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HIGHER-ORDER ESTIMATES FOR FULLY NONLINEAR DIFFERENCE EQUATIONS. I

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Abstract The purpose of this work is to establish *a priori* $C^{2,\alpha}$ estimates for mesh function solutions of nonlinear positive difference equations in fully nonlinear form on a uniform mesh, where the fully nonlinear finite-difference operator $\mathcal{F}_h$ is concave in the second-order variables. The estimate is an analogue of the corresponding estimate for solutions of concave fully nonlinear elliptic partial differential equations. We deal here with the special case that the operator does not depend explicitly upon the independent variables. We do this by discretizing the approach of Evans for fully nonlinear elliptic partial differential equations using the discrete linear theory of Kuo and Trudinger. The result in this special case forms the basis for a more general result in part II. We also derive the discrete interpolation inequalities needed to obtain estimates for the interior $C^{2,\alpha}$ semi-norm in terms of the C^0 norm.

Keywords: fully nonlinear difference equations; discrete *a priori* Hölder estimates; discrete seminorms; discrete interpolation inequalities

AMS 1991 *Mathematics subject classification:* Primary 35J60; 35J15; 39A12
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1. Introduction

The purpose of this work is to use results for linear finite-difference equations to derive certain Hölder estimates for centred second-order difference quotients of solutions of fully nonlinear positive finite-difference equations. Following [**14**], let E be an arbitrary set, which is called a *mesh*. A linear difference operator L acting on $\mathcal{M}(E)$, the set of real mesh functions, is given by

$$Lu(x) = \sum_{z\in E} A(x,z)u(z)$$

for any mesh function u, where A is a real-valued function on $E \times E$, which is non-zero for only a finite number of z values for each $x \in E$. The operator L is *monotone* if

$$A(x,z) \geqslant 0, \quad \forall (x,z) \in E \times E, \quad x \neq z,$$

and *positive* if, in addition, for all $x \in E$,

$$\sum_z A(x,z) \leqslant 0.$$

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If D is a subset of E, then the *interior* of D, relative to L, is defined by (cf. Definition 1.3)

$$\operatorname{int}_L(D) = \{x \in D \mid A(x,z) = 0,\ \forall z \notin D\},$$

and the *boundary* of D, relative to L, is defined by $\operatorname{bdy}_L(D) = D \setminus \operatorname{int}_L D$. With these definitions a simple maximum principle follows.

> If L is positive (monotone) and $Lu(x) > 0$ for all $x \in \operatorname{int}_L(D)$, then u cannot have a positive (zero) maximum in D at an interior point.

On a uniform mesh, we may write certain positive difference operators in a more familiar form. Let h be a positive parameter and let

$$\mathbb{Z}_h^n = \{x = h(l_1, \ldots, l_n) \mid l_i \in \mathbb{Z},\ i = 1, \ldots, n\}$$

denote the orthogonal lattice or *mesh*, with *mesh length* h, in Euclidean n-space $\mathbb{R}^n$. A real-valued function u on $\mathbb{Z}_h^n$ is called a mesh function, and for fixed $y (\neq 0) \in \mathbb{Z}_h^n$ we define the following difference operators acting on the linear space of mesh functions $\mathcal{M}$:

$$\left.\begin{aligned} \delta_y^+ u(x) &= \frac{1}{\|y\|_2}\{u(x+y) - u(x)\}, \\ \delta_y^- u(x) &= \frac{1}{\|y\|_2}\{u(x) - u(x-y)\}, \\ \delta_y u(x) = \tfrac{1}{2}(\delta_y^+ + \delta_y^-)u(x) &= \frac{1}{2\|y\|_2}\{u(x+y) - u(x-y)\}, \\ \delta_y^2 u(x) = \delta_y^+ \delta_y^- u(x) &= \frac{1}{\|y\|_2^2}\{u(x+y) - 2u(x) + u(x-y)\}, \end{aligned}\right\} \tag{1.1}$$

where $\|y\|_2$ is the Euclidean norm of y. Then we may consider second-order difference operators of the form

$$L_h u(x) = \sum_{y \neq 0} a(x,y)\delta_y^2 u(x) + \sum_{y \neq 0} b(x,y)\delta_y u(x) + c(x)u(x),$$

with real coefficients. This may be written as

$$\begin{aligned} L_h u(x) = \sum_{y \neq 0} &\left[\frac{a(x,y)}{\|y\|_2^2} + \frac{b(x,y)}{2\|y\|_2}\right] u(x+y) \\ &+ \sum_{y \neq 0}\left[\frac{a(x,y)}{\|y\|_2^2} - \frac{b(x,y)}{2\|y\|_2}\right] u(x-y) + \left[-2\sum_{y \neq 0}\frac{a(x,y)}{\|y\|_2^2} + c(x)\right] u(x). \end{aligned}$$

If we think of L_h in the form $L_h u(x) = \sum A(x,z)u(z)$, then we see that

$$a(x,y) - \tfrac{1}{2}\|y\|_2 |b(x,y)| \geqslant 0, \quad \forall x, y \in \mathbb{Z}_h^n, \quad y \neq 0, \tag{1.2}$$

is a sufficient condition for monotonicity of L_h. Letting $u \equiv 1$, we see that L_h is positive if, in addition to (1.2),

$$c(x) \leqslant 0, \quad \forall x \in \mathbb{Z}_h^n. \tag{1.3}$$

Under assumptions including that the real coefficients a, b and c satisfy (1.2) and (1.3) for all $x, y \in \mathbb{Z}_h^n$ and that $a(x, y)$ has compact support in y, and motivated by the desire for the approximation of viscosity solutions of nonlinear elliptic equations, Kuo and Trudinger began, in 1990, to establish analogues for the above positive difference operators of certain pointwise estimates for linear elliptic differential operators with bounded measurable coefficients [**12**]. They derived discrete versions of the Aleksandrov and Bakel'man maximum principle [**1, 2**], the Hölder estimates and Harnack inequality of Krylov and Safonov [**9**], and the local maximum principle and weak Harnack inequalities of Trudinger [**18**]. These results constitute sufficient linear theory for our study of difference equations in fully nonlinear form.

Following [**13**], consider a general nonlinear difference operator acting on mesh functions $u : \mathbb{Z}_h^n \to \mathbb{R}$, written in the form

$$F_h[u](x) = F_h(x, u(x), Tu(x)),$$

where, letting $E = \mathbb{Z}_h^n$, $Tu(x) = \{u(x + y) \mid y \in E' = E \setminus \{0\}\}$ is the set of non-trivial translates of $u(x)$, and F_h is a given real-valued function on $\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^{E'}$. We will assume that $F_h[u]$ is independent of $u(x + y)$ for $\|y\|_\infty = \sup_i |y_i| > Nh$ for some fixed $N \in \mathbb{N}$. We may then replace E' with the stencil [**8**]

$$Y_N = \{y \in E' \mid \|y\|_\infty \leqslant Nh\}.$$

We adopt the following definition of positivity of general nonlinear difference operators from Kuo and Trudinger [**13**].

Definition 1.1. The operator F_h is *positive* if

$$F_h(x, z, q + \eta) \geqslant F_h(x, z, q) \geqslant F_h(x, z + \tau, q + \eta), \tag{1.4}$$

for all $x \in \mathbb{R}^n$, $z, \tau \in \mathbb{R}$, $q, \eta \in \mathbb{R}^{Y_N}$ satisfying $0 \leqslant \eta_y \leqslant \tau$ for each $y \in Y_N$.

When F_h is differentiable with respect to z and q (which we henceforth assume), (1.4) may be written

$$\frac{\partial F_h}{\partial q_y} \geqslant 0, \quad \forall y \in Y_N, \qquad \frac{\partial F_h}{\partial z} + \sum_{y \in Y_N} \frac{\partial F_h}{\partial q_y} \leqslant 0, \tag{1.5}$$

pointwise. Notice that the first assumption here corresponds in the linear case to monotonicity $A(x, z) \geqslant 0$, $\forall z \neq x$, and the second inequality corresponds to the additional assumption $\sum A(x, z) \leqslant 0$ for positivity. If F is a fully nonlinear second-order differential operator of the general form

$$F[v](x) = F(x, v(x), Dv(x), D^2v(x)), \tag{1.6}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$, where F is a given real-valued function on the set $\tilde{\Gamma} = \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n$ ($\mathcal{S}^n$ being the linear space of real symmetric matrices), and $v \in C^2(\Omega)$, then we have the following definition.

Definition 1.2. The family of difference operators $\{F_h : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^{Y_N} \to \mathbb{R}, 0 < h \leqslant h_0\}$, where h_0 is a positive constant, is called *consistent* with the second-order differential operator F on the domain $\Omega \subset \mathbb{R}^n$ if for each $v \in C^2(\Omega)$

$$F_h[v](x) \to F[v](x), \quad \text{as } h \to 0,$$

uniformly on compact subsets of Ω.

We are interested in difference operators F_h defined in terms of the standard first- and second-order difference operators in (1.1). Writing

$$\delta u(x) = \{\delta_y u(x) \mid y \in Y_N\}, \qquad \delta^2 u(x) = \{\delta_y^2 u(x) \mid y \in Y_N\},$$

let us assume that F_h is of the form

$$F_h[u](x) = \mathcal{F}_h(x, u(x), \delta u(x), \delta^2 u(x)), \tag{1.7}$$

where $\mathcal{F}_h$ is a given function on $\Gamma = \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^{Y_N} \times \mathbb{R}^{Y_N}$. Denote points in Γ by (x, z, q, s). We assume that $\mathcal{F}_h$ is symmetric with respect to $s_{\pm y}$ (since $\delta_y^2 u \equiv \delta_{-y}^2 u$) and with respect to $\pm q_{\pm y}$ (since $\delta_y u \equiv -\delta_{-y} u$). One can show, then, that the conditions (1.5) are equivalent to

$$\tfrac{1}{2}\|y\|_2 \left|\frac{\partial \mathcal{F}_h}{\partial q_y}\right| \leqslant \frac{\partial \mathcal{F}_h}{\partial s_y}, \quad \forall y \in Y_N, \qquad \frac{\partial \mathcal{F}_h}{\partial z} \leqslant 0. \tag{1.8}$$

In [**13**], Kuo and Trudinger exhibit various families of nonlinear positive difference operators which are consistent with the operator (1.6) under appropriate conditions. They find that if the fully nonlinear second-order differential operator $F = F(x, z, p, r)$ is locally uniformly Lipschitz continuous in $\tilde{\Gamma}$, with respect to z, p, r, and satisfies the structure conditions

$$\lambda_0 I \leqslant [F_{r_{ij}}] \leqslant \Lambda_0 I, \quad \|F_p\|_2 \leqslant \mu_1, \quad -\mu_0 \leqslant F_z \leqslant 0, \quad |F(x, 0, 0, 0)| \leqslant k_1, \tag{1.9}$$

where λ_0, Λ_0, μ_1, μ_0 and k_1 are fixed positive constants, then there exists an $N \in \mathbb{N}$ and a consistent family F_h, $h \leqslant h_0$, of the form (1.7) satisfying, in addition to the conditions of positivity (1.8), the stronger condition

$$\frac{\partial \mathcal{F}_h}{\partial s_y} - \tfrac{1}{2}\|y\|_2 \left|\frac{\partial \mathcal{F}_h}{\partial q_y}\right| \geqslant \lambda, \tag{1.10}$$

for $y = y^i = he_i$, $i = 1, \ldots, n$, and some positive constant λ.

(The assumptions in (1.9) are modelled on linear uniformly elliptic equations. If $L[u] = a^{ij} D_{ij} + b^i D_i u + cu - f$, then these assumptions correspond to

$$\lambda_0 I \leqslant [a_{ij}] \leqslant \Lambda_0 I, \quad \|b\|_2 \leqslant \mu_1, \quad -\mu_0 \leqslant c \leqslant 0, \quad |f(x)| \leqslant k_1,$$

and there exist difference operators L_h satisfying (1.2) and (1.3) which are consistent with L (see [**8, 12, 15**]).)

