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The Transfer in the Invariant Theory of Modular Permutation Representations II

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Abstract. In this note we show that the image of the transfer for permutation representations of finite groups is generated by the transfers of special monomials. This leads to a description of the image of the transfer of the alternating groups. We also determine the height of these ideals.

Let $\mathbb{F}$ be a finite field of char $(\mathbb{F}) = p$. Let $\rho\colon G \hookrightarrow \mathrm{GL}(n, \mathbb{F})$ be a faithful representation of a finite group G. The group G acts via ρ on the n-dimensional vector space $V = \mathbb{F}^n$. This induces an action of G on the ring of polynomial functions $\mathbb{F}[x_1, \dots, x_n] = \mathbb{F}[V]$, where $x_1, \dots, x_n$ is the standard dual basis of V^*, via

$$gf(v) := f\big(\rho(g)^{-1}v\big) \quad \forall g \in G,\ f \in \mathbb{F}[x_1, \dots, x_n],\ v \in V.$$

Denote by $\mathbb{F}[V]^G$ the ring of polynomials invariant under the G-action, see [10] or [12] for an introduction to invariant theory of finite groups. The transfer

$$\mathrm{Tr}^G\colon \mathbb{F}[V] \to \mathbb{F}[V]^G; \quad f \mapsto \sum_{g\in G} gf$$

is an $\mathbb{F}[V]^G$-module homomorphism. It is surjective if and only if the characteristic of the ground field $\mathbb{F}$ does not divide the group order, *i.e.*, in the non-modular case, where it provides a tool for constructing the ring of invariants $\mathbb{F}[V]^G$, see Section 2.4 in [12]. On the other hand, in the modular case, where $p \mid |G|$, the transfer is never zero nor surjective, see Section 11.5 in [12]. This makes the transfer, resp., its image, an interesting object of study and inspired quite a number of (recent) research, *e.g.*, [2], [3], [6], [7], [8] and [11]. In this note we pursue the investigation of the image of the transfer of modular permutation representations started in [8].

1 A Generating Set

In this section we show that the image of the transfer for a permutation representation is generated by the transfers of special monomials—a result inspired by [4], as it is reworked by [5].

Let $\rho\colon G \hookrightarrow \mathrm{GL}(n, \mathbb{F})$ be a modular permutation representation of a finite group G permuting a basis $x_1, \dots, x_n$ for the dual vector space V^*. Denote by

$$x^E = x_1^{E_1} \cdots x_n^{E_n} \in \mathbb{F}[x_1, \dots, x_n]$$

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a monomial with multi-index $E = (E_1, \dots, E_n)$. We associate to E a partition

$$\lambda(E) = \bigl(\lambda_1(E), \dots, \lambda_n(E)\bigr)$$

where for every $i = 1, \dots, n$ there exists a $j \in \{1, \dots, n\}$ such that

$$\lambda_i(E) = E_{\sigma(j)},$$

for some $\sigma \in \Sigma_n$, *i.e.*, the partition $\lambda(E)$ is obtained from the exponent sequence E by reordering so that it is weakly decreasing. Call a monomial *special* if

$$\lambda_n(E) = 0 \quad \text{and} \quad \lambda_i(E) - \lambda_{i+1}(E) \leq 1 \quad \forall i = 1, \dots, n-1.$$

Theorem 1.1 *Let $\rho\colon G \hookrightarrow \mathrm{GL}(n, \mathbb{F})$ be a permutation representation of a finite group G. Then the image of the transfer is generated by*

$$\mathrm{Tr}^G(x^E) \quad \textit{for special monomials } x^E \in \mathbb{F}[V].$$

Proof Denote by $I \subseteq \mathbb{F}[V]^G$ the ideal generated by the transfers of special monomials. Let $x^E \in \mathbb{F}[V]$ be a non-special monomial. We have to show that

$$\mathrm{Tr}^G(x^E) \in I.$$

We assume that $\mathrm{Tr}^G(x^E) \neq 0$, for otherwise there is nothing to show. Denote the associated partition by $\lambda(E) = \bigl(\lambda_1(E), \dots, \lambda_n(E)\bigr)$. If $\lambda_n(E) \neq 0$, then

$$x^E = x^{E'} e_n,$$

where e_i denotes the i-th elementary symmetric function, $i = 1, \dots, n$, and, $E'_i = E_i - 1$. We have

