Canad. Math. Bull. Vol. **27** (1), 1984

(HM)-SPACES AND MEASURABLE CARDINALS

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ABSTRACT. A locally convex space E is called an (HM)-space if E has invariant nonstandard hulls. In this paper we prove that if E is an (HM)-space, then E is a $T(\mu)$-space, where μ is the first measurable cardinal. This is equivalent to say that in an (HM)-space, with $\dim(E) \geqq \mu$, does not exist a continuous norm. With this result, we prove that there exists an inductive semi-reflexive space E such that the bounded sets in E are finite-dimensional but E is not an (HM)-space. Thus, we answer negatively to an open problem raised up by Bellenot. In this paper, we do not use nonstandard analysis.

Let E be a locally convex space; a *nonstandard hull* of E is a standard locally convex space $\hat{E}$ constructed from a nonstandard model for E. If the nonstandard hulls do not depend on the used nonstandard models, E is said to be an *(HM)-space.* [5, 9].

In Section 1, we prove that every (HM)-space is a $T(\mu)$-space, where μ is the first measurable cardinal. That is equivalent to prove that it can not be defined a continuous norm on an (HM)-space with $\dim(E) \geqq \mu$. Applying this result, a part of a theorem of Henson and Moore [6] about the dimension of an (HM)-space is enlarged.

In Section 2, we prove assuming the existence of measurable cardinals, that there exists an inductive semi-reflexive [2] space E such that bounded sets in E are finite-dimensional but E is not an (HM)-space. This gives a negative answer to a question raised by Bellenot [1].

This is a part of the author's Ph.D. Thesis prepared at the University of Sevilla under the supervision of Professor Juan Arias de Reyna. I wish to thank Professor Arias de Reyna for his interest and advice.

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Notation. In the following, $\mathbb{N}$ will denote the set of the positive integers; E a separated locally convex space over $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$, real or complex numbers), E' the topological dual of E, E^* the algebraic dual and $\mathfrak{U}(E)$ the filter of all neighborhoods of 0 of the space E. If $A \subset E$ we denote by $\langle A \rangle$ the linear hull

Received by the editors August 20, 1982 and in revised form April 22, 1983.
AMS Subject Classification (1980): Primary 46A05, secondary 03E55.

of A. It is known that E is an (HM)-space if and only if every *almost-bounded ultrafilter* on E is a Cauchy ultrafilter [5], (an ultrafilter $\mathfrak{F}$ is said to be *almost-bounded* if for every U in $\mathfrak{U}(E)$, there exists a positive integer n such that nU is in $\mathfrak{F}$).

It is easy to see that (HM)-spaces are stable for projective constructions (for instance, subspaces and products).

Let α be an infinite cardinal. The space E is said to be a *$T(\alpha)$-space* if for every $U \in \mathfrak{U}(E)$, there exists a subspace M of E contained in U such that $\operatorname{cod}(M) < \alpha$[4]. A cardinal k is said to be *measurable* if it is uncountable and there exists a non-principal ultrafilter $\mathfrak{F}$ on k that is *k-complete*, i.e. if A_i belongs to $\mathfrak{F}$ for $i < k$, then $\bigcap \{A_i : i < k\}$ is in $\mathfrak{F}$ [3], p. 186. By ω we denote the least non-zero limit ordinal, and by μ the first measurable cardinal. A relationship between (HM)-spaces and measurable cardinals was obtained by Henson and Moore [6].

By $\omega(k)$, where k is an infinite cardinal we denote the space formed by all scalar families $\{x_\alpha : \alpha < k\}$ and by $\varphi(k)$ the space of all scalar families with finitely many non-zero coordinates. If $x = \{x_\alpha : \alpha < k\}$ is in $\omega(k)$ and $y = \{y_\alpha : \alpha < k\}$ is in $\varphi(k)$ we denote by $\langle x, y\rangle$ the canonical bilinear form, i.e. $\langle x, y\rangle = \sum_{\alpha<k} x_\alpha y_\alpha$ (see [7] p. 53, 56). We denote $\Omega = \omega(\aleph_0)$ and $\varphi = \varphi(\aleph_0)$. Finally, by $l^2(\mu)$ we denote the set of mappings x from μ to K such that $\sum_{\alpha<\mu} |x(\alpha)|^2$ is summable. This is a linear space under the usual operations, and an inner product is defined on it by $(x \mid y) = \sum_{\alpha<\mu} x(\alpha)\overline{y(\alpha)}$. With this product, $l^2(\mu)$ is a Hilbert space.

