_{E''}$ yields

$$S''B_{E''} \subset \overline{SB_E}^{w^*} \subset \overline{K + \lambda B_F}^{w^*} \subset K + \lambda B_{F''} \tag{1.2}$$

since K is w^*-compact in F''. Here $\bar{A}^{w^*}$ denotes the w^*-closure of A in F''. Suppose that $x'' \in E''$ satisfies $\|x'' + E\| = \text{dist}(x'', E) \le 1$ and let $\delta > 0$. We may assume that $\|x''\| \le 1 + \delta$, if necessary by passing to $x'' - y$ for some $y \in E$. There is by (1.2) an element $k(x'') \in K \subset F$ satisfying

$$\|S''x'' - k(x'')\| \le (1 + \delta)\lambda.$$

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One deduces that

$$\|R(S)(x''+E)\| \leq \|S''x''-k(x'')\| \leq (1+\delta)\lambda.$$

This gives the desired inequality upon letting δ approach 0.

It is well known that $S \in W(E,F)$ if and only if $R(S)=0$. The operator R induces the contractive representation $\tilde{R}: L(E,F)/W(E,F) \to L(E''/E, F''/F)$ considered in [**23**], in view of the inequality $\|R(S)\| \leq \|S\|_w$. This representation is not always bounded below.

Corollary 1.2. *Let E be a Banach space. If* $\operatorname{Im}\tilde{R}$ *is closed in* $L(E''/E)$, *then* ω *and* $\|\ \|_w$ *are equivalent seminorms in* $L(E)$. *In particular, there are Banach spaces E such that* $\operatorname{Im}\tilde{R}$ *fails to be closed in* $L(E''/E)$.

Proof. Evidently $\operatorname{Im}\tilde{R}$ is closed in $L(E''/E)$ if and only if $\|\ \|_w$ and $\|R(\ )\|$ are equivalent seminorms on $L(E)$. In this case ω is also equivalent to $\|\ \|_w$ according to the preceding theorem. It is known [**3**, Theorem 1 and Corollary 3] that this does not always hold.

ω and $\|R(\ )\|$ also fail in general to be comparable. See [**11**].

Let E be a complex Banach space and let $S \in L(E)$. The (symmetric) tauberian spectrum of S is the subset

$$\sigma_\tau(S) = \{\lambda \in \mathbb{C} : \lambda\,\mathrm{Id} - S \notin \tau(E) \cap \mathrm{co}\,\tau(E)\}$$

of $\sigma(S+W(E))$. Geometrically the tauberian spectrum consists of particular W-perturbed eigenvalues of S, since

$$\sigma_\tau(S) = \{\lambda \in \mathbb{C} : \text{there is } V \in W(E) \text{ such that either } \operatorname{Ker}(\lambda\,\mathrm{Id}-(S+V)) \text{ or } E/\overline{\operatorname{Im}(\lambda\,\mathrm{Id}-(S+V))} \text{ is non-reflexive}\}$$

by [**10**, Theorem 1]. Examples are later given where the tauberian spectrum of S is either the empty set or a non-closed subset of $\sigma(S+W(E))$.

Corollary 1.3. *Let E be a complex Banach space. Then*

$$r_\sigma(R(S)) \leq \lim_{n\to\infty} \omega(S^n)^{1/n} \leq r_\sigma(S+W(E)) \quad \textit{for all} \quad S \in L(E).$$

If E satisfies the condition

$$\Phi_w(E) = \Phi_i(E) \tag{1.3}$$

or if $\|R(\ )\|$ *and* $\|\ \|_w$ *are equivalent on* $L(E)$, *then* (0.1) *holds for all* $S \in L(E)$. *In addition,* $\lim_{n\to\infty} \omega(S^n)^{1/n} \geq \sup\{|\lambda| : \lambda \in \sigma_\tau(S)\}$ *whenever* $\sigma_\tau(S) \neq \emptyset$.