$$\mathrm{Tr}^G(x^E) = \mathrm{Tr}^G(x^{E'}) e_n.$$

Since the elementary symmetric functions are present in any ring of permutation invariants it is enough to show that

$$\mathrm{Tr}^G(x^{E'}) \in I.$$

So, without loss of generality, we assume that $\lambda_n(E) = 0$. We want to proceed by induction and choose for that the *dominance order* on monomials, *i.e.*,

$$x^E \leq_{\mathrm{dom}} x^F \iff \begin{cases} \lambda_1(E) \leq \lambda_1(F) \quad \text{and} \\ \lambda_1(E) + \lambda_2(E) \leq \lambda_1(F) + \lambda_2(F) \quad \text{and} \\ \quad \vdots \\ \lambda_1(E) + \cdots + \lambda_n(E) \leq \lambda_1(F) + \cdots + \lambda_n(F). \end{cases}$$

Assume to the contrary that $I \subsetneq \mathrm{Im}(\mathrm{Tr}^G)$, and let x^E be minimal with respect to the dominance ordering such that

$$\mathrm{Tr}^G(x^E) \notin I.$$

Since x^E is non-special, somewhere in the associated partition $\lambda(E)$ is a gap. Let t be the index of the first occurrence of a gap

$$t = \min\{i \mid \lambda_i(E) - \lambda_{i+1}(E) > 1\},$$

and define the *reduced monomial* $x^{\tilde{E}}$ to be the one where $\tilde{E}$ is obtained from E by lowering the largest t of the exponents E_i by 1. The reduced monomial $x^{\tilde{E}}$ is, by construction, strictly smaller in dominance order

$$x^{\tilde{E}} <_{\mathrm{dom}} x^E.$$

Write

$$\mathrm{Tr}^G(x^E) = \mathrm{Tr}^G(x^{\tilde{E}})e_t - R,$$

for some polynomial R. Note that

$$R = \mathrm{Tr}^G(x^{\tilde{E}})e_t - \mathrm{Tr}^G(x^E) \in \mathrm{Im}(\mathrm{Tr}^G)$$

shows that R is in the image of the transfer of G. By minimality of x^E we have that

$$\mathrm{Tr}^G(x^{\tilde{E}}) \in I.$$

The monomials occurring in R are strictly less in the dominance ordering than x^E. This is shown in the lemma below. Hence R is, by induction, also contained in I. Therefore

$$\mathrm{Tr}^G(x^E) = \mathrm{Tr}^G(x^{\tilde{E}})e_t - R \in I,$$

what contradicts our assumption and we are done. ■

The following lemma is a revised version of Lemma 10 in [5].

Lemma 1.2 *Every monomial x^F occurring in*

$$\mathrm{Tr}^G(x^{\tilde{E}})e_t\big(= \mathrm{Tr}^G(x^E) + R\big)$$

is smaller with respect to the dominance ordering than x^E, and equal if and only if x^F is a term in $\mathrm{Tr}^G(x^E)$.

Proof Write

$$x^F = x^{\tilde{F}} x_{j_1} \cdots x_{j_t}$$

where

$$\lambda(\tilde{E}) = \lambda(\tilde{F}) \quad \text{and} \quad J := \{j_1, \ldots, j_t\} \subseteq \{1, \ldots, n\}.$$

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We have
$$\lambda(\tilde{E}) = \big(\lambda_1(E) - 1, \dots, \lambda_t(E) - 1, \lambda_{t+1}(E), \dots, \lambda_n(E)\big).$$
The partition $\lambda(F)$ associated to F is obtained from $\lambda(\tilde{F})$ by adding one to some exponents according to the elements of J (and possibly reordering)

$$\lambda_i(F) = \begin{cases} \lambda_{\sigma(i)}(E) - 1 + \mathbf{1}_J(i) & \text{for } i = 1, \dots, t \\ \lambda_{\sigma(i)}(E) + \mathbf{1}_J(i) & \text{for } i = t+1, \dots, n, \end{cases}$$

where $\mathbf{1}_J$ is the function taking value one on J and zero elsewhere, and σ is a certain permutation corresponding to the possible reordering. Note that this element $\sigma \in \Sigma_n$ permutes the $\lambda_i(F)$ before and after t separately. We need to show that

$$\sum_{i=1}^{j} \lambda_i(F) \le \sum_{i=1}^{j} \lambda_i(E)$$

for $j = 1, \dots, n$. For $j \le t$ we get

$$\sum_{i=1}^{j} \lambda_i(F) = \sum_{i=1}^{j} \big(\lambda_{\sigma(i)}(E) - 1 + \mathbf{1}_J(i)\big)$$
$$\le \Big(\sum_{i=1}^{j} \lambda_i(E)\Big) - j + |J \cap \{1, \dots, j\}| \le \sum_{i=1}^{j} \lambda_i(E),$$

where the penultimate inequality follows because reordering lowers the sum of the λ_i.