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The family F_h can be chosen to satisfy the additional conditions

$$\forall y \in Y_N, \quad \frac{\partial \mathcal{F}_h}{\partial s_y} \leqslant \Lambda, \quad \left|\frac{\partial \mathcal{F}_h}{\partial q_y}\right| \leqslant \varrho_0, \quad -\frac{\partial \mathcal{F}_h}{\partial z} \leqslant \varsigma_0, \quad \mathcal{F}_h(x,0,0,0) \leqslant k_1, \tag{1.11}$$

where Λ, ϱ_0, ς_0 are constants depending on Λ_0, μ_1, μ_0, respectively, as well as the dimension n. The constant N (the size of the stencil) will depend upon n and Λ_0/λ_0, while h_0 depends in addition upon μ_1/λ_0 (see [**8**, **12**, **13**, **15**]).

Now we set up the Dirichlet problem for the nonlinear difference operator of the form (1.7). To do so we need the following definition.

Definition 1.3. If Ω is a subset of $\mathbb{R}^n$, then we let $\Omega_h = \Omega \cap \mathbb{Z}_h^n$ denote the subset of mesh points in Ω. We distinguish the interior of Ω_h and the boundary of Ω_h, relative to $\mathcal{F}_h$. The interior set, $\operatorname{int}_{\mathcal{F}_h}(\Omega_h)$, consists of those points $x \in \Omega_h$ such that for any mesh function u, $F_h[u](x)$ depends only upon values of u at points in Ω_h. The boundary set, $\operatorname{bdy}_{\mathcal{F}_h}(\Omega_h)$, is then defined by $\Omega_h \setminus \operatorname{int}_{\mathcal{F}_h}(\Omega_h)$.

Letting φ be a continuous function on $\bar{\Omega}$, we may then consider the discrete Dirichlet problem

$$F_h[u](x) = 0, \quad \forall x \in \operatorname{int}_{\mathcal{F}_h}(\Omega_h), \qquad u(x) = \varphi(x), \quad \forall x \in \operatorname{bdy}_{\mathcal{F}_h}(\Omega_h). \tag{1.12}$$

From Kuo and Trudinger [**13**] we have the following theorem.

Theorem 1.4 (see Lemma 3.2 in [13]). *Assuming (1.10) and (1.11) of $\mathcal{F}_h$, problem (1.12) is uniquely solvable when Ω is bounded.*

The proof is by the method of continuity (see [**13**, § 4.1]; see also [**5**, §§ 5.2, 17.2]) and relies upon the discrete maximum principle [**12**, Theorem 2.1].

Our purpose is to derive a discrete *a priori* $C^{2,\alpha}$ estimate for solutions u of (1.12) when $\mathcal{F}_h$ depends only upon $\{s_y\}$—that is, F_h is of the form $F_h[u] = \mathcal{F}_h(\delta^2 u)$ (we address the case of explicit dependence upon x in the next paper)—and there is $Y' \subset Y_N$ such that $\{he_1, \dots, he_n\} \subset Y'$ and $\mathcal{F}_h$ satisfies

$$0 < \lambda \leqslant \frac{\partial \mathcal{F}_h}{\partial s_y} \leqslant \Lambda, \quad \forall y \in Y', \qquad \frac{\partial \mathcal{F}_h}{\partial s_y} \equiv 0, \quad \forall y \in Y_N \setminus Y',$$

for some positive constants λ, Λ. Our major assumption in addition to the above is that $\mathcal{F}_h$ is concave in the second-order variables, $\{s_y\}$, analogous to the situation for partial differential equations, where the existence of classical solutions to boundary-value problems for fully nonlinear elliptic equations is only known, in more than two dimensions, when the function F is concave or convex with respect to the second-order partial derivatives of u. In fact, Nadirashvili has shown recently that for dimensions 12 and greater, solutions do not in general exist unless the operator is concave in the second-order variables [**16**].

We adopt an approach to finite-difference equations of the form $\mathcal{F}_h(\delta^2 u) = 0$ somewhat like the approach generally taken for fully nonlinear elliptic partial differential equations, using the discrete linear theory referred to above. Our main result is Theorem 3.2.

We have noted that the proof of existence for difference equations in fully nonlinear form is comparatively straightforward, relying only upon a discrete maximum principle. The $C^{2,\alpha}$ estimate is not essential for existence as in the case of elliptic partial differential equations. We are motivated to derive such primarily by the analogous partial differential equation theory. However, the estimate is expected to feature in future attempts at establishing error bounds for convergence.

The only such estimate known to the author is for the two-dimensional case by Hackbusch in [**6**].

1.1. Semi-norms

Assume that Ω is an open or closed domain in $\mathbb{R}^n$. Define

$$\Omega_h^i = \{x \in \Omega_h \mid x + y \in \Omega_h,\ \forall y \in Y_1\}, \qquad \Omega_h^b = \Omega_h \setminus \Omega_h^i,$$

noting that these are independent of $\mathcal{F}_h$. Let $u : \Omega_h \to \mathbb{R}$ be a mesh function. Let $\alpha \in (0,1)$. We define the following quantities, analogous to Hölder semi-norms of continuous functions:

$$|u|_{0;\Omega_h} = \sup_{x\in\Omega_h} |u(x)|;$$

$${}_N[u]_{1;\Omega_h} = \sup_{\substack{x\in\Omega_h,\ z\in Y_N\\ x\pm z\in\Omega_h}} |\delta_z^{\pm} u(x)|, \quad + \text{ throughout or } - \text{ throughout;}$$

$${}_N[u]_{2;\Omega_h} = \sup_{\substack{x\in\Omega_h,\ z\in Y_N\\ x\pm z\in\Omega_h}} |\delta_z^2 u(x)|;$$

$${}_N[u]_{2,\alpha;\Omega_h} = \sup_{\substack{x,y\in\Omega_h,\ x\neq y,\ z\in Y_N\\ x\pm z,\ y\pm z\in\Omega_h}} \frac{|\delta_z^2 u(x) - \delta_z^2 u(y)|}{\|x-y\|_2^{\alpha}}.$$

When Ω is properly contained in $\mathbb{R}^n$ and $x \in \Omega$ we reserve the symbol d_x for the quantity $\operatorname{dist}(x, \partial\Omega)$, and define $d_{xy} = \min\{d_x, d_y\}$. We use an underscore to indicate distances to the discrete boundary. For $x \in \Omega_h^i$, define the distance from x to the discrete boundary, Ω_h^b, by $\underline{d}_x = \operatorname{dist}(x, \Omega_h^b)$, and let $\underline{d}_{xy} = \min\{\underline{d}_x, \underline{d}_y\}$. Then, for $u : \Omega_h \to \mathbb{R}$, let us set

$${}_N[u]^*_{1;\Omega_h} = \sup_{\substack{x\in\Omega_h^i,\ z\in Y_N\\ x\pm z\in\Omega_h^i}} \underline{d}_x |\delta_z^{\pm} u(x)|;$$

$${}_N[u]^*_{1,\alpha;\Omega_h} = \sup_{\substack{x,y\in\Omega_h^i,\ x\neq y,\ z\in Y_N\\ x\pm z,\ y\pm z\in\Omega_h^i}} \underline{d}_{xy}^{1+\alpha} \frac{|\delta_z^{\pm} u(x) - \delta_z^{\pm} u(y)|}{\|x-y\|_2^{\alpha}},$$

in both cases taking '+' throughout or '−' throughout; and define

$${}_N[u]^*_{2;\Omega_h} = \sup_{\substack{x\in\Omega_h^i,\ z\in Y_N\\ x\pm z\in\Omega_h^i}} \underline{d}_x^2 |\delta_z^2 u(x)|;$$

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$$ {}_N[u]^*_{2,\alpha;\Omega_h} = \sup_{\substack{x,y\in\Omega^i_h,\ x\neq y,\ z\in Y_N\\ x\pm z,\ y\pm z\in\Omega^i_h}} \underline{d}^{2+\alpha}_{xy} \frac{|\delta_z^2 u(x) - \delta_z^2 u(y)|}{\|x-y\|_2^\alpha}. $$

Notice that these quantities are defined to be independent of $u|_{\Omega_h^b}$. If $[\cdot]$ is one of the quantities above, then we will denote the corresponding quantity with Y_N replaced with $\{\pm he_i\}_{i=1}^n$ by ${}_+[\cdot]$; Y_N replaced with Y' will be denoted by ${}_\star[\cdot]$. The '+' is to be suggestive of the fact that the quantity involves only the coordinate directions.

For $p > 0$, if $S \subset \mathbb{Z}^n_h$ and $u : S \to \mathbb{R}$, then define

$$ \|u\|_{p;S} = \left(\sum_{x\in S} h^n |u(x)|^p\right)^{1/p}. $$

Let $v : \Omega \to \mathbb{R}$ be a real-valued function where Ω is an open or closed domain properly contained in $\mathbb{R}^n$. We shall need the following semi-norms:

$$ |v|_{0;\Omega} = \sup_{x\in\Omega} |v(x)|; $$

$$ [v]^{(2)}_{0;\Omega} = \sup_{x\in\Omega} d_x^2 |v(x)|; $$

$$ [v]^{(2)}_{0,\alpha;\Omega} = \sup_{\substack{x,y\in\Omega\\ x\neq y\in\Omega}} d^{2+\alpha}_{xy} \frac{|v(x)-v(y)|}{\|x-y\|_2^\alpha}; $$

$$ |v|^{(2)}_{0,\alpha;\Omega} = [v]^{(2)}_{0;\Omega} + [v]^{(2)}_{0,\alpha;\Omega}; $$

$$ [v]^*_{2;\Omega} = \sup_{x\in\Omega,\ \|\beta\|_1=2} d_x^2 |D^\beta v(x)|; $$

$$ [v]^*_{2,\alpha;\Omega} = \sup_{\substack{x,y\in\Omega,\ x\neq y\\ \|\beta\|_1=2}} d^{2+\alpha}_{xy} \frac{|D^\beta v(x) - D^\beta u(y)|}{\|x-y\|_2^\alpha}. $$

Here β is a multi-index. If, for example, $\beta = (1,1,0,\dots,0)$, then $D^\beta u = \partial^2 u/\partial x_1 \partial x_2$.

