Teaching Note

Trigonometric identities from the mystic rose

Introduction

A regular n-gon with all its diagonals is commonly known as a mystic rose. In this Note we use its symmetries to derive trigonometric sums and products where the arguments are consecutive multiples of $\frac{\pi}{n}$. It is a companion piece to [1], where we projected regular polygons onto their diameters to obtain trigonometric sums with the same property. In what follows $k \in Z^{+}$, $k \geqslant 1$, $n \in Z^{+}$, $n \geqslant 3$. We shall use the notation we adopt in Figure 1 throughout the piece.

Tasks for students appear in italics. They will find a discussion, together with further results, at [2].

Results from Figure 1

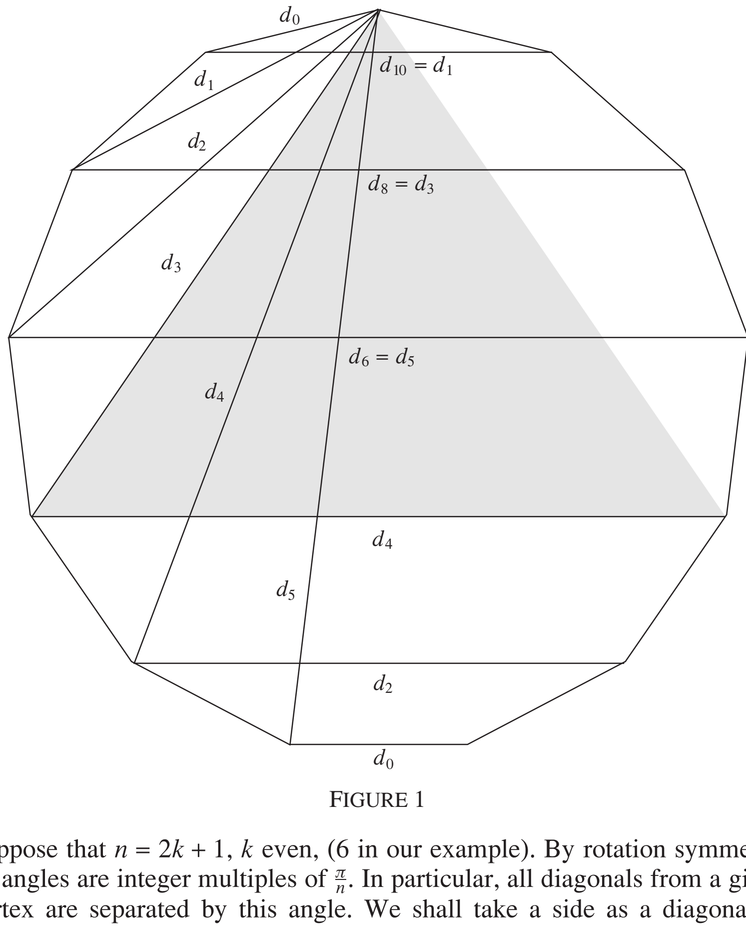

FIGURE 1

Suppose that $n = 2k + 1$, k even, (6 in our example). By rotation symmetry, all angles are integer multiples of $\frac{\pi}{n}$. In particular, all diagonals from a given vertex are separated by this angle. We shall take a side as a diagonal of

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length d_0 and label the $2k$ lengths in sequence d_0 to d_{2k-1}, a convention we shall adopt throughout the piece. By reflective symmetry there are just k lengths, running from d_0 to d_{k-1}. The lengths are paired so that $d_s = d_t$, where $s + t = 2k - 1$, so s and t are of opposite parity. We take the complete series of k isosceles triangles like the one shaded and write the base in terms of one of the equal sides:

$d_1 = 2d_0 \sin \frac{11\pi}{26}$, These k equations contain all k lengths both on
$d_3 = 2d_1 \sin \frac{9\pi}{26}$, the left and on the right. *How did we ensure this*?
$d_5 = 2d_2 \sin \frac{7\pi}{26}$, *Multiply, cancel, rearrange to obtain this identity*:
$d_7 = d_4 = 2d_3 \sin \frac{5\pi}{26}$, $\prod_{j=0}^{j=k-1} \sin \frac{(2j+1)\pi}{2(2k+1)} = 2^{-k}, k \geqslant 1.$ [A]
$d_9 = d_2 = 2d_4 \sin \frac{3\pi}{26}$ *Rewrite it in terms of cosine to obtain a*
$d_{11} = d_0 = 2d_5 \sin \frac{\pi}{26}$ *simpler expression*, [B].