Proof. It follows readily that

$$\sigma_\tau(S) = \{\lambda \in \mathbb{C} : \lambda\,\mathrm{Id} - R(S) \text{ is not injective or } \operatorname{Im}(\lambda\,\mathrm{Id}-R(S)) \text{ is not dense in } E''/E\} \subset \sigma(R(S)) \subset \sigma(S+W(E))$$

for any operator S. Theorem 1.1 yields

$$r_\sigma(R(S)) = \lim_{n\to\infty} \|R(S^n)\|^{1/n} \leq \lim_{n\to\infty} \omega(S^n)^{1/n} \leq r_\sigma(S+W(E))$$

for all $S \in L(E)$. Moreover, $\sup\{|\lambda| : \lambda \in \sigma_\tau(S)\} \leq r_\sigma(R(S))$ whenever $\sigma_\tau(S)$ is non-empty.

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If the Banach space E satisfies (1.3), then $\sigma(S+W(E))=\sigma(R(S))$ for all $S\in L(E)$ and (0.1) holds. In addition, if $\|R(\ )\|$ and $\|\ \|_w$ are comparable seminorms, then $r_\sigma(R(S))=r_\sigma(S+W(E))$ for all $S\in L(E)$.

It is important, for instance, in view of the previous result to determine the exact relations between classes such as Φ_w, Φ_i and $\tau\cap\operatorname{co}\tau$ on concrete Banach spaces. The R-representation has not been much studied from this point of view. Recall that $S\in\Phi_+(E)$ if S has closed range and finite dimensional kernel while $S\in\Phi_-(E)$ if $\operatorname{Im} S$ has finite codimension in E. The class of Fredholm operators is $\Phi(E)=\Phi_+(E)\cap\Phi_-(E)$. The Banach space E is *quasi-reflexive* if the canonical image of E has finite codimension in E''. The James space J (see [**15**, 1.d.2]) is the best known example. Let $J_n=J\oplus\ldots\oplus J$ (n copies) with the l^2-norm.

PROPOSITION 1.4. (i). *Suppose that E is a Banach space such that E and E' contain no closed infinite-dimensional reflexive subspaces. Then*

$$\Phi(E)=\Phi_w(E)=\Phi_i(E)=\tau(E)\cap\operatorname{co}\tau(E).$$

(ii). $\Phi_w(J_n)=\Phi_i(J_n)$ *for all* $n\in\mathbb{N}$.

Proof. (i). Assume that $S\in\tau(E)\cap\operatorname{co}\tau(E)\sim\Phi(E)$. If $S\notin\Phi_+(E)$, then there is an infinite-dimensional subspace M of E such that the restriction $S|_M$ is compact [**6**, 4.4.7]. On the other hand, $S\in\tau(E)$ implies that B_M is relatively weakly compact [**12**, 3.2]. This is not possible in view of the assumption on E. If $S\notin\Phi_-(E)$, then there is according to duality and [**6**, 4.4.7] an infinite-dimensional subspace $M\subset E'$ with $S'|_M$ compact. But $S'\in\tau(E')$ [**18**, p. 65] and one would deduce as before that M is a reflexive subspace of E'.

(ii). The spaces J_n are realized up to isomorphism as

$$\left\{(z_i): z_i\in l_n^2,\ \lim_{i\to\infty} z_i=0,\ \|(z_i)\|<\infty\right\},$$

equipped with the norm

$$\|(z_i)\|=\sup\left(\sum_{k=1}^{n-1}\|z_{p_k}-z_{p_{k+1}}\|^2+\|z_{p_n}\|^2\right)^{1/2};$$

see [**4**, 1.1]. The supremum is taken over all finite sequences $p_1<\ldots<p_n$ of natural numbers and l_n^2 denotes the n-dimensional Hilbert space. Here J_n'' consists of the sequences (z_i) with $z_i\in l_n^2$ and $\|(z_i)\|<\infty$. The isometry $\psi: J_n''/J_n\to l_n^2$ is given by $\psi((z_i)+J_n)=\lim_{i\to\infty} z_i$. Suppose that for $S\in L(J_n)$ there is $T\in L(l_n^2)$ with $R(S)T=TR(S)=\mathrm{id}_{l_n^2}$. Define $U: J_n\to J_n$ through $U(z_i)=(Tz_i)$. Evidently U is a bounded operator satisfying $R(U)=T$, since

$$\psi(R(U)((z_i)+J_n))=\lim_{i\to\infty} Tz_i\quad\text{for all}\quad (z_i)\in J_n''.$$

Consequently $R(SU-\mathrm{Id})=0$ and $(SU-\mathrm{Id})''J_n''\subset J_n$. This implies that $SU-\mathrm{Id}\in W(J_n)$. Similarly $US-\mathrm{Id}\in W(J_n)$ and hence $S\in\Phi_w(J_n)$.