2. The diagonal case, essentially

For $t > 0$ define $B_t(y) = \{x \in \mathbb{R}^n \mid \|x-y\|_2 < t\}$, the open ball in $\mathbb{R}^n$ of radius t, and then, of course, $B_t(y)_h = B_t(y) \cap \mathbb{Z}^n_h$. Recall that

$$ \operatorname*{osc}_V u = \sup_{y,z\in V} |u(y) - u(z)|. $$

Theorem 2.1. *Let $h > 0$. Assume that $\{he_1,\dots,he_n\} \subset Y' \subset Y_N$. Assume that $\mathcal{F}_h : \mathbb{R}^{Y'} \to \mathbb{R}$ is concave, and assume that for all $s \in \mathbb{R}^{Y'}$ and for all $y \in Y'$,*

$$ \lambda \leqslant \frac{\partial \mathcal{F}_h(s)}{\partial s_y} \leqslant \Lambda, $$

for some positive constants λ, Λ, *and*

$$\frac{\partial \mathcal{F}_h(s)}{\partial s_y} \equiv 0,$$

for all $y \in Y_N \setminus Y'$. *Assume that* Ω *is an open or closed domain in* $\mathbb{R}^n$. *Let* $v : \Omega_h \to \mathbb{R}$ *and* $\psi : \Omega_h \to \mathbb{R}$ *be mesh functions. Suppose that*

$$\mathcal{F}_h(\delta^2 v(x)) = \psi(x), \quad \forall x \in \operatorname{int}_{\mathcal{F}_h}(\Omega_h). \tag{2.1}$$

If $\tau \in (0,1)$, $R_0 > \sqrt{n}Nh/(1-\tau)$ *and* $x_0 \in \operatorname{int}_{\mathcal{F}_h}(\operatorname{int}_{\mathcal{F}_h}(\Omega_h))$ *are such that* $B_{R_0}(x_0)_h \subset \operatorname{int}_{\mathcal{F}_h}(\operatorname{int}_{\mathcal{F}_h}(\Omega_h))$, *then for any* $y \in Y'$ *and any* R *such that* $\sqrt{n}Nh/(1-\tau) < R \leqslant R_0$,

$$\begin{aligned} &\operatorname*{osc}_{B_R(x_0)_h} \delta_y^2 v \\ &\qquad \leqslant C\left(\frac{R}{R_0}\right)^{\alpha} \Big\{ \sup_{Y \in Y'} \operatorname*{osc}_{B_{R_0}(x_0)_h} \delta_Y^2 v + R_0[\psi]_{1;B_{R_0}(x_0)_h} + R_0^2[\psi]_{2;B_{R_0+\sqrt{n}Nh}(x_0)_h} \Big\}, \end{aligned}$$

where $C = C(n, N, \lambda, \Lambda, \tau)$ *is positive and* $\alpha \in (0,1)$ *with the same dependence.*

Proof. Here we emulate the continuous theory in [**5**, § 17.4]. In order to effectively differentiate the equation twice in an arbitrary direction $Y \in Y_N$, we use an idea from the proof of [**19**, Theorem 2.1] (that of adding the two inequalities below). Since $\mathcal{F}_h$ is concave, we have

$$\mathcal{F}_h(\delta^2 v(x \pm Y)) \leqslant \mathcal{F}_h(\delta^2 v(x)) + \sum_{y \in Y'} \frac{\partial \mathcal{F}_h(\delta^2 v(x))}{\partial s_y} [\delta_y^2 v(x \pm Y) - \delta_y^2 v(x)],$$

for all $x \in \operatorname{int}_{\mathcal{F}_h}(\operatorname{int}_{\mathcal{F}_h}(\Omega_h))$, and all $Y \in Y_N$. We restrict to $x \in \operatorname{int}_{\mathcal{F}_h}(\operatorname{int}_{\mathcal{F}_h}(\Omega_h))$ here so that $\delta_y^2 u(x \pm Y)$ is defined in Ω_h. Add the two inequalities represented here to find that

$$\psi(x+Y) - 2\psi(x) + \psi(x-Y) \leqslant \sum_{y \in Y'} \frac{\partial \mathcal{F}_h(\delta^2 v(x))}{\partial s_y} [\delta_y^2 v(x+Y) - 2\delta_y^2 v(x) + \delta_y^2 v(x-Y)].$$

Upon division by $\|Y\|_2^2$ this becomes

$$\delta_Y^2 \psi(x) \leqslant \sum_{y \in Y'} \frac{\partial \mathcal{F}_h(\delta^2 v(x))}{\partial s_y} \delta_Y^2 \delta_y^2 v(x).$$

Defining

$$L_h w = \sum_{y \in Y'} a(x,y) \delta_y^2 w(x),$$

where

$$a(x,y) = \frac{\partial \mathcal{F}_h(\delta^2 v(x))}{\partial s_y}$$

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and $w(x) = \delta_Y^2 u(x)$, the above inequality may be written

$$L_h w(x) \geqslant \delta_Y^2 \psi(x),$$

for all $x \in \operatorname{int}_{\mathcal{F}_h}(\operatorname{int}_{\mathcal{F}_h}(\Omega_h))$, for all $Y \in Y_N$.

We now apply the discrete weak Harnack inequality [**12**, Theorem 4.3]. Suppose $\tau \in (0,1)$, $x_0 \in \Omega_h$ and $R_0 > 0$ are such that $R_0 > \sqrt{n}Nh/(1-\tau)$ and $B_{R_0}(x_0)_h \subset \operatorname{int}_{\mathcal{F}_h}(\operatorname{int}_{\mathcal{F}_h}(\Omega_h))$. Choose R such that $\sqrt{n}Nh/(1-\tau) < R \leqslant R_0$.

For $\sigma \leqslant 1$ set

$$M_\sigma = \sup_{B_{\sigma R}(x_0)_h} w, \qquad m_\sigma = \inf_{B_{\sigma R}(x_0)_h} w.$$

Applying [**12**, Theorem 4.3] to $M_1 - w$ (which satisfies $L_h(M_1 - w) \leqslant -\delta_Y^2\psi(x)$ and $M_1 - w \geqslant 0$ in $B_R(x_0)_h$), we obtain a weak Harnack inequality; that is, there exists a positive number p (independent of τ) depending on n, N, λ and Λ, such that

$$\begin{aligned} \left\{ \left(\frac{h}{R}\right)^n \sum_{B_{\tau R}(x_0)_h} (M_1 - w)^p \right\}^{1/p} &\leqslant C\left\{ \min_{B_{\tau R}(x_0)_h} (M_1 - w) + \frac{R}{\lambda} \|\delta_Y^2 \psi\|_{n;B_R(x_0)_h} \right\} \\ &\leqslant C\left\{ M_1 - M_\tau + \frac{4R^2}{\lambda} [\psi]_{2;B_{R+\sqrt{n}Nh}(x_0)_h} \right\}, \qquad (2.2) \end{aligned}$$

where $C = C(n, N, \lambda, \Lambda, \tau)$. We emphasize that by definition $[\psi]_{2;B_{R+\sqrt{n}Nh}(x_0)_h}$ is independent of $\psi(x)$ for $x \notin B_{R+\sqrt{n}Nh}(x_0)_h$. The treatment here of $\|\delta_Y^2\psi\|_{n;B_R(x_0)_h}$ follows from the fact that

$$\begin{aligned} \|\delta_Y^2 \psi\|_{n;B_R(x_0)_h} &= \left(\sum_{x \in B_R(x_0)_h} h^n |\delta_Y^2 \psi(x)|^n \right)^{1/n} \\ &\leqslant \sup_{y \in B_R(x_0)_h} |\delta_Y^2 \psi(y)| \cdot h \cdot \left(\sum_{x \in B_R(x_0)_h} 1 \right)^{1/n} \\ &\leqslant \sup_{y \in Y_N,\ x, x \pm y \in B_{R+\sqrt{n}Nh}(x_0)_h} |\delta_y^2 \psi(x)| \cdot h \cdot \left(\left(\frac{2(R+h)}{h} \right)^n \right)^{1/n} \\ &\leqslant [\psi]_{2;B_{R+\sqrt{n}Nh}(x_0)_h} 2(R+h) \leqslant [\psi]_{2;B_{R+\sqrt{n}Nh}(x_0)_h} 4R, \end{aligned}$$

since

$$h < \frac{(1-\tau)}{\sqrt{n}N} R < R.$$

Following [**5**, § 17.4], to conclude a Hölder estimate for w from (2.2) we need a corresponding inequality for $-w$, which we obtain by considering (2.1) as a functional relationship between second-order difference quotients of v. In fact, it is a functional relationship between *pure* second-order difference quotients of v, and thus a discrete equivalent of [**5**, Lemma 17.13] is not necessary, although this is only because of our rather strong assumption $\partial\mathcal{F}_h/\partial s_y \equiv 0$ for all $y \in Y_N \setminus Y'$. (In fact, the formulation of an assumption for the quantities $\{\partial\mathcal{F}_h/\partial s_y \mid y \in Y_N\}$, analogous in some sense to the assumption of positive definiteness of the coefficient matrix in the continuous case, may well allow our assumptions to be weakened; we do not pursue this here.)

Using the concavity of $\mathcal{F}_h$ again, we have, for any $x, z \in \operatorname{int}_{\mathcal{F}_h}(\Omega_h)$,

$$\sum_{y \in Y'} \frac{\partial \mathcal{F}_h(\delta^2 v(z))}{\partial s_y}[\delta_y^2 v(z) - \delta_y^2 v(x)] \leqslant \mathcal{F}_h(\delta^2 v(z)) - \mathcal{F}_h(\delta^2 v(x));$$

that is,

$$\sum_{y \in Y'} \frac{\partial \mathcal{F}_h(\delta^2 v(z))}{\partial s_y}[w_y(z) - w_y(x)] \leqslant \psi(z) - \psi(x), \tag{2.3}$$

where $w_y(x) = \delta_y^2 v(x)$. Now set

$$M_{\sigma y} = \sup_{B_{\sigma R}(x_0)_h} w_y, \quad m_{\sigma z} = \inf_{B_{\sigma R}(x_0)_h} w_y, \quad 0 < \sigma \leqslant 1, \quad y \in Y'.$$

Each of the functions w_y satisfies (2.2), so that by summation over $y \in Y'$, $y \neq Y$ for some fixed $Y \in Y'$,

$$\begin{aligned}\left\{\left(\frac{h}{R}\right)^n \sum_{B_{\tau R}(x_0)_h} \left[\sum_{y \neq Y}(M_{1y} - w_y)\right]^p\right\}^{1/p} & \\ \leqslant ((2N+1)^n - 2)^{1/p} \sum_{y \neq Y} & \left\{\left(\frac{h}{R}\right)^n \sum_{B_{\tau R}(x_0)_h}(M_{1y} - w_y)^p\right\}^{1/p} \\ \leqslant C & \left\{\sum_{y \neq Y}(M_{1y} - M_{\tau y}) + \sum_{y \neq Y} \frac{R^2}{\lambda}[\psi]_{2;B_{R+\sqrt{n}Nh}(x_0)_h}\right\},\end{aligned}$$

by (2.2), where the first inequality follows by the Minkowski inequality, whether $0 < p < 1$, or $p \geqslant 1$. For $p \geqslant 1$ the constant is, of course, 1. For $0 < p < 1$, $\|\cdot\|_{p;S}$ is not a norm, but

$$\|f_1 + f_2 + \cdots + f_M\|_{p;S} \leqslant M^{1/p}(\|f_1\|_{p;S} + \|f_2\|_{p;S} + \cdots + \|f_M\|_{p;S}),$$

and the number of non-zero directions in Y' other than Y is at most $(2N+1)^n - 2$. Since $M_{1y} - M_{\tau y} \leqslant \operatorname{osc}_{B_R(x_0)_h} w_y - \operatorname{osc}_{B_{\tau R}(x_0)_h} w_y$, it follows that

$$\left\{\left(\frac{h}{R}\right)^n \sum_{B_{\tau R}(x_0)_h}\left[\sum_{y \neq Y}(M_{1y} - w_y)\right]^p\right\}^{1/p} \leqslant C\left\{W(R) - W(\tau R) + \frac{R^2}{\lambda}[\psi]_{2;B_{R+\sqrt{n}Nh}(x_0)_h}\right\}, \tag{2.4}$$

where, for $0 < \tau \leqslant 1$,

$$W(\tau R) = \sum_{y \in Y'} \operatorname*{osc}_{B_{\tau R}(x_0)_h} w_y = \sum_{y \in Y'} M_{\tau y} - m_{\tau y}.$$

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Rewriting (2.3) with $x \in B_R(x_0)_h$, $z \in B_{\tau R}(x_0)_h$, and singling out $Y \in Y'$, we obtain

$$\frac{\partial \mathcal{F}_h(\delta^2 v(z))}{\partial s_Y}[w_Y(z) - w_Y(x)] \leqslant \psi(z) - \psi(x) + \sum_{y \neq Y} \frac{\partial \mathcal{F}_h(\delta^2 v(z))}{\partial s_y}[w_y(x) - w_y(z)],$$

so that with $0 < \lambda \leqslant \partial \mathcal{F}_h / \partial s_y \leqslant \Lambda$ for all $y \in Y'$, we have

$$w_Y(z) - m_{1Y} \leqslant \frac{1}{\lambda}\left\{2nR[\psi]_{1;B_R(x_0)_h} + \Lambda \sum_{y \neq Y}(M_{1y} - w_y(z))\right\}. \tag{2.5}$$

The fact that

$$|\psi(x) - \psi(z)| \leqslant 2nR[\psi]_{1;B_R(x_0)_h}$$

is a result of it being no more than $2nR$ steps of size h in the coordinate directions from x to z.

