*J. Austral. Math. Soc.* (*Series A*) **59** (1995), 353–365

# COMMUTATORS IN PSEUDO-ORTHOGONAL GROUPS

**F. A. ARLINGHAUS, L. N. VASERSTEIN and HONG YOU**

(Received 6 November 1992; revised 11 April 1994)

Communicated by H. Lausch

**Abstract**

We study commutators in pseudo-orthogonal groups $O_{2n}R$ (including unitary, symplectic, and ordinary orthogonal groups) and in the conformal pseudo-orthogonal groups $GO_{2n}R$. We estimate the number of commutators, $c(O_{2n}R)$ and $c(GO_{2n}R)$, needed to represent every element in the commutator subgroup. We show that $c(O_{2n}R) \leq 4$ if $R$ satisfies the $\Lambda$-stable condition and either $n \geq 3$ or $n = 2$ and 1 is the sum of two units in $R$, and that $c(GO_{2n}R) \leq 3$ when the involution is trivial and $\Lambda = R^{\epsilon}$. We also show that $c(O_{2n}R) \leq 3$ and $c(GO_{2n}R) \leq 2$ for the ordinary orthogonal group $O_{2n}R$ over a commutative ring $R$ of absolute stable rank 1 where either $n \geq 3$ or $n = 2$ and 1 is the sum of two units in $R$.

1991 *Mathematics subject classification* (*Amer. Math. Soc.*): 20G35, 20H25.

## 1. Introduction

For any group $G$, let $c(G)$ be the least integer $s \geq 0$ such that every product of commutators is the product of $s$ commutators. If no such $s$ exists, we set $c(G) = \infty$.

The number $c(G)$ has been studied extensively for various groups $G$. For a survey of commutator results, see [7]. In [7], it was shown that $c(GL_n R) \leq 2$, where $R$ is a commutative ring of stable rank 1 and either $n \geq 3$ or $n = 2$ and 1 is the sum of two units in $R$. In [8], You obtained a similar result for symplectic groups, showing $c(Sp_{2n}R) \leq 4$ and $c(GSp_{2n}R) \leq 3$ where $R$ is a ring of stable rank 1 and $n \geq 3$.

The main goal of this paper is to study commutators in pseudo-orthogonal groups $O_{2n}R$ (including unitary, symplectic, and ordinary orthogonal groups) and in the conformal pseudo-orthogonal groups $GO_{2n}R$ and to estimate $c(O_{2n}R)$ and $c(GO_{2n}R)$. We show that if $R$ satisfies the $\Lambda$-stable condition and either $n \geq 3$ or $n = 2$ and 1 is the sum of two units in $R$, then $c(O_{2n}R) \leq 4$ and, when the involution is trivial and $\Lambda = R^{\epsilon}$, then $c(GO_{2n}R) \leq 3$. This result generalizes a previous result of [8]. We also





https://doi.org/10.1017/S1446788700037253 Published online by Cambridge University Press

show that $c(O_{2n}R) \leq 3$ and $c(GO_{2n}R) \leq 2$ for the ordinary orthogonal group $O_{2n}R$ over a commutative ring $R$ of absolute stable rank 1 where either $n \geq 3$ or $n = 2$ and 1 is the sum of two units in $R$.

We assume that an involution $* : x \mapsto x^*$ is given on an associative ring $R$ with 1. Thus $(x^*)^* = x$, $(x - y)^* = x^* - y^*$, and $(xy)^* = y^*x^*$ for any $x, y \in R$. The involution $*$ also determines an involution of the ring $M_nR$ of all $n$ by $n$ matrices by $(x_{ij})^* = x_{ji}^*$.

Let $\epsilon$ be an element of the center of $R$ such that $\epsilon\epsilon^* = 1$. Set $R_\epsilon = \{x - \epsilon x^* : x \in R\}$, $R^\epsilon = \{x \in R : x = -\epsilon x^*\}$. We fix an additive subgroup $\Lambda$ of $R$ with the following properties:

(i) $r\Lambda r^* \subset \Lambda$ for all $r \in R$;
(ii) $R_\epsilon \subset \Lambda \subset R^\epsilon$.

Let $\Lambda_n$ denote the set $\{(a_{ij})_{n\times n} : a_{ij} = -\epsilon a_{ji}^* \text{ for } i \neq j \text{ and } a_{ii} \in \Lambda\}$.