The condition of part (i) is satisfied for c_0 [**15**, 2.a.1]. More generally, if E is a Banach space such that E' is isometric to l^1, then E is c_0-hereditary (any infinite-

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dimensional subspace contains a copy of c_0). This follows for instance from a result of Fonf; compare [**7**, IX.12]. Thus part (i) holds for all $C(K)$-spaces, where K is a countable compact metric space. It is claimed in [**22**, 4.1] that $\Phi(L^1(0,1)) = \Phi_i(L^1(0,1))$, but the proof is incomplete. It would be interesting to determine whether this equality holds on the classical spaces with the Dunford–Pettis property; compare Section 2. The question whether $\Phi_w(E) = \Phi_i(E)$ gives rise to a lifting problem for operators on E''/E: is there an operator $U \in L(E)$ with $R(U) = S$ whenever $S \in L(E''/E)$? It is unclear to me whether this always holds even for quasi-reflexive E. However, it is possible to show as above that $\Phi_w(J(E,\mathrm{Id})) = \Phi_i(J(E,\mathrm{Id}))$ for the J-sum $J(E,\mathrm{Id})$ constructed in [**4**], which satisfies $J(E,\mathrm{Id})''/J(E,\mathrm{Id}) = E$ isometrically whenever E is a given reflexive space.

We close this section with two examples concerning operators on vector-valued sequence spaces that stress the analogy of the tauberian spectrum with the point spectrum. Let E be a non-reflexive Banach space. For $1 < p < \infty$ consider

$$l^p(E) = \left\{(x_n) : x_n \in E, n \in \mathbb{N}, \|(x_n)\|_p = \left(\sum_{n=1}^{\infty} \|x_n\|^p\right)^{1/p} < \infty\right\}.$$

Standard vector-valued duality yields canonical isometries $(l^p(E))' = l^q(E')$, where $\frac{1}{p} + \frac{1}{q} = 1$, and $(l^p(E))'' = l^p(E'')$. These identifications remain true for the spaces

$$l^p(\mathbb{Z}, E) = \left\{(x_n) : x_n \in E, n \in \mathbb{Z}, \|(x_n)\|_p = \left(\sum_{n\in\mathbb{Z}} \|x_n\|^p\right)^{1/p} < \infty\right\}.$$

Example 1.5. Let E be any complex non-reflexive Banach space. If S_+ is the vector-valued shift $S_+(x_n) = (x_{n+1})$ on $l^p(\mathbb{Z}, E)$, then $\sigma_\tau(S_+) = \emptyset$. Indeed, $\sigma(S_+) \subset \{z \in \mathbb{C} : |z| = 1\}$ (cf. [**8**, 1.31]) since S_+ is a bijective isometry on $l^p(\mathbb{Z}, E)$. Hence it suffices to verify that $\lambda\,\mathrm{Id} - S_+ \in \tau \cap \mathrm{co}\,\tau$ whenever $\lambda \in \mathbb{C}$ satisfies $|\lambda| = 1$. Assume that $(x_n'') \in l^p(\mathbb{Z}, E'')$ and that

$$(\lambda\,\mathrm{Id} - S_+)(x_n'') = (\lambda x_n'' - x_{n+1}'') \in l^p(\mathbb{Z}, E).$$

Consequently $\lambda x_n'' - x_{n+1}'' \in E$ for all $n \in \mathbb{Z}$. It follows that

$$\lambda^n x_0'' - x_n'' = \sum_{k=0}^{n-1} \lambda^{n-1-k}(\lambda x_k'' - x_{k+1}'') \in E,$$

for all $n \geq 1$. Similarly $\lambda^n x_0'' - x_n'' \in E$ for $n < 0$. This means that $\mathrm{dist}(x_n'', E) = \mathrm{dist}(x_0'', E)$ for $n \in \mathbb{Z}$, and hence that $(x_n'') \in l^p(\mathbb{Z}, E)$.