Applying Minkowski with $M = 2$ to (2.5) we have

$$\begin{aligned}
&\left\{\left(\frac{h}{R}\right)^n \sum_{B_{\tau R}(x_0)_h} (w_Y - m_{1Y})^p\right\}^{1/p} \\
&\qquad \leqslant \frac{2^{1/p}}{\lambda}\left[\left\{\left(\frac{h}{R}\right)^n \sum_{B_{\tau R}(x_0)_h} (2nR[\psi]_{1;B_R(x_0)_h})^p\right\}^{1/p} \right.\\
&\qquad\qquad \left. + \Lambda\left\{\left(\frac{h}{R}\right)^n \sum_{B_{\tau R}(x_0)_h} \left(\sum_{y \neq Y}(M_{1y} - w_y)\right)^p\right\}^{1/p}\right] \\
&\qquad \leqslant CR[\psi]_{1;B_R(x_0)_h} + C\{W(R) - W(\tau R) + R^2[\psi]_{2;B_{R+\sqrt{n}Nh}(x_0)_h}\},
\end{aligned}$$

by (2.4). It follows that

$$\begin{aligned}
&\left\{\left(\frac{h}{R}\right)^n \sum_{B_{\tau R}(x_0)_h} (w_Y - m_{1Y})^p\right\}^{1/p} \\
&\qquad \leqslant C\{W(R) - W(\tau R) + R[\psi]_{1;B_R(x_0)_h} + R^2[\psi]_{2;B_{R+\sqrt{n}Nh}(x_0)_h}\},
\end{aligned} \tag{2.6}$$

where C depends on n, N, λ, Λ and τ. Also, setting $w = w_Y$ in (2.2), we have

$$\left\{\left(\frac{h}{R}\right)^n \sum_{B_{\tau R}(x_0)_h} (M_{1Y} - w_Y)^p\right\}^{1/p} \leqslant C\{M_{1Y} - M_{\tau Y} + R^2[\psi]_{2;B_{R+\sqrt{n}Nh}(x_0)_h}\}. \tag{2.7}$$

Add (2.7) to (2.6), using the Minkowski inequality, to obtain

$$\begin{aligned}
&\left\{\left(\frac{h}{R}\right)^n \sum_{B_{\tau R}(x_0)_h} (M_{1Y} - m_{1Y})^p\right\}^{1/p} \\
&\qquad \leqslant C\{W(R) - W(\tau R) + R[\psi]_{1;B_R(x_0)_h} + R^2[\psi]_{2;B_{R+\sqrt{n}Nh}(x_0)_h}\}.
\end{aligned}$$

Summing over $Y \in Y'$ leads to

$$W(R) \leqslant C\{W(R) - W(\tau R) + R[\psi]_{1;B_R(x_0)_h} + R^2[\psi]_{2;B_{R+\sqrt{n}Nh}(x_0)_h}\},$$

and, hence,

$$W(\tau R) \leqslant \gamma W(R) + R[\psi]_{1;B_R(x_0)_h} + R^2[\psi]_{2;B_{R+\sqrt{n}Nh}(x_0)_h},$$

for $\gamma = 1 - 1/C$. The following modified version of [**5**, Lemma 8.23] then gives us a Hölder estimate for centred second-order difference quotients. The proof follows that of [**5**, Lemma 8.23] very closely.

Lemma 2.2. *Let $\mathcal{W}$ be a non-decreasing non-negative function on an interval $(0, R_0]$. Suppose there exist $\gamma, \tau \in (0,1)$ and a positive number η such that for all R such that $0 < \eta < R \leqslant R_0$, $\mathcal{W}$ satisfies the inequality*

$$\mathcal{W}(\tau R) \leqslant \gamma \mathcal{W}(R) + \sigma(R),$$

where σ is also non-decreasing and non-negative. Then, for any $\mu \in (0,1)$ and R such that $\eta < R \leqslant R_0$,

$$\mathcal{W}(R) \leqslant C\biggl(\biggl(\frac{R}{R_0}\biggr)^{\alpha} \mathcal{W}(R_0) + \sigma(R^{\mu} R_0^{1-\mu})\biggr),$$

where $C = C(\gamma)$ and $\alpha = (1-\mu)\log\gamma/\log\tau$ are positive constants.

Note that if $\mu = \log\gamma/(\log\tau + \log\gamma)$, then $\mu = \alpha$. Making this choice in Lemma 2.2 applied to W gives $C = C(\gamma)$ such that

$$W(R) \leqslant C\biggl(\frac{R}{R_0}\biggr)^{\alpha}\{W(R_0) + R_0[\psi]_{1;B_{R_0}(x_0)_h} + R_0^2[\psi]_{2;B_{R_0+\sqrt{n}Nh}(x_0)_h}\},$$

for $R_0 \geqslant R > \eta = \sqrt{n}Nh/(1-\tau)$. The result now follows. □

For $\Omega \subset \mathbb{R}^n$ define $\Omega(t) = \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) \geqslant t\}$, the subset of Ω consisting of all points whose Euclidean distance from the boundary of Ω is at least t. Of course, $\Omega(t)_h = \Omega(t) \cap \mathbb{Z}_h^n$.

Theorem 2.3. *Assume $\mathcal{F}_h : \mathbb{R}^{Y'} \to \mathbb{R}$ is as in Theorem 2.1. Let Ω be an open or closed domain properly contained in $\mathbb{R}^n$, and let $v : \Omega_h \to \mathbb{R}$ be a mesh function. Let $M > 0$ be such that $\Omega(Mh)_h \subset \operatorname{int}_{\mathcal{F}_h}(\Omega_h)$ (so $M > \sqrt{n}N$ is sufficient; $M > N$ is sufficient when Ω is the Cartesian product of n closed (not necessarily finite) intervals such that the corners of Ω are themselves lattice points). If*

$$\mathcal{F}_h(\delta^2 v(x)) = 0, \quad \forall x \in \Omega(Mh)_h,$$

then

$$_\star[v]^*_{2,\alpha;\Omega_h} \leqslant C|v|_{0;\Omega_h^i},$$

where $C = C(n, N, M, \lambda, \Lambda) > 0$, and $\alpha \in (0,1)$ has the same dependence. In particular,

$$_+[v]^*_{2,\alpha;\Omega_h} \leqslant C|v|_{0;\Omega_h^i}.$$

Proof. Let us first prove that

$$ {}_\star[v]^*_{2,\alpha;\Omega_h} \leqslant C(n,N,M,\lambda,\Lambda)_\star[v]^*_{2;\Omega_h}. \tag{2.8}$$

Note that the choice of M ensures that $\mathrm{int}_{\mathcal{F}_h}(\Omega(Mh)_h) \subset \mathrm{int}_{\mathcal{F}_h}(\mathrm{int}_{\mathcal{F}_h}(\Omega_h))$ (the left-hand set may be empty). If $x \in \Omega_h$ and $\underline{d}_x \geqslant Mh + \sqrt{n}Nh$, then $x \in \mathrm{int}_{\mathcal{F}_h}(\Omega(Mh)_h)$. Choose distinct $x, z \in \Omega_h^i$, and $y \in Y'$ such that $x \pm y$, $z \pm y \in \Omega_h^i$. Suppose, without loss of generality, that $\underline{d}_x \leqslant \underline{d}_z$. We discretize a standard argument as, for example, in the proof of [**5**, Theorem 4.8]. Let $\tau = \frac{1}{2}$ in Theorem 2.1, and let $\alpha = \alpha(n,N,\lambda,\Lambda) \in (0,1)$ be the associated constant in that theorem. Let $R_1 = \underline{d}_x/(6\sqrt{n}N + M)$, $R_0 = R_1 + 2\sqrt{n}Nh$, $R = \|x - z\|_2 (\geqslant h)$, and $\mathcal{R} = R + 2\sqrt{n}Nh$. Consider two cases:

(i) $z \in B_{R_1}(x)_h$, and

(ii) $z \notin B_{R_1}(x)_h$.

Case (i). Note that $\underline{d}_x \leqslant (6\sqrt{n}N + M)h$ would imply that $R_1 \leqslant h$, and $B_{R_1}(x)_h = \{x\}$, contradicting $x \neq z$. Hence, we may assume for now that $\underline{d}_{xz} > (6\sqrt{n}N + M)h$. Consequently, for any $p \in B_{R_0}(x)_h$,

$$\begin{aligned}\underline{d}_p \geqslant \underline{d}_x - R_0 = \underline{d}_x - \left(\frac{\underline{d}_x}{6\sqrt{n}N + M} + 2\sqrt{n}Nh\right) &= \left(1 - \frac{1}{6\sqrt{n}N + M}\right)\underline{d}_x - 2\sqrt{n}Nh \\ > (6\sqrt{n}N + M - 1)h - 2\sqrt{n}Nh &= 4\sqrt{n}Nh + Mh - h > (\sqrt{n}N + M)h.\end{aligned}$$

Hence, $B_{R_0}(x)_h \subset \mathrm{int}_{\mathcal{F}_h}(\Omega(Mh)_h)$, and

$$\begin{aligned}\underline{d}_{xz}^{2+\alpha}\frac{|\delta_y^2 v(x) - \delta_y^2 v(z)|}{\|x - z\|_2^\alpha} &\leqslant ((6\sqrt{n}N + M)R_1)^{2+\alpha}\frac{\mathrm{osc}_{B_{\mathcal{R}}(x)_h}\delta_y^2 v}{R^\alpha} \\ &\leqslant ((6\sqrt{n}N + M)R_0)^{2+\alpha}(3\sqrt{n}N)^\alpha\frac{\mathrm{osc}_{B_{\mathcal{R}}(x)_h}\delta_y^2 v}{\mathcal{R}^\alpha},\end{aligned}$$

since, with $h \leqslant R$, we have $\mathcal{R} = R + 2\sqrt{n}Nh \leqslant 3\sqrt{n}NR$. Now $R_0 > \mathcal{R} > 2\sqrt{n}Nh$, so Theorem 2.1 with $\psi \equiv 0$ implies that

$$\begin{aligned}\underline{d}_{xz}^{2+\alpha}\frac{|\delta_y^2 v(x) - \delta_y^2 v(z)|}{\|x - z\|_2^\alpha} &\leqslant CR_0^2 \sup_{y \in Y'} \operatorname*{osc}_{B_{R_0}(x)_h} \delta_y^2 v \\ &\leqslant 2CR_0^2 \sup_{y \in Y'} \sup_{p \in B_{R_0}(x)_h} |\delta_y^2 v(p)|.\end{aligned}$$

Now, with

$$h < \frac{\underline{d}_x}{6\sqrt{n}N + M}$$

we have

$$R_0 = \frac{\underline{d}_x}{6\sqrt{n}N + M} + 2\sqrt{n}Nh < \frac{\underline{d}_x}{6\sqrt{n}N + M} + \frac{2\sqrt{n}N\underline{d}_x}{6\sqrt{n}N + M} < \tfrac{1}{2}\underline{d}_x.$$

Then, for any $p \in B_{R_0}(x)_h$, we have $\underline{d}_p > \underline{d}_x - R_0 \geqslant 2R_0 - R_0 = R_0$. It now follows from above that

$$\underline{d}_{xz}^{2+\alpha} \frac{|\delta_y^2 v(x) - \delta_y^2 v(z)|}{\|x-z\|_2^\alpha} \leqslant C \sup_{p \in B_{R_0}(x)_h,\ y \in Y'} \underline{d}_p^2 |\delta_y^2 v(p)|$$
$$\leqslant C\,{}_\star[v]^*_{2;\Omega_h}.$$

The final inequality relies upon the fact that $p \pm y \in \Omega_h^i$ for all $p \in B_{R_0}(x)_h$ and all $y \in Y'$ because $B_{R_0}(x)_h \subset \mathrm{int}_{\mathcal{F}_h}(\Omega(Mh)_h)$.