As in [1], we define

$$O_{2n}R = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in GL_{2n}R : \alpha\delta^* + \epsilon\beta\gamma^* = I,\ \alpha\beta^*,\ \gamma\delta^* \in \Lambda_n \right\},$$

and

$$GO_{2n}R = \left\{ \begin{pmatrix} I & 0 \\ 0 & \zeta I \end{pmatrix} \psi : \psi \in O_{2n}R,\ \zeta \text{ a unit in the center of } R \text{ with } \zeta\Lambda = \Lambda \text{ and } \zeta = \zeta^* \right\}.$$

Set $\sigma k = k + n$ if $k \leq n$, $\sigma k = k - n$ if $k > n$. For $a \in R$, define

$$\rho_{ij}(a) = \begin{cases} I_{2n} + aE_{ij} - a^*E_{\sigma j,\sigma i} & (1 \leq i \neq j \leq n), \\ I_{2n} + aE_{i,j} - \epsilon^*a^*E_{\sigma j,\sigma i} & (i \neq \sigma j,\ 1 \leq i \leq n,\ n+1 \leq j \leq 2n), \\ I_{2n} + aE_{i,j} - \epsilon a^*E_{\sigma j,\sigma i} & (i \neq \sigma j,\ n+1 \leq i \leq 2n,\ 1 \leq j \leq n), \\ I_{2n} + aE_{i,\sigma i} & (1 \leq i = \sigma j \leq n,\ a^* \in \Lambda), \\ I_{2n} + aE_{i,\sigma i} & (n+1 \leq i = \sigma j \leq 2n,\ a \in \Lambda), \end{cases}$$

where $E_{ij}$ denotes the matrix with 1 in the $i$th row and the $j$th column and zeros elsewhere. We denote by $EO_{2n}R$ the subgroup of $O_{2n}R$ generated by the set of $\rho_{ij}(a)$ with $a \in R$.

A ring is said to satisfy the $\Lambda$-stable condition (see [3]) if when $Ra + Rb = R$, then there is an $x^* \in \Lambda$ such that $R(a + xb) = R$. If an associative ring $R$ satisfies the $\Lambda$-stable condition, we can show that the ring $R' = M_nR$ of $n$ by $n$ matrices satisfies the $\Lambda_n$-stable condition; that is, if $a, b \in R'$ with $R'a + R'b = R'$, then $a + xb$ is invertible for some $x^* \in \Lambda_n$ (see Lemma 6).

When $\Lambda = R$ (in which case $O_{2n}R$ is the ordinary symplectic group), the $\Lambda$-stable condition is equivalent to the first Bass stable range condition. Some examples of rings satisfying the $\Lambda$-stable condition can be found in [3, pp. 218-223].

For a subset $S$ of $R$, we denote by $J(S)$ the intersection of all left maximal ideals of $R$ which contain $S$. We say a sequence $a_0, a_1, \dots, a_n$ in $R$ can be shortened if there are $t_0, t_1, \dots, t_{n-1}$ in $R$ such that $a_n \in J(a_0 + t_0 a_n, \dots, a_{n-1} + t_{n-1}a_n)$.

If every sequence containing $n+1$ elements in $R$ can be shortened, we say that the ring satisfies the asr($n$) condition. When $n$ is the least integer such that $R$ satisfies the asr($n$) condition, we say that $n$ is the absolute stable rank of $R$, denoted by $\mathrm{asr}(R) = n$ (see [5]). In general, the stable rank of $R$ (denoted by $\mathrm{sr}(R)$) is less than or equal to the absolute stable rank of $R$ (see [5]).

EXAMPLES. (See [5]):

(i) If $R \neq 0$ is semilocal, then $\mathrm{asr}(R) = \mathrm{sr}(R) = 1$.

(ii) If $R$ is commutative and the maximal spectrum of $R$ is Noetherian of finite dimension $n$, then any module-finite $R$-algebra $A$ has absolute stable rank at most $n+1$.

We denote by $B^+$ (respectively $B^-$) the subgroup of $O_{2n}R$ consisting of the matrices of the form $\begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{*^{-1}} \end{pmatrix}$ (respectively $\begin{pmatrix} \alpha & 0 \\ \beta & \alpha^{*^{-1}} \end{pmatrix}$) where $\alpha$ is an $n$ by $n$ upper (respectively lower) triangular matrix and $\alpha\beta^* \in \Lambda_n$ (respectively $\alpha^*\beta \in \Lambda_n$). The subgroup of $B^+$ (respectively $B^-$) formed by the above matrices such that the diagonal entries of $\alpha$ are 1 is denoted by $U^+$ (respectively $U^-$). We use the symbol $I_{2(n-k)} \oplus O_{2k}R$ to denote the subgroup of $O_{2n}R$ formed by the matrices

$$\begin{pmatrix} I & & & \\ & \alpha & & \beta \\ & & I & \\ & \gamma & & \delta \end{pmatrix}, \text{ where } \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in O_{2k}R.$$

In Section 2, we obtain two decomposition results involving $B^+$ and $B^-$ or $U^+$ and $U^-$.