Secondly, if $j > t$ then

$$\sum_{i=1}^{j} \lambda_i(F) = \sum_{i=1}^{t} \big(\lambda_{\sigma(i)}(E) - 1 + \mathbf{1}_J(i)\big) + \sum_{i=t+1}^{j} \big(\lambda_{\sigma(i)}(E) + \mathbf{1}_J(i)\big)$$
$$= \sum_{i=1}^{t} \lambda_i(E) + \sum_{i=t+1}^{j} \lambda_{\sigma(i)}(E) - t + |J \cap \{1, \dots, j\}| \le \sum_{i=1}^{j} \lambda_i(E).$$

Finally,[1]
$$x^F =_{\mathrm{dom}} x^E \Longleftrightarrow \lambda(F) = \lambda(E).$$
If x^F is a term in the transfer of x^E, then x^F and x^E differ only by a permutation. Therefore $\lambda(F) = \lambda(E)$.

Conversely, assume that $\lambda(F) = \lambda(E)$. Note that

$$e_t x^{\tilde{E}} = x^E + f,$$

[1] $=_{\mathrm{dom}}$ means, of course, that both inequalities hold.

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for some polynomial f. Without loss of generality we assume that

$$x_1 \cdots x_t x^{\tilde{E}} = x^E.$$

The only term in $e_t x^{\tilde{E}}$, whose exponent sequence has the same partition $\lambda(E)$ as x^E is $x_1 \cdots x_t x^{\tilde{E}}$ $(= x^E)$, because all other terms have a gap at a different index. Moreover, by construction, $x^{\tilde{F}}$ is a term in the transfer of $x^{\tilde{E}}$, *i.e.*, there exists an element $g \in G$ such that

$$gx^{\tilde{E}} = x^{\tilde{F}}.$$

Hence

$$g(x^E) + g(f) = g(e_t x^{\tilde{E}}) = e_t x^{\tilde{F}} = x^F + \textit{other terms}.$$

Since $\lambda(F) = \lambda(E)$ and the partitions of the exponent sequences of all other terms involved (*i.e.*, the terms of $g(f)$ and the terms occurring in *other terms*) are different from these, it follows that

$$x^F = g(x^E)$$

as claimed. ∎

Remark We could derive the preceding result also in the following way: By [2] the ring of polynomials $\mathbb{F}[V]$ is generated by special monomials as a module over $\mathbb{F}[V]^{\Sigma_n}$, and, *a fortiori*, as a module over $\mathbb{F}[V]^G$ for any permutation group G. Hence in the modular situation the image of the transfer is given by applying the transfer to these module generators.

Since special monomials have at most degree $\frac{n(n-1)}{2}$, we have proved the following.

Corollary 1.2 *The image of the transfer of a permutation representation is generated by the polynomials of degree at most* $\frac{n(n-1)}{2}$.

It is worthwhile noting, that the statement of this corollary can be obtained independently of the preceding theorem, [13]:

Proof The ring generated by the elementary symmetric functions is present in every invariant ring of a permutation group G, *i.e.*,

$$\mathbb{F}[V]^{\Sigma_n} = \mathbb{F}[e_1, \dots, e_n] \hookrightarrow \mathbb{F}[V]^G \hookrightarrow \mathbb{F}[V],$$

where Σ_n denotes the symmetric group in n letters. The elementary symmetric functions form a homogeneous system of parameters for $\mathbb{F}[V]$. Since the ring of polynomials $\mathbb{F}[V]$ is Cohen-Macaulay, it is free finitely generated over $\mathbb{F}[e_1, \dots, e_n]$. Hence the maximal degree of a module generator of $\mathbb{F}[V]$ over $\mathbb{F}[e_1, \dots, e_n]$ can be obtained by dividing the respective Poincaré series. The degree $\binom{n}{2}$ of this polynomial

$$\frac{\mathcal{P}(\mathbb{F}[V], t)}{\mathcal{P}(\mathbb{F}[V]^{\Sigma_n}, t)} = \frac{\prod_{i=1}^{n}(1-t^i)}{(1-t)^n} = \prod_{i=1}^{n-1}(1 + t + \cdots + t^i)$$