Case (ii). When $\|x-z\|_2 \geqslant R_1$, we have

$$\underline{d}_{xz}^{2+\alpha} \frac{|\delta_y^2 v(x) - \delta_y^2 v(z)|}{\|x-z\|_2^\alpha} \leqslant ((6\sqrt{n}N + M)R_1)^{2+\alpha} \frac{|\delta_y^2 v(x)| + |\delta_y^2 v(z)|}{R_1^\alpha}$$
$$\leqslant (6\sqrt{n}N + M)^\alpha (\underline{d}_x^2 |\delta_y^2 v(x)| + \underline{d}_z^2 |\delta_y^2 v(z)|) \leqslant C\,{}_\star[v]^*_{2;\Omega_h}.$$

The final inequality follows since x, z and y were chosen to satisfy $x \pm y, z \pm y \in \Omega_h^i$. Combining the two cases and taking the supremum on the left-hand side over distinct $x, z \in \Omega_h^i$, and $y \in Y'$ satisfying $x \pm y, z \pm y \in \Omega_h^i$, we have (2.8). Interpolation via Lemma A 2 gives the result, since α depends only upon n, N, λ and Λ. □

3. A Poisson promise and the main result

Theorem 3.1. *Let $\Omega = \prod_{i=1}^n [a_i h, b_i h]$, where $a_i, b_i \in \mathbb{Z}$ and $a_i < b_i$ for each $i = 1, \ldots, n$. Let $\mathcal{F}_h$ be as in Theorem 2.1. If $u : \Omega_h \to \mathbb{R}$ satisfies*

$$\mathcal{F}_h(\delta^2 u(x)) = 0, \quad \forall x \in \mathrm{int}_{\mathcal{F}_h}(\Omega_h),$$

then

$${}_N[u]^*_{2,\alpha;\Omega_h} \leqslant C|u|_{0;\Omega_h^i},$$

where $C = C(n, N, \lambda, \Lambda) > 0$ and $\alpha = \alpha(n, N, \lambda, \Lambda) \in (0,1)$.

Any second-order difference quotient $\delta_y^2 u(x)$ with $y \in Y_N$ may be expressed as a linear combination of difference quotients $\{\delta_Y^2 u(x') \mid Y \in Y_1,\ x' \text{ in a mesh neighbourhood of } x\}$, with the size of the mesh neighbourhood uniform in $x \in \Omega_h$. It follows that it is sufficient to estimate ${}_1[u]^*_{2,\alpha;\Omega_h}$ in order to establish Theorem 3.1. We defer the proof to the next section.

From Theorem 3.1 we now derive the key theorem of this paper, which will be used in our discretization of Safonov's derivation of a $C^{2,\alpha}$ estimate in the general case that the difference operator depends explicitly upon the independent variables, to appear in a subsequent paper. We denote the closed n-cube in $\mathbb{R}^n$ with centre x, side length $2R$ and edges parallel to the coordinate axes by

$$K_R(x) = \{y \in \mathbb{R}^n \mid \|y - x\|_\infty \leqslant R\},$$

and, of course, $K_R(x)_h = K_R(x) \cap \mathbb{Z}_h^n$.

Theorem 3.2. *With $x_0 \in \mathbb{Z}_h^n$, assume that $R \geqslant (2N+1)h$, and let $\Omega_h = K_R(x_0)_h$. Let $u : \Omega_h \to \mathbb{R}$ satisfy*

$$F_h[u](x) \equiv \mathcal{F}_h(\delta^2 u(x)) = 0, \quad \forall x \in \operatorname{int}_{\mathcal{F}_h}(\Omega_h),$$

where $\mathcal{F}_h$ is as in Theorem 2.1. It is further assumed that $F_h[0] \equiv 0$. Then

$$_N[u]_{2,\alpha;K_{R/2}(x_0)_h} \leqslant CR^{-2-\alpha} \max_{\operatorname{bdy}_{\mathcal{F}_h}(K_R(x_0)_h)} |u|,$$

where $\alpha = \alpha(n, N, \lambda, \Lambda) \in (0,1)$ and $C = C(n, N, \lambda, \Lambda) > 0$.

Proof. By Theorem 3.1 we have $\alpha \in (0,1)$ and positive C such that

$$_N[u]^*_{2,\alpha;K_R(x_0)_h} \leqslant C|u|_{0;K_R(x_0)_h^i}.$$

Choose distinct x, $z \in K_{R/2}(x_0)_h$, $y \in Y_N$, such that $x \pm y$ and $z \pm y \in K_{R/2}(x_0)_h$. It follows that

$$\underline{d}_{xz}^{2+\alpha} \frac{|\delta_y^2 u(x) - \delta_y^2 u(z)|}{\|x - z\|_2^\alpha} \leqslant C|u|_{0;K_R(x_0)_h^i}.$$

Now

$$\underline{d}_{xz} > d_{xz} - h \geqslant \tfrac{1}{2}R - \frac{R}{2N+1} = \frac{2N-1}{2(2N+1)}R,$$

so we have

$$\left(\frac{(2N-1)R}{2(2N+1)}\right)^{2+\alpha} \frac{|\delta_y^2 u(x) - \delta_y^2 u(z)|}{\|x - y\|_2^\alpha} \leqslant C|u|_{0;K_R(x_0)_h^i},$$

and, hence,

$$_N[u]_{2,\alpha;K_{R/2}(x_0)_h} \leqslant CR^{-2-\alpha}|u|_{0;K_R(x_0)_h^i}. \tag{3.1}$$

In order to estimate the right-hand side we apply the discrete maximum principle to the linearization for arbitrary mesh functions v_1, v_2,

$$F_h[v_1](x) - F_h[v_2](x) = L_h(v_1 - v_2)(x), \quad \forall x \in \Omega_h,$$

where

$$L_h = \sum_{y \in Y_N} a(x,y)\delta_y^2,$$

$$a(x,y) = \int_0^1 \frac{\partial \mathcal{F}_h}{\partial s_y}(\delta^2 w_t(x))\,\mathrm{d}t$$

and

$$w_t(x) = tv_1(x) + (1-t)v_2(x), \quad 0 \leqslant t \leqslant 1.$$

Taking $v_1 \equiv u$, the solution, and $v_2 \equiv 0$, we have

$$L_h(u)(x) = 0, \quad \forall x \in \operatorname{int}_{\mathcal{F}_h}(\Omega_h).$$

Using the 'discrete maximum principle' [**12**, Theorem 2.1], we have

$$\max_{\operatorname{int}_{\mathcal{F}_h}(\Omega_h)} |u| \leqslant \max_{\operatorname{bdy}_{\mathcal{F}_h}(\Omega_h)} |u|.$$

This together with (3.1) gives the result. □

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4. A promise fulfilled

In this section we fulfil our promise of proving Theorem 3.1. We do this using the continuous theory for Poisson's equation (a suggestion of Neil Trudinger) applied to a continuous extension of u, due to Kunkle [**10**].

To present Kunkle's result, consider the closed n-cube of side length h given by the Cartesian product of n intervals: $[0,h]^n \equiv [0,h] \times \cdots \times [0,h]$. This lies at the centre of the n-cube of side length $3h$: $[-h,2h]^n \equiv [-h,2h] \times \cdots \times [-h,2h]$, and, of course, the set of mesh points in this cube is $[-h,2h]^n \cap \mathbb{Z}^n_h$. This set could be thought of as a mesh neighbourhood of $[0,h]^n$. For a multi-index $\beta = (\beta_1, \beta_2, \dots, \beta_n)$ we use the notation $\delta^\beta u = (\delta^+_{he_1})^{\beta_1}(\delta^+_{he_2})^{\beta_2} \cdots (\delta^+_{he_n})^{\beta_n} u$. We denote the set of points used in the definition of a difference quotient by supp, the support of the difference quotient. Then, for any set of mesh points $S \subset \mathbb{Z}^n_h$, we define

$$|\delta^\beta u|_{0;S} = \sup_{x\in S,\ \operatorname{supp}\delta^\beta u(x)\subset S} |\delta^\beta u(x)|.$$

With these understandings we may quote a result of Kunkle's.

Theorem 4.1 (see Theorem 13.2 in [11]). *Let Ω be the Cartesian product of n closed (not necessarily finite) intervals such that the corners of Ω are themselves lattice points, and assume that in each of the orthogonal directions e_j, Ω has diameter at least $3h$. With $\Omega_h = \Omega \cap \mathbb{Z}^n_h$, let $u : \Omega_h \to \mathbb{R}$ be a mesh function on Ω_h. Then there exists an interpolant $u_e \in C^\infty(\mathbb{R}^n)$ (so $u_e(x) = u(x)$ for all $x \in \Omega_h$) and a constant C depending only upon n and the choice of a particular compactly supported function $\psi : \mathbb{R} \to \mathbb{R}$ (we fix a choice for the duration of this paper) such that for all β satisfying $0 \leqslant \beta_i \leqslant 3$ for each $i = 1, \dots, n$, we have*

$$|D^\beta u_e|_{0;\Omega} \leqslant C|\delta^\beta u|_{0;\Omega_h}.$$

In fact, if $z \in \Omega_h$ and $\operatorname{dist}(z, \partial\Omega) \geqslant 2h$, then

$$|D^\beta u_e|_{0;z+[0,h]^n} \leqslant C|\delta^\beta u|_{0;z+[-h,2h]^n\cap\mathbb{Z}^n_h}.$$

If $\delta^\beta u$ is not bounded (for instance if Ω is not compact), then for every compact subset K of Ω there is a compact $K' \subset \Omega$ such that

$$|D^\beta u_e|_{0;K} \leqslant C|\delta^\beta u|_{0;K'\cap\mathbb{Z}^n_h}.$$

(This is a straightforward modification of Kunkle's result, which has an important consequence for the proof of the general estimate for $\mathcal{F}_h$ depending explicitly upon x, to feature in a subsequent paper. See [**7**] for an exposition of Kunkle's construction and details of the modification. Kunkle's work generalizes that of Favard [**4**] and DeBoor [**3**] from mesh functions of one variable to mesh functions of several variables.) We will also need the following two Lemmas.

Lemma 4.2. *Let Ω be as in the statement of Theorem 3.1. There exists C dependent only upon n such that for all $\alpha \in (0,1)$ and, for all $f \in C^2(\Omega)$:*

$$_1[f|_{\Omega_h}]^*_{2,\alpha;\Omega_h} \leqslant C[f]^*_{2,\alpha;\Omega}.$$

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Lemma 4.3. *Let Ω be as in the statement of Theorem 3.1. Define*

$$\Omega^+ = \prod_{i=1}^{n}[(a_i - 1)h, (b_i + 1)h],$$

a closed domain $2h$ greater in diameter than Ω in each direction. We have, for example, $\Omega^+(h) = \Omega$. Denote $\Omega^+ \cap \mathbb{Z}_h^n$ by Ω_h^+. Let $v : \Omega_h^+ \to \mathbb{R}$ be a mesh function on Ω_h^+, and let v_e be the extension of $v|_{\Omega_h}$ to Ω provided by Theorem 4.1. Let $\alpha \in (0, 1)$. There exists a positive constant C dependent only upon n and α such that

$$[\Delta v_e]^{(2)}_{0,\alpha;\Omega} \leqslant C\, {}_+[v]^*_{2,\alpha;\Omega_h^+},$$

where Δv_e is the Laplacian of v_e.

These will be proved at the end of this section.