PROPOSITION 1. *Assume that $R$ satisfies the $\Lambda$-stable condition. Then every matrix $\theta \in EO_{2n}R$ can be written in the form*

(i) *$\psi_1\lambda_1\psi_2$, where $\psi_i$ are in $B^+$ and $\lambda_1$ is in $B^-$,*

(ii) *$\psi_1\lambda_1\psi_2\lambda_2$, where $\psi_i$ are in $U^+$ and $\lambda_i$ are in $U^-$.*

*Therefore any matrix $\theta \in EO_{2n}R$ is similar to the product $\psi\lambda$, where $\psi$ is in $B^+$ and $\lambda$ is in $B^-$.*

PROPOSITION 2. *Assume that $n \geq \operatorname{asr}(R) + 2$. Then every matrix $\theta \in O_{2n}R$ can be written in the form*

(i) $\psi_1\lambda_1\theta_1\psi_2$, *where $\psi_i$ are in $B^+$, $\lambda_1$ is in $B^-$, and $\theta_1 \in I_{2(n-k)} \oplus O_{2k}R$ for $k = \operatorname{asr}(R) + 1$,*

(ii) $\psi_1\lambda_1\theta_1\psi_2\lambda_2$, *where $\psi_i$ are in $U^+$, $\lambda_i$ are in $U^-$, and $\theta_1 \in I_{2(n-k)} \oplus O_{2k}R$ for $k = \operatorname{asr}(R) + 1$.*

*Therefore any matrix $\theta \in O_{2n}R$ is similar to the product $\psi\lambda\theta_1$, where $\psi$ is in $B^+$, $\lambda$ is in $B^-$, and $\theta_1 \in I_{2(n-k)} \oplus O_{2k}R$. If $R$ is commutative and $*$ is the trivial involution, the conclusions of* (i) *and* (ii) *hold also for $k = \operatorname{asr}(R)$.*

We also obtain the following corollary to Proposition 2.

COROLLARY 3. *For the ordinary orthogonal group, assume $\operatorname{asr}(R) \leq 1$ for a commutative ring $R$, and $n \geq 2$. Then every matrix $\theta \in EO_{2n}R$ can be written in the form*

(i) $\psi_1\lambda_1\psi_2$, *where $\psi_i$ are in $B^+$ and $\lambda_1$ is in $B^-$,*

(ii) $\psi_1\lambda_1\psi_2\lambda_2$, *where $\psi_i$ are in $U^+$ and $\lambda_i$ are in $U^-$.*

*Therefore any matrix $\theta \in EO_{2n}R$ is similar to the product $\psi\lambda$, where $\psi$ is in $B^+$ and $\lambda$ is in $B^-$.*

In Section 3, we will use Propositions 1 and 2 and Corollary 3 to prove the following results.

THEOREM 4. *Let $R$ be a commutative ring with 1 satisfying the $\Lambda$-stable condition. Assume that either $n \geq 3$ or $n = 2$ and 1 is the sum of two units in $R$. Then*

(i) $c(EO_{2n}R) \leq 4$, *hence* $c(O_{2n}R) \leq 4$,

(ii) *when the involution is trivial and $\Lambda = R^\epsilon$, $c(GO_{2n}R) \leq 3$.*

THEOREM 5. *Let $R$ be a commutative ring with 1 of absolute stable rank 1, and let $O_{2n}R$ be the ordinary orthogonal group. Assume that either $n \geq 3$ or $n = 2$ and 1 is the sum of two units in $R$. Then*

(i) $c(EO_{2n}R) \leq 3$, *hence* $c(O_{2n}R) \leq 3$,

(ii) $c(GO_{2n}R) \leq 2$.

## 2. Preliminary results

LEMMA 6. *If $R$ satisfies the $\Lambda$-stable condition, then $R' = M_nR$ satisfies the $\Lambda_n$-stable condition.*

PROOF. Let $a, b \in R'$ with $R'a + R'b = R'$. Our problem is to find $\zeta^* \in \Lambda_n$ such that $a + \zeta b \in GL_n R$. If we replace $\begin{pmatrix} a \\ b \end{pmatrix}$ by $\begin{pmatrix} \eta & \alpha \\ 0 & \eta^{*^{-1}} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$ with $\eta \in GL_n R$ and $\eta\alpha^* \in \Lambda_n$, we obtain an equivalent problem.