The cases $k = 2, 3$ can be confirmed by applying the cosine rule in figures A, B respectively in Figure 2. *Confirm this*. In each case one isosceles triangle has been divided into two [3].

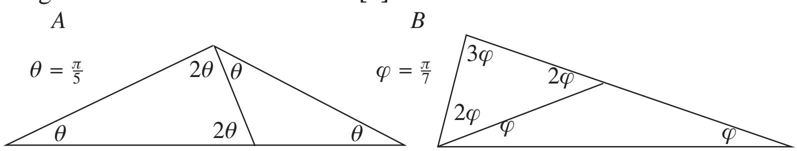

FIGURE 2

Results from Figure 3

In [4] the author published a figure which shows that the isosceles triangle on d_0 as base and its apex at the opposite vertex of a regular $(2k + 1)$-gon can be dissected into k isosceles triangles, each with side-length d_0 and base angles in arithmetic progression from $\frac{\pi}{2k+1}$ to $\frac{k\pi}{2k+1}$, in the manner shown in Figure 3. The lengths of the bases are labelled l_1 to l_k. As before, $\theta = \frac{\pi}{2k+1}$.

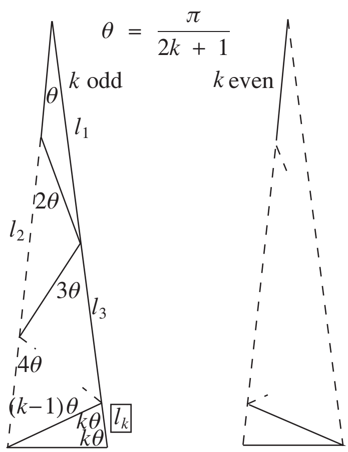

FIGURE 3

Equate equal sides of the triangle to derive this identity:

$$\sum_{j=0}^{j=k} (-1)^j \cos \frac{j\pi}{2k+1} = 0.5, \quad k \geqslant 1. \qquad \text{[C]}$$

Interpret the identity in terms of the area under the cosine curve.

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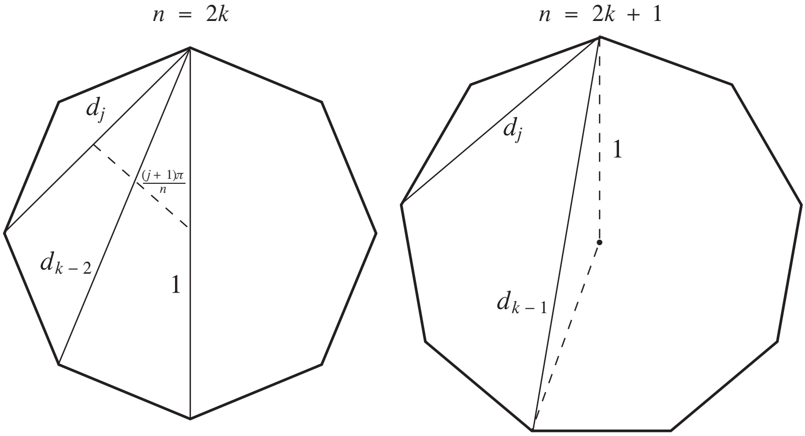
Results from Figure 4

Including sides with diagonals, the product of the lengths of the diagonals from a given vertex of a regular n-gon inscribed in the unit circle is n. (This surprising result is readily shown with complex numbers. See, for example, [5].) We shall call this the product-of-diagonals theorem.

See Figure 4. The length of the general diagonal d_j, defined above, is

$$2 \sin \frac{(j + 1)\pi}{n}.$$



$k - 1$ paired diagonals $\quad$ k paired diagonals

FIGURE 4

(a) If $n = 2k$, we have $(k - 1)$ paired diagonals and a single diagonal of length 2. Using the product-of-diagonals theorem, substituting the above value for d_j, we have

$$\prod_{j=1}^{k-1} \sin \frac{j\pi}{2k} = \sqrt{k} \cdot 2^{1-k}, \ k \geqslant 2. \qquad [D]$$

When $k = 2^{2t}$, $t \in Z^+$, $t \geqslant 1$, the right-hand side equals $2^{1+t-2^{2t}}$, a negative integer power of 2. Thus

$$\prod_{j=1}^{k-1} \sin \frac{j\pi}{2k} = 2^p = \prod_{j=1}^{k-1} \cos \frac{(k-j)\pi}{2k}, \text{where } k = 2^{2t}, p + 1 + t - 2^{2t}, t \geqslant 1. \quad [E]$$

For these negative integer powers of 2, we can equate a product from [E] with the corresponding expression from [A]. Taking $t = 1$, we have

$$\cos \frac{\pi}{5} \cos \frac{2\pi}{5} = \cos \frac{\pi}{8} \cos \frac{\pi}{4} \cos \frac{3\pi}{8} = 2^{-2}.$$

Write the corresponding equation for $t = 2$.