Proof of Theorem 3.1. As noted after the statement of Theorem 3.1, it suffices to estimate ${}_1[u]^*_{2,\alpha;\Omega_h}$. Let α be as provided by Theorem 2.3. Define the mesh function $v : \Omega_h^+ \to \mathbb{R}$ by

$$v(x) = \begin{cases} u(x), & x \in \Omega_h^i, \\ 0, & x \in \Omega_h^b \cup (\Omega_h^+)^b. \end{cases}$$

Then, since v coincides with u on Ω_h^i, we have ${}_1[v]^*_{2,\alpha;\Omega_h} = {}_1[u]^*_{2,\alpha;\Omega_h}$. Note that we can only be sure v satisfies the difference equation on $\Omega((N+1)h)_h = \Omega^+((N+2)h)_h$. However, Theorem 2.3 addresses just such a predicament, and so, applied to v on Ω_h^+, it yields

$${}_+[v]^*_{2,\alpha;\Omega_h^+} \leqslant C|v|_{0;\Omega_h}, \tag{4.1}$$

since $(\Omega_h^+)^i = \Omega_h$.

Let v_e be the C^3 interpolant of v on Ω provided by Theorem 4.1. Then, by Lemma 4.2,

$${}_1[u]^*_{2,\alpha;\Omega_h} = {}_1[v]^*_{2,\alpha;\Omega_h} \leqslant C[v_e]^*_{2,\alpha;\Omega}. \tag{4.2}$$

We estimate the right-hand side using the continuous theory for Poisson's equation. In particular, Theorem 4.8 from [**5**] gives us

$$\begin{aligned}
[v_e]^*_{2,\alpha;\Omega} &\leqslant C(n,\alpha)(|v_e|_{0;\Omega} + |\Delta v_e|^{(2)}_{0,\alpha;\Omega}) \\
&= C(n,\alpha)(|v_e|_{0;\Omega} + [\Delta v_e]^{(2)}_{0;\Omega} + [\Delta v_e]^{(2)}_{0,\alpha;\Omega}) \\
&\leqslant C(n,\alpha)(|v_e|_{0;\Omega} + [v_e]^*_{2;\Omega} + [\Delta v_e]^{(2)}_{0,\alpha;\Omega}).
\end{aligned}$$

Interpolating the second summand on the right-hand side, for example using [**5**, inequality (6.8)], this becomes

$$[v_e]^*_{2,\alpha;\Omega} \leqslant C(n,\alpha)(|v_e|_{0;\Omega} + [\Delta v_e]^{(2)}_{0,\alpha;\Omega}). \tag{4.3}$$

This together with (4.2) allows us to proceed. We use, in turn, (4.3), Theorem 4.1, Lemma 4.3 and (4.1) to conclude:

$$\begin{aligned} {}_1[u]^*_{2,\alpha;\Omega_h} &\leqslant C(|v_{\mathrm{e}}|_{0;\Omega} + [\Delta v_{\mathrm{e}}]^{(2)}_{0,\alpha;\Omega}) \\ &\leqslant C(|v|_{0;\Omega_h} + {}_+[v]^*_{2,\alpha;\Omega_h^+}) \\ &\leqslant C|v|_{0;\Omega_h} = C|u|_{0;\Omega_h^i}. \end{aligned}$$

□

To conclude this section, we must, of course, prove Lemmas 4.2 and 4.3.

Proof of Lemma 4.2. Note that for all $x \in \Omega_h^i$ we have $\underline{d}_x = d_x$, so we dispense with the underscore notation for this proof. Choose distinct x, $y \in \Omega_h^i$ and $z \in Y_1$ such that $x \pm z, y \pm z \in \Omega_h^i$ and $\delta_z^2 f(x) \neq \delta_z^2 f(y)$. Since Ω is convex, the mean value theorem implies that there exist $s, t \in (-1, 1)$ such that with $x' = x + sz$, $y' = y + tz$ and $\hat{z} = z/\|z\|_2$,

$$\begin{aligned} d_{xy}^{2+\alpha} \frac{|\delta_z^2 f(x) - \delta_z^2 f(y)|}{\|x-y\|_2^\alpha} &= d_{xy}^{2+\alpha} \frac{|D_{\hat{z}\hat{z}} f(x') - D_{\hat{z}\hat{z}} f(y')|}{\|x-y\|_2^\alpha} \\ &\leqslant d_{xy}^{2+\alpha} n \frac{|D^\beta f(x') - D^\beta f(y')|}{\|x-y\|_2^\alpha}, \end{aligned}$$

for β with $\|\beta\|_1 = 2$ giving the maximum of $|D^\beta f(x') - D^\beta f(y')|$ over all such multi-indices, and using the fact that $\|\hat{z}\|_1 \leqslant \sqrt{n}$. Note that

$$\frac{\|x'-y'\|_2}{\|x-y\|_2} \leqslant \frac{\|x'-x\|_2 + \|x-y\|_2 + \|y-y'\|_2}{\|x-y\|_2} \leqslant \frac{\sqrt{n}h}{h} + 1 + \frac{\sqrt{n}h}{h}.$$

From above we then have

$$d_{xy}^{2+\alpha} \frac{|\delta_z^2 f(x) - \delta_z^2 f(y)|}{\|x-y\|_2^\alpha} \leqslant d_{xy}^{2+\alpha} n (2\sqrt{n}+1)^\alpha \frac{|D^\beta f(x') - D^\beta f(y')|}{\|x'-y'\|_2^\alpha}.$$

If $d_x = h$, then z is parallel to the boundary and $d_{x'} = h = d_x$. If $d_x \geqslant 2h$, then $d_{x'} \geqslant d_x - h \geqslant d_x - d_x/2 = d_x/2$, and hence $d_x \leqslant 2d_{x'}$. So, in either case, $d_x \leqslant 2\underline{d}_{x'}$. Likewise, $d_y \leqslant 2d_{y'}$, and hence $d_{xy} \leqslant 2d_{x'y'}$. Continuing from above, then, we have

$$\begin{aligned} d_{xy}^{2+\alpha} \frac{|\delta_z^2 f(x) - \delta_z^2 f(y)|}{\|x-y\|_2^\alpha} &\leqslant 2^{2+\alpha} d_{x'y'}^{2+\alpha} n (2\sqrt{n}+1)^\alpha \frac{|D^\beta f(x') - D^\beta f(y')|}{\|x'-y'\|_2^\alpha} \\ &\leqslant 8n(2\sqrt{n}+1)[f]^*_{2,\alpha;\Omega}. \end{aligned}$$

Taking the supremum on the left-hand side over distinct x, $y \in \Omega_h^i$ and $z \in Y_1$ such that $x \pm z, y \pm z \in \Omega_h^i$ establishes the lemma. □

Proof of Lemma 4.3. Choose $i \in \{1, \dots, n\}$, and distinct x, $y \in \Omega$. Consider two cases:

(i) $\|x-y\|_\infty \leqslant 2h$, and

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(ii) $\|x-y\|_\infty > 2h$.

Case (i). $\|x-y\|_\infty \leqslant 2h$. By the mean value theorem, for some $\xi = tx + (1-t)y$, where $t \in (0,1)$, and $\hat{z} = (x-y)/\|x-y\|_2$, we have

$$d_{xy}^{2+\alpha}\frac{|D_{ii}v_{\mathrm{e}}(x) - D_{ii}v_{\mathrm{e}}(y)|}{\|x-y\|_2^\alpha} = d_{xy}^{2+\alpha}|D_{\hat{z}}D_{ii}v_{\mathrm{e}}(\xi)|\|x-y\|_2^{1-\alpha}$$
$$\leqslant d_{xy}^{2+\alpha}\sum_{j=1}^{n}|D_{jii}v_{\mathrm{e}}(\xi)|\|x-y\|_2^{1-\alpha}. \tag{4.4}$$

By Theorem 4.1 there is a constant $C > 0$ such that for any $i,j,k \in \{1,\dots,n\}$,

$$|D_{ijk}v_{\mathrm{e}}(\xi)| \leqslant C|\delta^+_{he_i}\delta^+_{he_j}\delta^+_{he_k}v(x_h)|,$$

for some $x_h \in \Omega_h$ such that $\operatorname{supp}\delta^+_{he_i}\delta^+_{he_j}\delta^+_{he_k}v(x_h) \subset \Omega_h$. In addition, Theorem 4.1 gives us that if $\operatorname{dist}(\xi,\partial\Omega) \geqslant 2h$, then

$$\operatorname{supp}\delta^+_{he_i}\delta^+_{he_j}\delta^+_{he_k}v(x_h) \subset K_{2h}(\xi). \tag{4.5}$$

We now consider two subcases:

(i)(a) $d_\xi = \operatorname{dist}(\xi,\partial\Omega) \geqslant 2h$, and

(i)(b) $d_\xi < 2h$;

and show that in either subcase, defining ${}_+d_x = \operatorname{dist}(x,(\Omega_h^+)^b) = \operatorname{dist}(x,\partial\Omega^+)$, we have that for all $z_h \in \operatorname{supp}\delta^+_{he_j}\delta^+_{he_i}\delta^+_{he_i}v(x_h)$,

$$d_{xy} \leqslant 2\,{}_+d_{z_h}. \tag{4.6}$$

Subcase (i)(a). $d_\xi \geqslant 2h$. If $z_h \in \operatorname{supp}\delta^+_{he_i}\delta^+_{he_j}\delta^+_{he_k}v(x_h)$, then we have by (4.5) that

$${}_+d_{z_h} \geqslant \operatorname{dist}(\xi,\partial\Omega^+) - 2h = \operatorname{dist}(\xi,\partial\Omega) + h - 2h \geqslant d_{xy} - h \geqslant d_{xy} - {}_+d_{z_h},$$

since $z_h \in \Omega_h$ and $d_\xi \geqslant d_{xy}$, and (4.6) follows.

Subcase (i)(b). $d_\xi < 2h$. We must have $d_{xy} < 2h$, and $h \leqslant {}_+d_{z_h}$ for all

$$z_h \in \operatorname{supp}\delta^+_{he_i}\delta^+_{he_j}\delta^+_{he_k}v(x_h),$$

since we still have

$$\operatorname{supp}\delta^+_{he_i}\delta^+_{he_j}\delta^+_{he_k}v(x_h) \subset \Omega_h.$$

Inequality (4.6) is now obvious.