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is the maximal degree of a module generator. Since the transfer $\mathrm{Tr}^G\colon \mathbb{F}[V] \to \mathbb{F}[V]^G$ is an $\mathbb{F}[V]^G$-module homomorphism, it is, *a fortiori*, an $\mathbb{F}[V]^{\Sigma_n}$-module homomorphism. It follows that $\mathrm{Im}(\mathrm{Tr}^G)$ is generated as an $\mathbb{F}[V]^G$-module, *i.e.*, as an ideal, by polynomials of degree at most $\binom{n}{2}$. ■

Finally we calculate the height of the image of the transfer.

Theorem 1.4 *Let $\rho\colon G \hookrightarrow \mathrm{GL}(n, \mathbb{F})$ be a permutation representation of a finite group G. Let the characteristic p of the groundfield $\mathbb{F}$ divide the group order of G. Then the height $\mathrm{ht}\left(\mathrm{Im}(\mathrm{Tr}^G)\right)$ is divisible by $p-1$. Moreover, we have the following (in-)equalities:*

$$\frac{n}{p}(p-1) \geq \mathrm{ht}\left(\mathrm{Im}(\mathrm{Tr}^G)\right)$$
$$= \min\{k(p-1) \mid g \in G, |g| = p, g \text{ is a product of } k\ p\text{-cycles}\} \geq p-1.$$

Proof By M. Feshbach's transfer theorem, [10] Theorem 6.4.7, the transfer variety

$$\sqrt{\mathrm{Im}(\mathrm{Tr}^G)} = \Big(\bigcap_{|g|=p, g\in G} I_g\Big) \cap \mathbb{F}[V]^G,$$

where $I_g \subseteq \mathbb{F}[V]$ is the ideal generated by the image of

$$1 - g\colon V^* \to V^*,$$

$g \in G$ of order p. The ideals I_g are generated by linear forms, and therefore prime. By the Krull relations, the height is preserved when contracting an ideal I_g to $I_g \cap \mathbb{F}[V]^G$. Hence

$$\begin{aligned}\mathrm{ht}\left(\mathrm{Im}(\mathrm{Tr}^G)\right) &= \min\{\mathrm{ht}\,(I_g) \mid g \in G, |g| = p\}\\ &= \min\{n - \dim(V^g) \mid g \in G, |g| = p\}.\end{aligned}$$

An element $g \in G$ is a product of p-cycles, and hence

$$\dim(V^g) = k + (n - kp),$$

where k denotes the number of p-cycles. Since we assume that $p \mid |G|$, we have that $k \geq 1$. Therefore,

$$\begin{aligned}\frac{n}{p}(p-1) &\geq \mathrm{ht}\left(\mathrm{Im}(\mathrm{Tr}^G)\right)\\ &= \min\{k(p-1) \mid g \in G, |g| = p, g \text{ is a product of } k\ p\text{-cycles}\}\\ &\geq p-1.\end{aligned}$$

■

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2 The Alternating Group

We apply our results to find the image of the transfer of the alternating groups A_n. First recall from [1] and Section 1.3 [12], or Section 14.2 in [9] that

$$\mathbb{F}[V]^{A_n} = \mathbb{F}[e_1, \dots, e_n, \nabla_n] \big/ (r),$$

where

$$\nabla_n = \begin{cases} \Delta_n = \prod_{i<j}(x_i - x_j) & \text{for } p \text{ odd } (\textit{i.e.}, \text{the discriminant}), \\ \operatorname{Tr}^{A_n}(x_1 x_2^2 \cdots x_{n-1}^{n-1}) & \text{for } p = 2, \end{cases}$$

and r is quadratic in ∇_n.