1. The main theorem

THEOREM. *Let E be an* (HM)-*space. Then the following conditions are satisfied*:

(i) *E is a $T(\mu)$-space.*

(ii) *If* $\dim(E) \geq \mu$, *it can not be defined a continuous norm on E.*

Proof. We first prove that (ii) implies (i). Assume that E is not a $T(\mu)$-space. Then there exists $U \in \mathfrak{U}(E)$ such that for every subspace M of E contained in U, one has $\operatorname{cod}(M) \geq \mu$. Let N be the greatest subspace of E contained in U, and F an algebraic supplement of N. It is clear that F is an (HM)-space and $\dim(F) \geq \mu$. However, the Minkowski functional of $U \cap F \in \mathfrak{U}(F)$ is a continuous norm on F.

Let us prove (ii). Assume that p is a continuous norm on E. We construct, by transfinite induction, a family $\{e_\alpha : \alpha < \mu\}$ of elements of E such that for $\xi \neq \eta$, η, $\xi < \mu$, $p(e_\xi - e_\eta) \geq 1$.

Suppose that for $\alpha < \mu$, we have a family $\{e_\beta : \beta < \alpha\}$ satisfying $p(e_\xi - e_\eta) \geq 1$, for $\xi \neq \eta$, ξ, $\eta < \alpha$. Put $F_\alpha = \overline{\langle\{e_\beta : \beta < \alpha\}\rangle}$. Since every element in F_α is the limit for the topology on E defined by the norm p, of a sequence in $\langle\{e_\beta : \beta < \alpha\}\rangle$, we have:

$$\dim(F_\alpha) \le (\mathrm{card}(\alpha))^{\aleph_0} < (2^{\mathrm{card}(\alpha)})^{\aleph_0} = 2^{\mathrm{card}(\alpha)} < \mu$$

because μ is strongly inaccessible [3], p. 193. Hence there exists an element $x_\alpha \in E \setminus F_\alpha$. Writing $e_\alpha = \lambda x_\alpha$ for an adequate real number λ, the family $\{e_\beta : \beta \le \alpha\}$ satisfies $p(e_\xi - e_\eta) \ge 1$ for $\xi \ne \eta$, $\xi, \eta \le \alpha$.

Since μ is the first measurable cardinal, there exists a countably complete non-principal ultrafilter $\mathfrak{F}$ on E such that the family $\{e_\alpha : \alpha < \mu\}$ is in $\mathfrak{F}$. By the given construction, $\mathfrak{F}$ is not a Cauchy ultrafilter. Nevertheless, $\mathfrak{F}$ is almost-bounded, because if $V \in \mathfrak{U}(E)$ is absolutely convex, $E = \bigcup nV$ and there exists an $n \in \mathbb{N}$ such that $nV \in \mathfrak{F}$.

Remark. In this proof we have seen that (ii) implies (i). It is easy to see that (i) also implies (ii).

Recall that a nonempty class of locally convex spaces is said to be a variety if it closed under the operations of taking subspaces, quotient spaces, arbitrary cartesian products and isomorphic images. The variety generated by a class $\mathfrak{C}$ of locally convex spaces is the intersection of all varieties containing $\mathfrak{C}$. If $\mathfrak{C}$ consists of a single locally convex space, then the variety is said to be singly generated ([4]). It is proved in [4], theorem 2.7 that a variety $\mathfrak{B}$ is singly generated if and only if there exists an infinite cardinal m such that every space in $\mathfrak{B}$ is a $T(m)$-space.