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(b) If $n = 2k + 1$, we have k paired diagonals, leading to

$$\prod_{j=1}^{k} \sin \frac{j\pi}{2k+1} = \sqrt{2k+1} \cdot 2^{-k}, \qquad k \geqslant 1. \qquad \text{[F]}$$

For what values of k is the product rational?

We can identify nested products in [D] and [F]. For example,

$$\text{[D]}, k = 9: \sin \tfrac{\pi}{18} \cdot \sin \tfrac{2\pi}{18} \cdot \sin \tfrac{3\pi}{18} \cdot \sin \tfrac{4\pi}{18} \cdot \sin \tfrac{5\pi}{18} \cdot \sin \tfrac{6\pi}{18} \cdot \sin \tfrac{7\pi}{18} \cdot \sin \tfrac{8\pi}{18} = 3 \cdot 2^{-8},$$

$$\text{[E]}, k = 4: \quad \sin \tfrac{\pi}{9} \cdot \quad \cdot \sin \tfrac{2\pi}{9} \cdot \quad \cdot \sin \tfrac{3\pi}{9} \cdot \quad \cdot \sin \tfrac{4\pi}{9} = 3 \cdot 2^{-4}.$$

Divide the first equation by the second. What leads you to expect the result?

Results from Figure 5

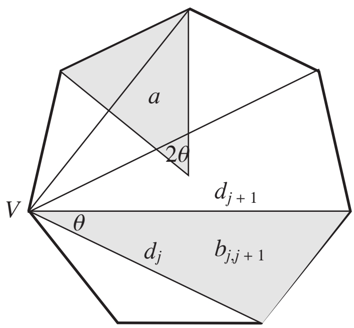

FIGURE 5

We express the area of the n-gon in two ways:

as the sum of n elements like a and as the sum of the sequence of elements like $b_{j,j+1}$ sharing a vertex V. In the case of an odd-sided polygon, equating these sums yields

$$\sum_{j=1}^{2k-1} \sin j\theta \sin (j+1)\theta = \frac{2k+1}{2} \cos\theta,$$

$$k \geqslant 1, \theta = \frac{\pi}{2k+1}. \qquad \text{[G]}$$

Derive the corresponding result for an even-sided polygon, [H]

Results from Figure 6

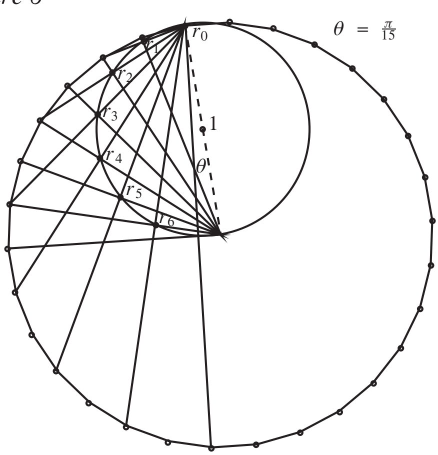

FIGURE 6

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Figure 6 shows a regular n-gon where n has the form $2(2k+1)$. Here $k = 7$. Within it is shown an n-gon where $n = 2k + 1$. The circumdiameter of the second is the radius of the first, 1. The segment r_j has length $\cos\frac{j\pi}{2k+1}$. These segments bisect at right angles diagonals of a larger $(2k+1)$-gon inscribed in that with $2(2k+1)$ sides. Enlarging by factor $\frac{1}{2}$ from the vertex shown, we have a diagonal d_j of the small polygon, whose size is given by

$$d_j = r_{j+1} \tan\frac{(j+1)\pi}{2k+1}.$$

From the product-of-diagonals theorem we take half the diagonals and correct for the fact that the circumradius of the small polygon is $\frac{1}{2}$:

$$\prod_{j=0}^{k-1} d_j = \sqrt{2k+1} \cdot 2^{-k}.$$

We then substitute for d_j, noting a change of limiting values:

$$\prod_{j=1}^{k} r_j \tan\frac{j\pi}{2k+1} = \prod_{j=1}^{k} r_j \prod_{j=1}^{k} \tan\frac{j\pi}{2k+1} = \sqrt{2k+1} \cdot 2^{-k}.$$

But we know from [A] that $\prod_{j=1}^{k} r_j = 2^{-k}$. Substituting and cancelling gives

$$\prod_{j=1}^{k} \tan\frac{j\pi}{2k+1} = \sqrt{2k+1},\ k \geqslant 1. \qquad \text{[I]}$$

Check this result by dividing [F] *by* [B].