Case $p \neq 2$ If the characteristic p of $\mathbb{F}$ is not 2, then the index of A_n in Σ_n is prime to p, so Theorem 5.1 in [11] gives

$$\operatorname{Im}(\operatorname{Tr}^{A_n}) \cap \mathbb{F}[V]^{\Sigma_n} = \operatorname{Im}(\operatorname{Tr}^{\Sigma_n}),$$

and hence we have

$$\big(\operatorname{Im}(\operatorname{Tr}^{\Sigma_n})\big)^e \subseteq \operatorname{Im}(\operatorname{Tr}^{A_n}) \subset \mathbb{F}[V]^{A_n},$$

where $(-)^e$ denotes the extended ideal. The discriminant Δ_n is a sum of orbit sums of monomials x^E with partition

$$\lambda(E) = \{0, \dots, n-1\}.$$

Any such monomial has trivial isotropy group, *i.e.*, the orbit of any such monomial has maximal length. This in turn means that

$$\Delta_n = \operatorname{Tr}^{A_n}(x_1 x_2^2 \cdots x_{n-1}^{n-1}) - \operatorname{Tr}^{A_n}(x_2 x_1^2 x_3^3 \cdots x_{n-1}^{n-1}) \in \operatorname{Im}(\operatorname{Tr}^{A_n}),$$

is the image of the transfer of a sum of special monomials of highest degree. So we have

$$\Big(\big(\operatorname{Im}(\operatorname{Tr}^{\Sigma_n})\big)^e, \Delta_n\Big) \subseteq \operatorname{Im}(\operatorname{Tr}^{A_n}).$$

We claim that these two ideals are equal. To this end take a polynomial $f \in \mathbb{F}[V]$ such that $\operatorname{Tr}^{A_n}(f) \neq 0$. Then

$$\operatorname{Tr}^{A_n}(f) = f_1 \Delta_n + f_0$$

for suitable polynomials $f_0, f_1 \in \mathbb{F}[V]^{\Sigma_n}$, because $\mathbb{F}[V]^{A_n}$ is, as a module over $\mathbb{F}[V]^{\Sigma_n}$, generated by 1 and Δ_n. So,

$$\operatorname{Tr}^{\Sigma_n}(f) = \operatorname{Tr}^{\Sigma_n}_{A_n} \operatorname{Tr}^{A_n}(f) = f_1 \operatorname{Tr}^{\Sigma_n}_{A_n}(\Delta_n) + 2 f_0.$$

Hence

$$f_0 \in \operatorname{Im}(\operatorname{Tr}^{\Sigma_n}),$$

and therefore

$$\mathrm{Tr}^{A_n}(f) = f_1\Delta_n + f_0 \in \Big(\big(\mathrm{Im}(\mathrm{Tr}^{\Sigma_n})\big)^e, \Delta_n \Big).$$

Finally note that the alternating group A_n contains a p-cycle, whenever $p \mid |A_n|$, $p \geq 3$. Therefore by applying Theorem 1.4 we find that

$$\mathrm{ht}\left(\mathrm{Im}(\mathrm{Tr}^{A_n})\right) = \min\{\mathrm{ht}\,(I_g) \mid g \in A_n, |g| = p\} = p - 1.$$

Remark Note that this description is valid in both, the non-modular and the modular situation (except, of course, the calculation of the height: this is always maximal in the nonmodular case).

Remark Consider the modular case, *i.e.*, $p \leq n$. Then the image of the transfer of the full symmetric group can be found in Theorem 9.18 of [2]: Denote by $x^E \in \mathbb{F}[V]$ a monomial with $E_1 \geq \cdots \geq E_n$. Rewrite this monomial as

$$x^E = x_1^{E_1} \cdots x_n^{E_n} = (x_1 \cdots x_{\epsilon_k})^k (x_{\epsilon_k+1} \cdots x_{\epsilon_k+\epsilon_{k-1}})^{k-1} \cdots (x_{\epsilon_k+\cdots+\epsilon_1+1} \cdots x_n)^0,$$

and $\epsilon_0 = n - (\epsilon_1 + \cdots + \epsilon_k)$. Then $\mathrm{Im}(\mathrm{Tr}^{\Sigma_n})$ is generated by $\mathrm{Tr}^{\Sigma_n}(x^E)$, where x^E satisfies

(1) $\epsilon_i < p$ for $i = 0, \ldots, k$, and
(2) $\epsilon_i + \epsilon_{i+1} \geq p$ for $i = 0, \ldots, k-1$.

Note that this is a set of special monomials.