Now

$$|D_{jii}v_{\mathrm{e}}(\xi)| \leqslant C|\delta^+_{he_j}\delta^+_{he_i}\delta^+_{he_i}v(x_h)|$$
$$= C|\delta^+_{he_j}\delta^2_{he_i}v(x_h + he_i)|$$
$$= C|\delta^+_{he_j}\delta^2_{he_i}v(z_h)|,$$

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where $z_h \in \operatorname{supp} \delta^+_{he_j} \delta^+_{he_i} \delta^+_{he_i} v(x_h) \subset \Omega_h$, and satisfies (4.6). Continuing from (4.4) we have

$$\begin{aligned} d_{xy}^{2+\alpha} |D_{jii} v_e(\xi)| \|x-y\|_2^{1-\alpha} &\leqslant C d_{xy}^{2+\alpha} \frac{|\delta^2_{he_i} v(z_h + he_j) - \delta^2_{he_i} v(z_h)|}{h} h^{1-\alpha} (2\sqrt{n})^{1-\alpha} \\ &\leqslant C (2\sqrt{n})^{1-\alpha} 2^{2+\alpha} {}_+d^{2+\alpha}_{z_h, z_h + he_j} \frac{|\delta^2_{he_i} v(z_h + he_j) - \delta^2_{he_i} v(z_h)|}{h^\alpha} \\ &\leqslant C {}_+[v]^*_{2,\alpha;\Omega_h^+}. \end{aligned}$$

Case (ii). $\|x-y\|_\infty > 2h$. Choose mesh neighbours $x_h, y_h \in (\Omega_h^i)^i$ of x and y, respectively, so that $\|x_h - x\|_\infty < 2h$ and $\|y_h - y\|_\infty < 2h$. We have $d_{x_h} \geqslant 2h$, and, therefore, using the geometry of Ω we have $d_{x_h} \geqslant d_x - 2h \geqslant d_x - d_{x_h}$, and so $d_x \leqslant 2 d_{x_h}$. Likewise, $d_y \leqslant 2 d_{y_h}$. It is not difficult to show that $\|x_h - y_h\|_2 \leqslant (2\sqrt{n} + 1)\|x-y\|_2$. Set $w = he_i$, and then write by the triangle inequality

$$\begin{aligned} & d_{xy}^{2+\alpha} \frac{|D_{ii} v_e(x) - D_{ii} v_e(y)|}{\|x-y\|_2^\alpha} \\ &\quad \leqslant d_{xy}^{2+\alpha} \bigg[(\sqrt{n})^\alpha \frac{|D_{ii} v_e(x) - D_{ii} v_e(x_h)|}{\|x - x_h\|_2^\alpha} + \frac{|D_{ii} v_e(x_h) - \delta^2_w v(x_h)|}{\|x-y\|_2^\alpha} \\ &\qquad + (2\sqrt{n}+1)^\alpha \frac{|\delta^2_w v(x_h) - \delta^2_w v(y_h)|}{\|x_h - y_h\|_2^\alpha} + \frac{|\delta^2_w v(y_h) - D_{ii} v_e(y_h)|}{\|x-y\|_2^\alpha} \\ &\qquad + (\sqrt{n})^\alpha \frac{|D_{ii} v_e(y_h) - D_{ii} v_e(y)|}{\|y_h - y\|_2^\alpha} \bigg]. \end{aligned} \tag{4.7}$$

We address the summands on the right-hand side in turn, reducing the first, second, fourth and fifth to case (i) $\|x-y\|_\infty \leqslant 2h$ addressed above. If $x_h \neq x$, then we have for the first summand on the right-hand side that $d_{xy} \leqslant d_x \leqslant 2 d_{x,x_h}$, since $d_x \leqslant 2 d_{x_h}$, and

$$\begin{aligned} d_{xy}^{2+\alpha} \frac{|D_{ii} v_e(x) - D_{ii} v_e(x_h)|}{\|x - x_h\|_2^\alpha} &\leqslant 2^{2+\alpha} d_{x,x_h}^{2+\alpha} \frac{|D_{ii} v_e(x) - D_{ii} v_e(x_h)|}{\|x - x_h\|_2^\alpha} \\ &\leqslant C {}_+[v]^*_{2,\alpha;\Omega_h^+}, \end{aligned}$$

by case (i), since $\|x - x_h\|_\infty \leqslant 2h$. The last summand on the right-hand side of (4.7) is dealt with similarly.

With $w = he_i$, the second summand on the right-hand side of (4.7) satisfies

$$d_{xy}^{2+\alpha} \frac{|D_{ii} v_e(x_h) - \delta^2_w v(x_h)|}{\|x-y\|_2^\alpha} = d_{xy}^{2+\alpha} \frac{|D_{ii} v_e(x_h) - D_{ii} v_e(x')|}{\|x-y\|_2^\alpha},$$

by the mean value theorem, where $x' = x_h + the_i$, $t \in (-1, 1)$, and, hence, $\|x_h - x'\|_2 < h < \|x-y\|_\infty \leqslant \|x-y\|_2$. Also, since we have restricted x_h to $(\Omega_h^i)^i$, we have $d_{x'} \geqslant h$ and $d_{xy} \leqslant d_x \leqslant d_{x'} + 3h \leqslant 4 d_{x'}$. Therefore, if $x' \neq x_h$, then, remembering that $d_x \leqslant 2 d_{x_h}$, we have

$$\begin{aligned} d_{xy}^{2+\alpha} \frac{|D_{ii} v_e(x_h) - \delta^2_w v(x_h)|}{\|x-y\|_2^\alpha} &\leqslant 4^{2+\alpha} d_{x_h,x'}^{2+\alpha} \frac{|D_{ii} v_e(x_h) - D_{ii} v_e(x')|}{\|x_h - x'\|_2^\alpha} \\ &\leqslant C {}_+[v]^*_{2,\alpha;\Omega_h^+}, \end{aligned}$$

the final step following again by reduction to case (i), since $\|x_h - x'\|_\infty \leqslant 2h$. The fourth summand on the right-hand side of (4.7) is dealt with similarly.

Finally, for the third summand on the right-hand side of (4.7), by choice of x_h and y_h,

$$d_{xy} \leqslant 2d_{x_h,y_h} < 2\,{}_+d_{x_h,y_h},$$

and then, if $x_h \neq y_h$, we have

$$d_{xy}^{2+\alpha} \frac{|\delta_{he_i}^2 v(x_h) - \delta_{he_i}^2 v(y_h)|}{\|x_h - y_h\|_2^\alpha} \leqslant 2^{2+\alpha}\,{}_+d_{x_h,y_h}^{2+\alpha} \frac{|\delta_{he_i}^2 v(x_h) - \delta_{he_i}^2 v(y_h)|}{\|x_h - y_h\|_2^\alpha} \leqslant 2^{2+\alpha}\,{}_+[v]^*_{2,\alpha;\Omega_h^+},$$

by definition.

The arbitrariness of i, x and y allows us to infer the result. □

Appendix A. Interpolation inequalities

Lemma A 1. *Let $\Omega_h = \{a, a+h, a+2h, \dots, b-2h, b-h, b\} \subset \mathbb{Z}_h^1$. We define the boundary $\Omega_h^b = \{a, b\}$, and the interior $\Omega_h^i = \{a+h, \dots, b-h\}$. Let $u : \Omega_h \to \mathbb{R}$ be a mesh function on Ω_h. Assume that $b - a \geqslant 5h$, ensuring the existence of at least two distinct points $x, y \in \Omega_h^i$ such that $x \pm h, y \pm h \in \Omega_h^i$, so that ${}_1[u]^*_{2,\alpha;\Omega_h} > -\infty$. Suppose $1 \leqslant j \leqslant k \leqslant 2$, where $j, k \in \mathbb{N}$ and $0 \leqslant \alpha < 1$. Then, for any $\varepsilon > 0$, there exists a constant $C = C(\varepsilon, \alpha, j, k) > 0$ such that*

$$ {}_1[u]^*_{j;\Omega_h} \leqslant C|u|_{0;\Omega_h^i} + \varepsilon\,{}_1[u]^*_{k,\alpha;\Omega_h}, \tag{A 1}$$

where ${}_1[u]^*_{k,0;\Omega_h}$ is defined to be ${}_1[u]^*_{k;\Omega_h}$. Note that each term in this inequality is independent of the values of u on the boundary Ω_h^b.

Proof. For notational convenience we omit the presubscript '1', and the subscript Ω_h, the set Ω_h being implicitly understood, except that $|u|_0$ will denote $|u|_{0;\Omega_h^i}$. We loosely trace the proof of [**5**, Lemma 6.32], and consider several cases.

Case (1). $j = 1$, $k = 2$; $\alpha = 0$. We wish to show that

$$[u]_1^* \leqslant C|u|_0 + \varepsilon[u]_2^*, \tag{A 2}$$

for any $\varepsilon > 0$. Let x be any point in Ω_h^i, $\underline{d}_x$ its distance from Ω_h^b, and $\mu \leqslant \frac{1}{2}$ a positive constant to be specified later. Set $d = \mu \underline{d}_x$ and $K = K_d(x)_h = \{y \in \Omega_h \mid |y - x| \leqslant d\}$. If $d < h$ and $x \pm h \in \Omega_h^i$, then

$$\underline{d}_x |\delta_h^{\pm} u(x)| < \frac{h}{\mu} \frac{|u(x \pm h) - u(x)|}{h} \leqslant \frac{2}{\mu}|u|_0. \tag{A 3}$$

If, however, $d \geqslant h$, then let $d_1 = [\![d/h]\!]h$, the largest integer multiple of h less than or equal to d. (Here, $[\![z]\!]$ denotes the 'greatest integer function'.) Note, then, that $d_1 > d - h \geqslant d - d_1$, since $d_1 \geqslant h$, which implies that $2d_1 > d$, so

$$d_1 \leqslant d < 2d_1 \tag{A 4}$$

and

$$\frac{1}{d_1} < \frac{2}{d} \leqslant \frac{2}{d_1}. \tag{A 5}$$

Instead of the differential calculus used in [**5**] at this point, we use the following identity from [**17**, p. 638]:

$$u(x+Mh) = u(x) + Mh\delta_h^+ u(x) + h\sum_{k=1}^{M-1}[\delta_h^+ u(x+kh) - \delta_h^+ u(x)], \tag{A 6}$$

for $x, x+Mh \in \Omega_h$, $M \in \mathbb{N}$, where the sum on the right-hand side is equal to zero if $M = 1$. By symmetry we also have

$$u(x-Mh) = u(x) - Mh\delta_h^- u(x) - h\sum_{k=1}^{M-1}[\delta_h^- u(x-kh) - \delta_h^- u(x)], \tag{A 7}$$

for $x, x - Mh \in \Omega_h$. Then (A 6) and (A 7) imply that

$$\begin{aligned} \delta_h^\pm u(x) &= \frac{\pm[u(x\pm Mh) - u(x)]}{Mh} - \frac{1}{M}\sum_{k=1}^{M-1}[\delta_h^\pm u(x\pm kh) - \delta_h^\pm u(x)] \\ &= \frac{\pm[u(x\pm Mh) - u(x)]}{Mh} \mp \frac{1}{M}\sum_{k=1}^{M-1}\sum_{l=1}^{k} h\delta_h^2 u(x\pm lh). \end{aligned} \tag{A 8}$$

Let M be such that $Mh = d_1$. To show that the right-hand side of (A 8) depends only on values of u on Ω_h^i, it will suffice to show that $x \pm Mh \in \Omega_h^i$. By definition of d, d_1 and M, we have $\underline{d}_{x\pm Mh} \geqslant \underline{d}_x - d = \underline{d}_x - \mu\underline{d}_x \geqslant \frac{1}{2}\underline{d}_x$. Then, since $d \geqslant h$, we have $\underline{d}_x \geqslant 2h$, and hence $\underline{d}_{x\pm Mh} \geqslant h$; that is, $x \pm Mh \in \Omega_h^i$. In fact (noting that $Mh \leqslant d < (M+1)h$), $x \pm Mh \in K \subset \Omega_h^i$. In particular, $x \pm h \in K$, which is to say, $x \in K^i$. It follows, using (A 4) and (A 5), that

$$\begin{aligned} |\delta_h^\pm u(x)| &\leqslant \frac{|u(x\pm Mh) - u(x)|}{Mh} + \frac{1}{M}\sum_{k=1}^{M-1}\sum_{l=1}^{k} h|\delta_h^2 u(x\pm lh)| \\ &\leqslant \frac{2|u|_0}{d_1} + \frac{h}{M}\frac{(M-1)M}{2}\sup_{y\pm h\in K}|\delta_h^2 u(y)| \\ &\leqslant \frac{4|u|_0}{d} + \tfrac{1}{2}(M-1)h\sup_{y\pm h\in K}|\delta_h^2 u(y)| \\ &\leqslant \frac{4|u|_0}{d} + \tfrac{1}{2}d\sup_{y\pm h\in K}|\delta_h^2 u(y)| \\ &\leqslant \frac{4|u|_0}{d} + \tfrac{1}{2}d\sup_{y\pm h\in K}\underline{d}_y^{-2}\sup_{y\pm h\in K}\underline{d}_y^2|\delta_h^2 u(y)|. \end{aligned}$$

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Since $\underline{d}_y \geqslant \underline{d}_x - d = (1-\mu)\underline{d}_x \geqslant \underline{d}_x/2$ for all $y \in K$, it follows that

$$\begin{aligned}\underline{d}_x|\delta_h^{\pm}u(x)| &\leqslant \underline{d}_x \frac{4}{\mu \underline{d}_x}|u|_0 + \underline{d}_x \tfrac{1}{2}\mu \underline{d}_x \sup_{y\pm h\in K} \underline{d}_y^{-2} \sup_{y\pm h\in K} \underline{d}_y^2 |\delta_h^2 u(y)| \\ &\leqslant \frac{4}{\mu}|u|_0 + \tfrac{1}{2}\mu \sup_{y\pm h\in K}\left(\frac{\underline{d}_x}{\underline{d}_y}\right)^2 [u]_2^* \\ &\leqslant \frac{4}{\mu}|u|_0 + 2\mu[u]_2^*,\end{aligned}$$

since $\underline{d}_x^2/\underline{d}_y^2 \leqslant 4$. Hence, considering (A 3),

$$[u]_1^* = \sup_{\substack{x\in\Omega_h^i \\ x\pm h\in\Omega_h^i}} \underline{d}_x|\delta_h^{\pm}u(x)| \leqslant \frac{4}{\mu}|u|_0 + 2\mu[u]_2^*.$$

Choosing $\mu = \mu(\varepsilon)$ less than $\varepsilon/2$, we conclude (A 2) with $C = 4\mu^{-1}$.