Thus we can state:

COROLLARY. *The variety generated by the* (HM)-*spaces, is singly generated.*

DEFINITION. Let $\mathfrak{P}$, $\mathfrak{Q}$ two families of seminorms on E. We say that $\mathfrak{P}$ is equivalent to $\mathfrak{Q}$ ($\mathfrak{P} \sim \mathfrak{Q}$) if they define the same topology on E. We call minimal cardinal of $\mathfrak{P}$ (see [8]) to the cardinal:

$$\alpha(\mathfrak{P}) = \inf\{\mathrm{card}(J) : \{p_j : j \in J\} \sim \mathfrak{P}\}.$$

Then we have the next result:

PROPOSITION. *Let E be an* (HM)-*space and $\mathfrak{P}$ a family of continuous seminorms on E such that $\alpha(\mathfrak{P}) < \mu$. Then* $\mathrm{cod}(N) < \mu$, *where*

$$N = \{x \in E : p(x) = 0, \forall p \in \mathfrak{P}\}$$

Proof. We may suppose that $\alpha(\mathfrak{P}) = \mathrm{card}(\mathfrak{P})$. For every $p \in \mathfrak{P}$ denote $N_p = \{x \in E : p(x) = 0\}$. Then $N = \bigcap \{N_p : p \in \mathfrak{P}\}$. Denote by G_p an algebraic supplement of N_p. Then G_P is an (HM)-space and p restricted to G_p is a continuous norm. Therefore $\dim(G_p) < \mu$ for every $p \in \mathfrak{P}$. Since $\alpha(\mathfrak{P}) < \mu$ we have $\mathrm{cod}(N) < \mu$.

Applying this proposition we prove a corollary that enlarges a part of a result of Henson and Moore [6], Theorem 2.

COROLLARY. *Assume that the topology of an* (HM)-*space E is defined by a family of seminorms $\mathfrak{P}$ such that $\alpha(\mathfrak{P})<\mu$. Then* $\dim(E)\leq\mu$.

2. **A counter-example.** Assuming the existence of measurable cardinals we answer to a problem raised up by Bellenot [1].

THEOREM. *Assuming the existence of measurable cardinals, there exists an inductive semi-reflexive space E, such that bounded sets in E are finite-dimensional but E is not an* (HM)-*space.*

Proof. Let E be the space $\varphi(\mu)$ and F the linear space of all families $\{a_\alpha : \alpha<\mu\}$ where a_α is a scalar and $\mathrm{card}(\{\alpha<\mu : a_\alpha\neq 0\})$ is countable. The bilinear form $\langle x, a\rangle=\sum x_\alpha a_\alpha$ where $a=\{a_\alpha : \alpha<\mu\}$ is in F and $x=\{x_\alpha : \alpha<\mu\}$ is in E defines a duality between E and F. For every a in F, we define the seminorm on E:

$$p_a(x)=\sum |a_\alpha x_\alpha|\,(a=\{a_\alpha : \alpha<\mu\}; \qquad x=\{x_\alpha : \alpha<\mu\})$$

Let T_0 be the topology defined on E by the seminorms $\{p_a : a\in F\}$. It is easy to see that F is the dual of $E(T_0)$.

1. Every bounded set in $E(T_0)$ is finite-dimensional.

For every $x\in E$ (resp. $a\in F$) define $\mathrm{supp}(x)=\{\alpha<\mu : x_\alpha\neq 0\}$ (resp. $\mathrm{supp}(a)=\{\alpha<\mu : a_\alpha\neq 0\}$). Let B be a bounded set in $E(T_0)$. We will prove that $A=\bigcup\{\mathrm{supp}(x): x\in B\}$ is a finite set. Otherwise there exists a sequence $\{x^n : n\in\mathbb{N}\}$ in B such that $\mathrm{supp}(x^n)$ is not contained in $\bigcup\{\mathrm{supp}(x^i): 1\leq i\leq n-1\}$.

Choose $\alpha_n\in\mathrm{supp}(x^n)\setminus\bigcup\{\mathrm{supp}(x^i): 1\leq i\leq n-1\}$ and $a\in F$ defined by $a_\alpha=0$ if $\alpha\neq\alpha_n$; $a_{\alpha_n}=n/x^n_{\alpha_n}$.

It is clear that $p_a(x^n)\geq n$. Thus B is not a bounded set.

Let T be the topology defined on E by the seminorms $\{p_a : a\in F\}$ and p; where p is the usual norm of the Hilbert space $l^2(\mu)$. It is easy to prove that F is the dual space of $E(T)$.