Case $p = 2$ Let $x^E = x_1^{E_1} \cdots x_n^{E_n}$ be a special monomial. Denote, similarly as above, by ϵ_i the number of exponents equal to i. This means we can rewrite

$$x^E = x_1^{E_1} \cdots x_n^{E_n} = (x_{j_1} \cdots x_{j_{\epsilon_k}})^k (x_{j_{\epsilon_k+1}} \cdots x_{j_{\epsilon_k+\epsilon_{k-1}}})^{k-1} \cdots (x_{j_{\epsilon_k+\cdots+\epsilon_1+1}} \cdots x_{j_n})^0,$$

where $\{j_1, \ldots, j_n\} = \{1, \ldots, n\}$. Note that $\epsilon_0 + \cdots + \epsilon_k = n$. We have that the image of the transfer of a monomial x^E is trivial if and only if

$$|\,\mathrm{Iso}_{A_n}(x^E)| \equiv 0 \bmod 2,$$

where $\mathrm{Iso}_{A_n}(x^E) \subseteq A_n$ denotes the isotropy subgroup of x^E in A_n. Therefore, in order to conclude that $\mathrm{Tr}^{A_n}(x^E) = 0$, it is enough to find an element of order 2 in $\mathrm{Iso}_{A_n}(x^E)$:

If one of the ϵ_i's, say ϵ_k, is greater than 3, then

$$(j_1 j_2)(j_3 j_4) \in \mathrm{Iso}_{A_n}(x^E)$$

is an element of order 2 in the isotropy group of x^E, and $\mathrm{Tr}^{A_n}(x^E) = 0$. This leaves to consider monomials x^E such that $\epsilon_0, \ldots, \epsilon_k \leq 3$. If two (or more) of the ϵ_i's, say ϵ_k and ϵ_{k-1}, are greater than one, then we find

$$(j_1 j_2)(j_{\epsilon_k+1} j_{\epsilon_k+2}) \in \mathrm{Iso}_{A_n}(x^E).$$

Again, we can conclude that $\mathrm{Tr}^{A_n}(x^E) = 0$. Hence, the image of the transfer of the alternating group A_n is generated by the images $\mathrm{Tr}^{A_n}(x^E)$ of special monomials x^E such that

$$\{E_1, \ldots, E_n\} = \begin{cases} \{0, \ldots, n-1\} & \text{or} \\ \{0, \ldots, n-2\} & \text{or} \\ \{0, \ldots, n-3\} \text{ and } \epsilon_i = 3 & \text{for some } i = 0, \ldots, k. \end{cases}$$

Note that we find precisely two monomials of the first type with different orbits, namely

$$\nabla_n = \mathrm{Tr}^{A_n}(x_1 x_2^2 \cdots x_{n-1}^{n-1}),$$

and

$$\nabla_n' = \mathrm{Tr}^{A_n}(x_2 x_1^2 x_3^3 \cdots x_{n-1}^{n-1}) = \Big(\prod_{i<j}(x_i + x_j)\Big) + \nabla_n = \Delta_n + \nabla_n,$$

where the discriminant

$$\Delta_n = \prod_{i<j}(x_i + x_j) = \mathrm{Tr}^{\Sigma_n}(x_1 x_2^2 \cdots x_{n-1}^{n-1})$$

generates the image of the transfer of Σ_n, see [6].

Observe that for $n = 3$ (*i.e.*, the non-modular situation) the above given set of monomials gives

$$\mathrm{Tr}^{A_3}(x_1^2 x_2) = \nabla_3, \quad \mathrm{Tr}^{A_3}(x_1^2 x_3) = \nabla_3' = e_1 e_2 + e_3 + \nabla_3,$$
$$\mathrm{Tr}^{A_3}(x_1 x_2) = e_2, \quad \text{and} \quad \mathrm{Tr}^{A_3}(x_1) = e_1,$$

and we once more see that the transfer is surjective.

Finally, we can apply, as in the case of odd characteristic, Theorem 1.4 and find that the height $\mathrm{ht}\left(\mathrm{Im}(\mathrm{Tr}^{A_n})\right) = 2$ for $n \geq 4$, *i.e*, for the modular situation. However, we could also prove this, without making use of M. Feshbach's transfer theorem, by direct calculation: Consider the subgroup

$$\mathbb{Z}/2 < A_n,$$

generated by $(12)(34) \in A_n$. By Lemma 1.3 of [7] we have

$$\left(\mathrm{Im}(\mathrm{Tr}^{A_n})\right)^e \subseteq \mathrm{Im}(\mathrm{Tr}^{\mathbb{Z}/2}) \subseteq \mathbb{F}[V]^{\mathbb{Z}/2}.$$