Case (2). $j=2$, $k=2$; $\alpha > 0$. As before let $x \in \Omega_h^i$, $0 < \mu \leqslant \frac{1}{2}$, $d = \mu\underline{d}_x$, $K = K_d(x)_h$, and $d_1 = [\![d/h]\!] h$. If $d < h$ and $x \pm h \in \Omega_h^i$, then

$$\underline{d}_x^2|\delta_h^2 u(x)| < \frac{h^2}{\mu^2}\frac{|u(x+h) - 2u(x) + u(x-h)|}{h^2} \leqslant \frac{4}{\mu^2}|u|_0. \tag{A 9}$$

If, however, $d \geqslant h$, then $d_1 \geqslant h$ and (A 4) and (A 5) hold. Again using the identity (A 6) with $M \in \mathbb{N}$ chosen such that $Mh = d_1$, this time applied to $\delta_h^+ u$, we obtain

$$\begin{aligned}\delta_h^+ u(x+Mh) &= \delta_h^+ u(x) + Mh(\delta_h^+)^2 u(x) + h\sum_{k=1}^{M-1}[(\delta_h^+)^2 u(x+kh) - (\delta_h^+)^2 u(x)] \\ &= \delta_h^+ u(x) + Mh\delta_h^2 u(x+h) + h\sum_{k=1}^{M-1}[\delta_h^2 u(x+(k+1)h) - \delta_h^2 u(x+h)].\end{aligned}$$

It follows, since $M \geqslant 1$, that replacing x with $x-h$,

$$\delta_h^+ u(x+(M-1)h) = \delta_h^+ u(x-h) + Mh\delta_h^2 u(x) + h\sum_{k=1}^{M-1}[\delta_h^2 u(x+kh) - \delta_h^2 u(x)],$$

in which case

$$\delta_h^2 u(x) = \frac{\delta_h^+ u(x+(M-1)h) - \delta_h^+ u(x-h)}{Mh} - \frac{1}{M}\sum_{k=1}^{M-1}[\delta_h^2 u(x+kh) - \delta_h^2 u(x)]. \tag{A 10}$$

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As in the paragraph after (A 8), one can show that $x \pm Mh \in K \subset \Omega_h^i$, and therefore the left-hand side of (A 10) depends only on the values of u on Ω_h^i. Therefore,

$$
\begin{aligned}
&|\delta_h^2 u(x)| \\
&\leqslant \frac{2}{d_1} \sup_{y,y+h\in K} |\delta_h^+ u(y)| + \frac{1}{M}\sum_{k=1}^{M-1} |\delta_h^2 u(x+kh) - \delta_h^2 u(x)| \\
&\leqslant \frac{4}{d} \sup_{y,y+h\in K} \underline{d}_y^{-1} \sup_{y,y+h\in K} \underline{d}_y |\delta_h^+ u(y)| + \frac{(M-1)}{M} \sup_{y\pm h\in K} \underline{d}_{xy}^{-2-\alpha} \underline{d}_{xy}^{2+\alpha} |\delta_h^2 u(y) - \delta_h^2 u(x)| \\
&\leqslant \frac{4}{d} \sup_{y,y+h\in K} \underline{d}_y^{-1} \sup_{y,y+h\in K} \underline{d}_y |\delta_h^+ u(y)| + d^\alpha \sup_{y,y\pm h\in K} \underline{d}_{xy}^{-2-\alpha} \sup_{\substack{y\pm h\in K\\ y\neq x}} \underline{d}_{xy}^{2+\alpha} \frac{|\delta_h^2 u(y) - \delta_h^2 u(x)|}{|x-y|^\alpha}.
\end{aligned}
$$

As in case (1), $\underline{d}_y$ (and therefore $\underline{d}_{xy}$) is greater than $\underline{d}_x/2$ for all $y \in K$. It follows that

$$
\begin{aligned}
\underline{d}_x^2 |\delta_h^2 u(x)| &\leqslant \underline{d}_x^2 \frac{4}{\mu \underline{d}_x} \sup_{y,y+h\in K} \underline{d}_y^{-1} \sup_{y,y+h\in K} \underline{d}_y |\delta_h^+ u(y)| \\
&\qquad + \underline{d}_x^2 \mu^\alpha \underline{d}_x^\alpha \sup_{y\pm h\in K} \underline{d}_{xy}^{-2-\alpha} \sup_{\substack{y\pm h\in K\\ y\neq x}} \underline{d}_{xy}^{2+\alpha} \frac{|\delta_h^2 u(y) - \delta_h^2 u(x)|}{|x-y|^\alpha} \\
&\leqslant \frac{4}{\mu} 2 \sup_{y,y+h\in K} \underline{d}_y |\delta_h^+ u(y)| + \mu^\alpha 2^{2+\alpha} \sup_{\substack{y\pm h\in K\\ y\neq x}} \underline{d}_{xy}^{2+\alpha} \frac{|\delta_h^2 u(y) - \delta_h^2 u(x)|}{|x-y|^\alpha} \\
&\leqslant \frac{8}{\mu}[u]_1^* + \mu^\alpha 2^{2+\alpha} [u]_{2,\alpha}^*.
\end{aligned}
$$

Hence, considering (A 9),

$$
[u]_2^* = \sup_{\substack{x\in\Omega_h^i\\ x\pm h\in\Omega_h^i}} \underline{d}_x^2 |\delta_h^2 u(x)| \leqslant \frac{4}{\mu^2}|u|_0 + \frac{8}{\mu}[u]_1^* + 2^{2+\alpha}\mu^\alpha [u]_{2,\alpha}^*.
$$

Now, if $\mu = \mu(\varepsilon, \alpha)$ is chosen so that $8\mu^\alpha \leqslant \varepsilon/2$, then, with $C' = 8/\mu$ and $C'' = 4/\mu^2$, we have

$$
[u]_2^* \leqslant C''|u|_0 + C'[u]_1^* + \tfrac{1}{2}\varepsilon [u]_{2,\alpha}^*.
$$

By (A 2) there exists C''' such that

$$
[u]_1^* \leqslant C'''|u|_0 + \frac{1}{2C'}[u]_2^*,
$$

and so finally we arrive at

$$
[u]_2^* \leqslant C|u|_0 + \varepsilon [u]_{2,\alpha}^*,
$$

as desired.

If instead of (A 10) we use (A 8) in the argument of case (2), modifying the details accordingly, then we obtain (A 1) for $j = 1$, $k = 1$, $\alpha > 0$.

Finally, the case $j = 1$, $k = 2$, $\alpha > 0$ follows from cases (1) and (2). □

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Lemma A 2. *Let Ω be an open or closed domain properly contained in $\mathbb{R}^n$. Let $h > 0$ be sufficiently small that $\Omega_h^i \neq \emptyset$, and such that there exist $x \in \Omega_h^i$ and $z \in Y'$ such that $x \pm z \in \Omega_h^i$. Let $u : \Omega_h \to \mathbb{R}$ be a mesh function on Ω_h. Suppose $1 \leqslant j \leqslant k \leqslant 2$, where $j, k \in \mathbb{N}$, and $0 \leqslant \alpha < 1$. Then for any $\varepsilon > 0$ there exists a constant $C = C(\varepsilon, \alpha, j, k, N, n) > 0$ such that*

$$_\star[u]^*_{j;\Omega_h} \leqslant C|u|_{0;\Omega_h^i} + \varepsilon\, _\star[u]^*_{k,\alpha;\Omega_h}.$$

Proof. Let $\varepsilon > 0$. Let $x \in \Omega_h^i$.

If $\underline{d}_x < 3\sqrt{n}Nh$ and $z \in Y'$ such that $x \pm z \in \Omega_h^i$, then

$$\underline{d}_x^2|\delta_z^2 u(x)| \leqslant 9nN^2h^2 \frac{|u(x+z) - 2u(x) + u(x-z)|}{h^2} \leqslant 36nN^2|u|_{0;\Omega_h^i},$$

and, similarly,

$$\underline{d}_x|\delta_z^\pm u(x)| \leqslant C|u|_{0;\Omega_h^i}.$$

However, if $\underline{d}_x \geqslant 3\sqrt{n}Nh$ and $z \in Y'$, then consider $B_h = \bar{B}_{\underline{d}_x}(x) \cap \mathbb{Z}_h^n$, the 'closed' mesh ball of radius $\underline{d}_x$ centred at x. Let

$$S = \{x \pm kz \mid k \in \mathbb{N}\} \cap B_h.$$

Note that S consists of at least seven points. We will apply Lemma A 1 to S. With this in mind, define $S^i = \{x \in S \mid x+z, x-z \in S\}$ and $S^b = S \setminus S^i$ (so S^b consists of the two endpoints of S), and for $y \in S^i$, let $_S\underline{d}_y = \operatorname{dist}(y, S^b)$. We have, for example, that for all $y \in S$, $_S\underline{d}_y \leqslant \underline{d}_x$. Note that $\underline{d}_x - \sqrt{n}Nh \leqslant {_S\underline{d}_x} \leqslant \underline{d}_x$. Since $\underline{d}_x \geqslant 3\sqrt{n}Nh$, we have

$$_S\underline{d}_x \geqslant \underline{d}_x - \sqrt{n}N\left(\frac{\underline{d}_x}{3\sqrt{n}N}\right) = \tfrac{2}{3}\underline{d}_x.$$

Application of Lemma A 1 gives

$$\max\{\underline{d}_x|\delta_z^\pm u(x)|,\ \underline{d}_x^2|\delta_z^2 u(x)|\} \leqslant C|u|_{0;S^i} + \varepsilon\, _z[u]^*_{k,\alpha;S},$$

where $k = 1$ or 2 accordingly, $0 \leqslant \alpha < 1$, and the semi-norm $_z[u]^*_{\dots}$ is the same as $_\star[u]^*_{\dots}$ except that z is fixed in its definition. It follows that

$$\max\{\underline{d}_x|\delta_z^\pm u(x)|,\ \underline{d}_x^2|\delta_z^2 u(x)|\} \leqslant C|u|_{0;\Omega_h^i} + \varepsilon\, _z[u]^*_{k,\alpha;\Omega_h},$$

since, for any $\bar{x} \in S^i$, $_S\underline{d}_{\bar{x}} \leqslant \underline{d}_{\bar{x}}$. In the light of the first part of the proof, this is true for all $x \in \Omega_h^i$ and all $z \in Y'$ satisfying $x \pm z \in \Omega_h^i$, and the result follows. □

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