The fact that $\lambda\,\mathrm{Id} - S_+$ is cotauberian for the same values of λ is verified in a similar manner since $S_+' = S_-$, where $S_-(x_n') = (x_{n-1}')$ on $l^q(\mathbb{Z}, E')$. This establishes the claim.

Example 1.6. Let E be a complex non-reflexive Banach space. Suppose that $\{r_n : n \in \mathbb{N}\}$ is an enumeration of the set $\{\alpha + i\beta \in \mathbb{C} : \alpha,\ \beta \text{ rational},\ 0 < \alpha^2 + \beta^2 < 1\}$. Let $S \in L(l^p(E))$ be defined by $S(x_n) = (r_n x_n)$. Then $\sigma_\tau(S) = \{r_n : n \in \mathbb{N}\}$ and $\sigma(S + W(l^p(E))) = \sigma(S) = \{z \in \mathbb{C} : |z| \leq 1\}$.

Indeed, here $S'(x_n') = (r_n x_n')$, $S''(x_n'') = (r_n x_n'')$ for all $(x_n') \in l^q(E')$ respectively $(x_n'') \in l^p(E'')$. Clearly $r_n\,\mathrm{Id} - S$ fails to be tauberian for all $n \in \mathbb{N}$, since the non-reflexive space $E \subset \mathrm{Ker}(r_n\,\mathrm{Id} - S)$. It follows similarly by duality that $r_n\,\mathrm{Id} - S' \notin \tau(l^q(E'))$ and consequently $\{r_n : n \in \mathbb{N}\} \subset \sigma_\tau(S)$. There remains to verify that $\lambda \operatorname{Id} - S \in \tau \cap \operatorname{co} \tau$ whenever $\{z \in \mathbb{C} : |z| \leq 1\} \sim \{r_n : n \in \mathbb{N}\}$. In fact, if $(x''_n) \in l^p(E'')$, then

$$(\lambda I - S'')(x''_n) = ((\lambda - r_n)x''_n) \in l^p(E)$$

if and only if $x''_n \in E$ for all $n \in \mathbb{N}$. The verification that $\lambda I - S$ is cotauberian is formally similar using duality. This yields the claim.

2. Further results. The equality (0.1) is certainly valid on a given complex space E if ω and $\| \ \|_w$ are equivalent seminorms on $L(E)$ because of (0.2). Equivalence holds if E has a certain weakly compact approximation property, but it does not hold in general [**3**, Theorem 1].

Recall that a Banach space E has the *Dunford–Pettis property* if all weakly compact operators $S : E \to F$ map relatively weakly compact sets $B \subset E$ to relatively compact sets SB. Standard examples of spaces with this property are the $\mathscr{L}^1$- and $\mathscr{L}^\infty$-spaces, such as l^1, $L^1(0,1)$, c_0, $C(0,1)$, l^∞ and $M(0,1)$ [**14**, II.4.30]. It is known that $\Phi_w(E) = \Phi(E)$ whenever E has the Dunford–Pettis property. Indeed, suppose that $T_1, T_2 \in L(E)$, $V_1, V_2 \in W(E)$ satisfy $T_1 S = \operatorname{Id} + V_1$ and $ST_2 = \operatorname{Id} + V_2$. Then $\operatorname{Id} - V_i^2 \in \Phi(E)$ $(i = 1, 2)$, since V_i^2 is compact in view of the Dunford–Pettis property of E. Thus $\operatorname{Id} + V_i$ and S are Fredholm operators by [**6**, 3.2.6]. In this event $\sigma(S + W(E)) = \sigma(S + K(E))$ for all $S \in L(E)$.

A Banach space E has the *Schur property* if all relatively weakly compact subsets of E are relatively compact. This property clearly passes to subspaces. The canonical example is $l^1(I)$ for all index sets I. Recall that E has the *λ-extension property* if for all subspaces M of F and all $S \in L(M, E)$ there is an extension T of S to F with $\|T\| \leq \lambda \|S\|$.