For what values of k is the product an integer?

Look for nested products as you did for [D] *and* [F]. *For example, taking the value* $k = 4$, *we have*

$$\tan\frac{\pi}{9} \tan\frac{2\pi}{9} \tan\frac{3\pi}{9} \tan\frac{4\pi}{9} = 3.$$

But we know that $3 = \left(\tan\frac{3\pi}{9}\right)^2$, so we can write

$$\tan\frac{3\pi}{9} = \tan\frac{\pi}{3} = \tan\frac{\pi}{9} \tan\frac{2\pi}{9} \tan\frac{4\pi}{9}.$$

Results from Figure 7

In Figure 7 we add unit vectors, $\cos m\theta + i \sin m\theta$, which complete a circuit in which each is rotated an equal angle with respect to that preceding. Each symbol combines the previous vertex with the next, so we have a sequence like 03 - 36 - 61. A zero sum results from the addition of either the real or imaginary components, providing a sum in cosine and sine respectively. In Figure 7A the circuit follows the polygon perimeter; in Figures 7B, 7C it traces star polygons through the diagonals, (whether simple, or compound as in the case of the hexagon).

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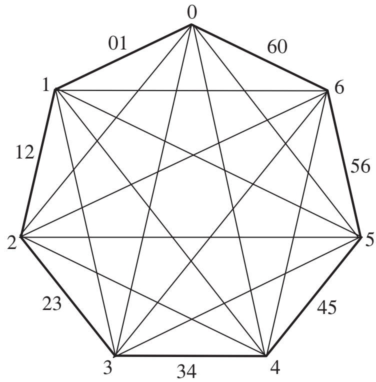

FIGURE 7A

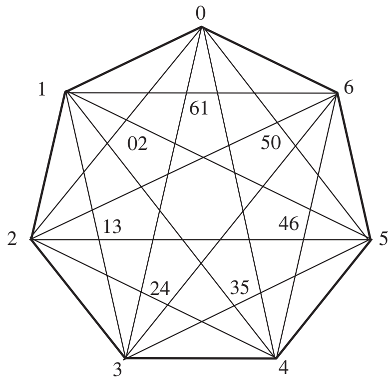

FIGURE 7B

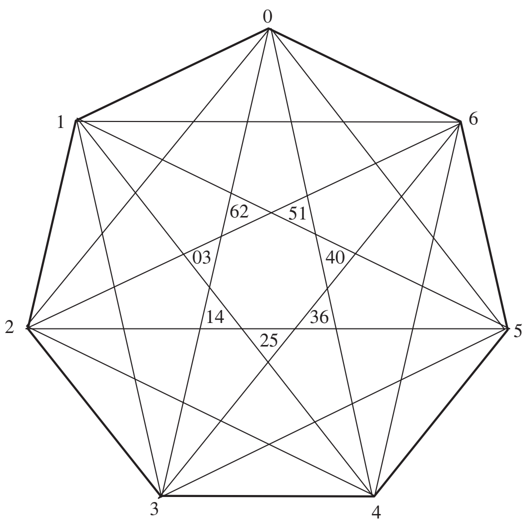

FIGURE 7C

We have these identities, the same for sine and cosine:

$$\sum_{j=1}^{n} \cos\left(2j\frac{\pi}{n}\right) = \sum_{j=1}^{n} \cos\left((2j-1)\frac{\pi}{n}\right) = 0, \; n \geqslant 2, \qquad [J]$$

$$\sum_{j=1}^{n} \sin\left(2j\frac{\pi}{n}\right) = \sum_{j=1}^{n} \sin\left((2j-1)\frac{\pi}{n}\right) = 0, \; n \geqslant 2. \qquad [K]$$

Considering the unit circle, how do you interpret these results?

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PAUL STEPHENSON
Bohmerstrrasse 66, 45144,
Essen, Germany
e-mail: *pstephenson1@me.com*

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