Hence, we have, by going-up and down, that

$$\mathrm{ht}\left(\mathrm{Im}(\mathrm{Tr}^{A_n})\right) \leq \mathrm{ht}\left(\mathrm{Im}(\mathrm{Tr}^{\mathbb{Z}/2})\right).$$

By Theorem 2.4 in [8]

$$\mathrm{ht}\left(\mathrm{Im}(\mathrm{Tr}^{\mathbb{Z}/2})\right) \leq 2.$$

Therefore, also the image of the transfer of A_n has height at most 2. We claim that its height is precisely 2. Since the image of the transfer is non-trivial, its height is positive, and we assume to the contrary, that

$$\mathrm{ht}\left(\mathrm{Im}(\mathrm{Tr}^{A_n})\right) = 1.$$

Then also the ideals

$$\mathrm{Im}(\mathrm{Tr}^{A_n}) \cap \mathbb{F}[V]^{\Sigma_n} \supseteq \mathrm{Im}(\mathrm{Tr}^{\Sigma_n})$$

have height 1. Recall that the image of the transfer of the symmetric group Σ_n is a prime ideal generated by the discriminant, see [6]. This implies

$$\mathrm{Im}(\mathrm{Tr}^{A_n}) \cap \mathbb{F}[V]^{\Sigma_n} = \mathrm{Im}(\mathrm{Tr}^{\Sigma_n}).$$

However, take

$$\nabla_n, \nabla_n' \in \mathrm{Im}(\mathrm{Tr}^{A_n}).$$

Then the orbit of, say, ∇_n under Σ_n is $\{\nabla_n, \nabla_n'\}$, so

$$\Delta_n = \nabla_n + \nabla_n', \nabla_n \cdot \nabla_n' \in \mathbb{F}[V]^{\Sigma_n}.$$

The second polynomial is not in the image of the transfer of Σ_n

$$\nabla_n \cdot \nabla_n' \notin (\nabla_n + \nabla_n') = (\Delta_n) = \mathrm{Im}(\mathrm{Tr}^{\Sigma_n}) = \mathbb{F}[V]^{\Sigma_n},$$

because $\nabla_n \cdot \nabla_n' \in \mathbb{F}[V]^{\Sigma_n}$ is irreducible. Since

$$\nabla_n \cdot \nabla_n' \in \mathrm{Im}(\mathrm{Tr}^{A_n}) \cap \mathbb{F}[V]^{\Sigma_n},$$

this is a contradiction. So, $\mathrm{ht}\left(\mathrm{Im}(\mathrm{Tr}^{A_n})\right) = 2$ for $n \geq 4$.

We summarize the results of this section in a theorem.

Theorem 2.1 *We assume that the characteristic divides the order of the alternating group, for otherwise the transfer is surjective. Set*

$$x^E = x_1^{E_1} \cdots x_n^{E_n} = (x_{j_1} \cdots x_{j_{\epsilon_k}})^k (x_{j_{\epsilon_k+1}} \cdots x_{j_{\epsilon_k+\epsilon_{k-1}}})^{k-1} \cdots (x_{j_{\epsilon_k+\cdots+\epsilon_1+1}} \cdots x_{j_n})^0$$

and

$$\epsilon_0 = n - (\epsilon_1 + \cdots + \epsilon_k) \quad \text{and} \quad \{j_1, \ldots, j_n\} = \{1, \ldots, n\}.$$

If the characteristic p is odd then the image of the transfer of the alternating group A_n is generated by the discriminant Δ_n and $\mathrm{Tr}^{A_n}(x^E)$, where

(2) $\epsilon_i < p$ *for* $i = 0, \ldots, k$, *and*
(3) $\epsilon_i + \epsilon_{i+1} \geq p$ *for* $i = 0, \ldots, k-1$.

Moreover its height is precisely $p - 1$.

If the characteristic is even, then the image of the transfer is generated by $\mathrm{Tr}^{A_n}(x^E)$, where

$$\{E_1, \ldots, E_n\} = \begin{cases} \{0, \ldots, n-1\} & \text{or} \\ \{0, \ldots, n-2\} & \text{or} \\ \{0, \ldots, n-3\} \text{ and } \epsilon_i = 3 & \text{for some } i = 0, \ldots, k. \end{cases}$$

Moreover, this ideal has height precisely 2.

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