THEOREM 2.1. *The seminorms ω and $\| \ \|_w$ are equivalent on $L(E)$, and thus the equality* (0.1) *holds, in the following cases.*

(i) *E has the weakly compact approximation property of* [**3**], *for instance if E has the Schur and the bounded approximation property.*

(ii) *There is a projection $P : E'' \to E$ and E' has the λ-extension property for some λ. These conditions are satisfied by $L^1(0,1)$, $(l^\infty)'$, $M(0,1)$ or by any further even dual of these.*

(iii) c_0.

(iv) *E is quasi-reflexive.*

Proof. (i) See [**3**, Theorem 1].

(ii) Observe first that

$$\|S'\|_w \leq \|S\|_w \leq \|P\| \, \|S'\|_w \tag{2.1}$$

for all $S \in L(E)$. Indeed, if $\mu > \|S'\|_w$ and $V \in W(E')$ is such that $\|S' - V\| < \mu$, then $PV'K_E$ is weakly compact on E while

$$\|S - PV'K_E\| \leq \|P\| \, \|S'' - V'\| < \|P\| \, \mu,$$

where K_E denotes the natural embedding of E into its bidual. Thus (2.1) follows with the general inequality $\|S'\|_w \leq \|S\|_w$ (by Gantmacher's theorem).

Moreover, it follows from the proof of [**2**, 5.2] and the above that

$$\omega(S'') \leq \|S''\|_w = \|S'\|_w \leq \lambda \omega(S'')$$

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whenever $S \in L(E)$, since E' has the λ-extension property. Consequently one obtains after combining with the general inequality $\omega(S'') \le \omega(S)$ [**2**, 5.1] that

$$\omega(S) \le \|S\|_w \le \|P\| \, \|S'\|_w \le \lambda \, \|P\| \, \omega(S)$$

for $S \in L(E)$.

It is well known that there is a projection $P: (L^1(0,1))'' \to L^1(0,1)$ with norm 1. The existence of the required projection in the other cases follows since they are dual spaces. Finally, all duals E' have the extension property since the spaces E considered here are $\mathcal{L}^1$-spaces; see [**14**, II.5.7].

(iii) We claim that

$$\omega(S) = \gamma(S) = \|S\|_w = \operatorname{dist}(S, K(c_0)), S \in L(c_0).$$

The argument is based on block-basis techniques.

Let (e_i) be the coordinate basis of c_0, let $\varepsilon > 0$ be small and assume that $S \in L(c_0)$ is normalized by

$$1 = \operatorname{dist}(S, K(c_0)) \le \|S\| \le 1 + \varepsilon.$$

Since $W(c_0) = K(c_0)$ and since $\gamma(R) = \operatorname{dist}(R, K(c_0))$ for all R on c_0 [**13**, 3.6], it is enough to verify that

$$\omega(S) > \beta \tag{2.2}$$

for all $0 < \beta < 1$. This is achieved by showing that the restriction of S to some subspace isometric to c_0 is a nice isomorphism. Assume that $0 < \mu < 1$ is given. According to [**20**, 1.2] there are block basic sequences (x_n) and (z_n) with respect to the basis (e_i) such that for all $n \in \mathbb{N}$:

$$\|x_n\| = 1, \qquad \|Sx_n\| > \mu, \tag{2.3}$$

$$\|Sx_n - z_n\| < \delta/2^n. \tag{2.4}$$

Here the images (Sx_n) are almost disjoint and the blocks (z_n) are corresponding truncations, so that it is possible to make the difference in (2.4) arbitrarily small, given any preassigned $\delta > 0$. The closed linear span $[x_n]$ is isometric to c_0 and we estimate $\omega(SB_{[x_n]})$ from below. Since (z_n) are disjoint finite blocks formed from (Sx_n) one may ensure from the bimonotonicity of the unit basis that

$$\mu < \|z_n\| \le \|Sx_n\| \le 1 + \varepsilon \quad (n \in \mathbb{N}). \tag{2.5}$$