2. Since $T\geq T_0$, every bounded set in $E(T)$ is finite-dimensional. But by (ii), $E(T)$ is not an (HM)-space.

3. $E(T)$ is an inductive semi-reflexive space.

It suffices to prove that $E(T_0)$ is an inductive semi-reflexive space. Let $u\in E'^*$ be a linear form that is bounded on the equicontinuous subsets of E'. For every $a\in F$, consider a normal covering $M_a=\{\{b_\alpha : \alpha<\mu\}: |b_\alpha|\leq|a_\alpha|, \forall\alpha<\mu\}$. Denoting $U_a=\{x\in E: p_a(x)\leq 1\}$, it is known that $M_a=U_a^0$ [7], 30.2(3). Therefore $\{M_a : a\in F\}$ is a basis of equicontinuous sets of E'.

Assume that $A\subset\mu$ is countable and let $\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots$ be a enumeration of A. Then the space $\{a\in F: \mathrm{supp}(a)\subset A\}$ is isomorphic to Ω. To see that consider a mapping $f:\Omega\to F$ such that $f(\{x_n : n\in\mathbb{N}\})=\{a_\alpha : \alpha<\mu\}$ where $a_\alpha=0$ if $\alpha\notin A$ and $a_\alpha=x_n$ if $\alpha=\alpha_n$. Define a mapping $g:\varphi\to E$ such that

$g(\{z_n : n \in \mathbb{N}\}) = \{y_\alpha : \alpha < \mu\}$ where $y_\alpha = 0$ if $\alpha \notin A$ and $y_\alpha = z_n$ if $\alpha = \alpha_n$. For every $z \in \varphi$ and $y \in \Omega$ it is clear that $\langle z, y \rangle = \langle g(z), f(y) \rangle$. The space φ, endowed with the finest locally convex topology is complete and nuclear, hence it is an inductive semi-reflexive space [2].

Since $u \circ f \in \varphi'^*$ is bounded on equicontinuous subsets of φ', there exists $z^A \in \varphi$ such that $(u \circ f)(x) = \langle z^A, x \rangle = \langle g(z^A), f(x) \rangle$ for every $x \in \Omega$, i.e. $u(a) = \langle x^A, a \rangle$ for every $a \in F$ such that $\operatorname{supp}(a) \subset A$ and $x^A \in E$.

Let μ be equal to $\bigcup \{A_\alpha : \alpha < \mu\}$ where each A_α is a countable set (for instance, $A_\alpha = [\alpha\omega, (\alpha+1)\omega)$). There exists for every $\alpha < \mu$ an element $x^{(\alpha)}$ in E such that $u(a) = \langle x^{(\alpha)}, a \rangle$ for every $a \in F$ with $\operatorname{supp}(a) \subset A_\alpha$. We prove that all $x^{(\alpha)}$ are zero except at most finitely many of them. Otherwise let $\alpha_1, \ldots, \alpha_n, \ldots$ be a sequence such that $x^{(\alpha_i)} \neq 0$ and $N = \bigcup \{A_{\alpha_i} : i \in \mathbb{N}\}$. Since N is a countable set there exists $x^N \in E$ such that $u(a) = \langle x^N, a \rangle$ for every $a \in F$ with $\operatorname{supp}(a) \subset N$. Since x^N restricted to A_{α_i} is equal to $x^{(\alpha_i)}$, x^N has infinitely many coordinates non-zero.

Consider $x = \sum x^{(\alpha)} \in E$. For every $a \in F$ we have $u(a) = \langle x, a \rangle$. Indeed, choose $M = \operatorname{supp}(a)$ and $x^M \in E$ such that $u(b) = \langle x^M, b \rangle$ for every $b \in F$ with $\operatorname{supp}(b) \subset M$. If $\alpha < \mu$ and $\operatorname{supp}(b) \subset M \cap A_\alpha$ we have $u(b) = \langle x, b \rangle$. Thus x is equal to x^M in $M \cap A_\alpha$ for every $\alpha < \mu$. Therefore x is equal to x^M in M and $u(a) = \langle x^M, a \rangle = \langle x, a \rangle$.

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