Evidently

$$\mu \max_{n\in\mathbb{N}} |\lambda_n| \le \min_{n\in\mathbb{N}} \|z_n\| \max_{n\in\mathbb{N}} |\lambda_n| \le \left\| \sum_{n=1}^{\infty} \lambda_n z_n \right\| \le (1+\varepsilon) \max_{n\in\mathbb{N}} |\lambda_n|$$

for all $(\lambda_n) \in c_0$. If $\delta > 0$ is chosen small enough one ensures from (2.5) and perturbation results for basic sequences [**15**, 1.a.9(i)] that

$$\nu \max_{n\in\mathbb{N}} |\lambda_n| \le \left\| \sum_{n=1}^{\infty} \lambda_n Sx_n \right\| \le (1+\varepsilon) \max_{n\in\mathbb{N}} |\lambda_n|$$

for all $0 < \nu < \mu$ and all $(\lambda_n) \in c_0$. Consequently the restriction $S|_{[x_n]}$ is an isomorphism onto $[Sx_n]$ with $\|S|_{[x_n]}\| \, \|(S|_{[x_n]})^{-1}\| \le \nu^{-1}(1+\varepsilon)$. Observe further according to disjoint-

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ness and the proof of the perturbation result in [**15**, 1.a.9(ii)] that there are projections $P: c_0 \to [x_n]$ and $Q: c_0 \to [Sx_n]$ such that $\|P\| = 1$ and $\|Q\| < \lambda$, for any $\lambda > \nu^{-2}(1+\varepsilon)^2$, as soon as $\delta > 0$ is small enough. It is easily estimated that

$$\omega(S) \geq \frac{1}{\|Q\|}\,\omega(SB_{[x_n]}) \geq \frac{1}{\|Q\|}\,\nu^2(1+\varepsilon)^{-2}\omega(B_{[x_n]}).$$

Here $\omega(B_{[x_n]}) = 1$ while $\omega(SB_{[x_n]})$ is computed in the subspace $[Sx_n]$. Eventually this yields (2.2) after appropriate choices of μ, ν, ε and λ.

(iv) Recall that $R \in W(E)$ if and only if $R''E'' \subset E$. It follows from the finite-dimensionality of E''/E that $W(E)$ has finite codimension in $L(E)$. The claim is seen since all norms are equivalent in the finite-dimensional space $L(E)/W(E)$.

Remarks 2.2. (i) In the case $L^1(0,1)$ there is a different proof of the equality $\omega(S) = \|S\|_w$ by combining [**1**, 3.6] and [**21**, Theorem 1].

(ii) Relative to the spaces $E = L^1(0,1)$ or c_0 there are Banach spaces F such that ω and $\|\ \|_w$ are not comparable on $L(F, E)$, since E fails the approximation property which ensures equivalency [**3**, Theorem 1 and Corollary 3]. It is surprising that the situation is different on $L(E)$. Also, for $E = C(0,1)$ or l^∞ there is a subspace $F = \left(\bigoplus_{n\in\mathbb{N}} E_n\right)_{l^p}$ such that ω and $\|\ \|_w$ fail to be equivalent in $L(F, E)$. In the construction of [**3**] the sum F actually embeds into E since F is separable for $C(0,1)$, while F' has a countable total subset in the case of l^∞. Unfortunately it is not clear whether ω and $\|\ \|_w$ are comparable on $L(C(0,1))$ or $L(l^\infty)$.

We conclude by applying a representation of Buoni and Klein [**5**] of the generalized Calkin algebra $L(E)/W(E)$ in order to obtain a formula for the inner radius of a subset of the spectrum. It is referred to [**25**] or [**19**] for an analogous result in the Calkin algebra setting. If E is a non-reflexive Banach space, let

$$l^\infty(E) = \left\{(x_n) : x_n \in E, n \in \mathbb{N} \text{ and } \|(x_n)\| = \sup_{n\in\mathbb{N}} \|x_n\| < \infty\right\}$$

and

$$w(E) = \{(x_n) \in l^\infty(E) : \{x_n : n \in \mathbb{N}\} \text{ is relatively weakly compact in } E\}.$$

Consider $Q(E) = l^\infty(E)/w(E)$, where the quotient norm satisfies

$$\|(x_n) + w(E)\| = \omega(\{x_n : n \in \mathbb{N}\}) \quad \text{for all} \quad (x_n) + w(E) \in Q(E) \tag{2.6}$$

by [**3**, Lemma 9]. Any $S \in L(E, F)$ induces $Q(S) \in L(Q(E), Q(F))$ through $Q(S)((x_n) + w(E)) = (Sx_n) + w(F)$ for $(x_n) + w(E) \in Q(E)$. The subclass

$$\tau_+(E, F) = \left\{S \in L(E, F) : \omega_+(S) = \inf_B \frac{\omega(SB)}{\omega(B)} > 0\right\}$$

of the tauberian operators was studied in [**3**]. The infimum in the definition is taken over all bounded non-relatively weakly compact sets $B \subset E$. Clearly ω_+ is supermultiplicative and the limit $\lim_{n\to\infty} \omega_+(S^n)^{1/n}$ exists for any $S \in L(E)$. We require some facts in order to give a spectral interpretation of the limit.

LEMMA 2.3. *Let E and F be Banach spaces and let $S \in L(E, F)$. Then the injection modulus $j(Q(S))$ of $Q(S)$ satisfies*

$$j(Q(S)) = \inf\{\|(Sx_n) + w(F)\| : \|(x_n) + w(E)\| = 1\} \geq \omega_+(S). \tag{2.7}$$

Equality holds in (2.7) *whenever E is separable.*

Proof. If $\|(x_n) + w(E)\| = \omega(\{x_n : n \in \mathbb{N}\}) = 1$, then

$$\|(Sx_n) + w(F)\| = \omega(\{Sx_n : n \in \mathbb{N}\}) \geq \omega_+(S)$$

in view of (2.6) and this entails (2.7). Let E be separable and assume that $\lambda > \omega_+(S)$. Pick a bounded subset $B \subset E$ satisfying $\omega(B) = 1$ and $\omega(SB) < \lambda$. By assumption there is a sequence (x_n) in B such that $\overline{\{x_n : n \in \mathbb{N}\}} = B$. Then $\|(x_n) + w(E)\| = \omega(B) = 1$ and consequently

$$j(Q(S)) \leq \omega(\{Sx_n : n \in \mathbb{N}\}) = \omega(SB) < \lambda.$$

This establishes the claim.

The τ_+-spectrum of $S \in L(E)$ on a complex non-reflexive Banach space E is $\sigma_\tau^+(S) = \{\lambda \in \mathbb{C} : \lambda\,\mathrm{Id} - S \notin \tau_+(E)\}$. If E is separable, then $\sigma_\tau^+(S) \subset \sigma(S)$ is closed and non-empty. The fact $\sigma_\tau^+(S) \neq \varnothing$ follows from $\partial\sigma(Q(S)) \subset \sigma_\tau^+(S)$ for the boundary of the spectrum (cf. [**8**, 1.16]), since $\sigma_\tau^+(S)$ coincides with the approximate point spectrum of $Q(S)$ in this case.

PROPOSITION 2.4. *Let E be a complex, separable non-reflexive Banach space. Then*

$$\lim_{n\to\infty} \omega_+(S^n)^{1/n} = \min\{|\lambda| : \lambda \in \sigma_\tau^+(S)\}, \quad S \in L(E). \tag{2.8}$$

Proof. The asymptotic formula of Makai and Zemanek [**16**, Theorems 1 and 3] for the injection modulus states that

$$\lim_{n\to\infty} j(Q(S^n))^{1/n} = \min\{|\lambda| : \lambda\,\mathrm{Id} - Q(S) \text{ is not bounded below}\} \quad (S \in L(E)).$$

According to Lemma 2.3 one has $j(Q(S^n)) = \omega_+(S^n)$ and

$$\sigma_\tau^+(S) = \{\lambda \in \mathbb{C} : \lambda\,\mathrm{Id} - Q(S) \text{ is not bounded below}\}$$

whenever E is separable. This yields (2.8).

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