Consider the first column of the matrix $(a, b)^t$. Since $R$ satisfies the $\Lambda$-stable condition, $R$ has stable rank 1. Thus we may find a suitable matrix $\eta_1 \in GL_n R$ and replace $(a, b)^t$ by $\mathrm{diag}(\eta_1, \eta_1^{*^{-1}})$ $(a, b)^t$ such that $b_{11}$ and the first column of $a$ form a unimodular vector. By the hypothesis there is an $x^* \in \Lambda$ such that $(a_{11} + xb_{11}, a_{21}, \dots, a_{n1})^t$ is unimodular. We multiply $b_{11}$ by $x \in \Lambda^*$ and add $xb_{11}$ to $a_{11}$. Again, replacing $(a, b)^t$ by $\mathrm{diag}(\eta_2, \eta_2^{*^{-1}})$ $(a, b)^t$, we may assume that $a_{11} = 1$, $a_{21} = \dots = a_{n1} = 0$, that is, we have

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 & u \\ 0 & a_1 \\ * & * \\ v & b_1 \end{pmatrix},$$

where $a_1, b_1$, and $b_1 - vu$ are $n - 1$ by $n - 1$ matrices. Note that the matrix $(a_1, *, b_1 - vu)^t$ is unimodular. In this matrix, we can add multiples of the $n$-th row to the $(n + 1)$th through $(2n - 1)$th rows without changing $a_1$. Thus we can assume that $(a_1, b_1 - vu)^t$ is unimodular.

Then we can use induction on $n$ to complete the proof. When $n = 1$, it is trivial. Assume it is true for $n-1$. Then there is an $x_1^* \in \Lambda_{n-1}$ such that $a_1 + x_1(b_1 - vu) \in GL_{n-1}R$. We can take $x_2^* = \begin{pmatrix} 0 & 0 \\ 0 & x_1^* \end{pmatrix} \in \Lambda_n$ such that the first $n$ by $n$ block in

$$\begin{pmatrix} I & x_2 \\ 0 & I \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & a_1 \\ * & * \\ 0 & b_1 - vu \end{pmatrix}$$

lies in $GL_n R$, and thus we're done.

PROOF OF PROPOSITION 1. (i) Let $\theta = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in EO_{2n}R$. By Lemma 6, there is $\zeta^* \in \Lambda_n$ such that $\eta = \alpha + \zeta\gamma \in GL_n R$ and $\eta^*\gamma \in \Lambda_n$. Then

$$\begin{pmatrix} I & \zeta \\ 0 & I \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \eta & \beta + \zeta\delta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} I & 0 \\ \gamma\eta^{-1} & I \end{pmatrix} \begin{pmatrix} I & (\beta + \zeta\delta)\eta^* \\ 0 & I \end{pmatrix} \begin{pmatrix} \eta & 0 \\ 0 & \eta^{*^{-1}} \end{pmatrix}.$$

By [7, Theorem 1], $\eta = \psi_1'\lambda_1'\psi_2'$, where the $\psi_i'$ are upper triangular matrices in $GL_n R$ and $\lambda_1'$ is a lower triangular matrix in $GL_n R$. Then

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \psi_1' & \omega_1 \\ 0 & \psi_1'^{*^{-1}} \end{pmatrix} \begin{pmatrix} \lambda_1' & 0 \\ \omega_2 & \lambda_1'^{*^{-1}} \end{pmatrix} \begin{pmatrix} \psi_2' & \omega_3 \\ 0 & \psi_2'^{*^{-1}} \end{pmatrix} = \psi_1\lambda_1\psi_2.$$

https://doi.org/10.1017/S1446788700037253 Published online by Cambridge University Press

(ii) The proof is similar, using the factorization $\eta = \psi_1'\lambda_1'\psi_2'\lambda_2'$ from Lemma 9 of [4].

LEMMA 7. ([5]). *For any ring $R$ and positive integer $n$,* $\operatorname{asr}(R) \le n$ *if and only if for every sequence $a_0, a_1, \dots, a_n$ in $R$, there are $t_0, t_1, \dots, t_{n-1}$ in $R$ such that $R(1 + ha_n) + R(a_0 + t_0 a_n) + \cdots + R(a_{n-1} + t_{n-1}a_n) = R$ for every $h$ in $R$.*

LEMMA 8.

(i)

$$\prod_{j=n+1}^{2n} \rho_{1,j}(*) \begin{pmatrix} 1 & & & \\ & I_{n-1} & & \\ & & 1 & \\ & \chi & & I_{n-1} \end{pmatrix} = \prod_{j=2}^{n} \rho_{1j}(*) \begin{pmatrix} 1 & & & \\ & I_{n-1} & & \\ & & 1 & \\ & \chi & & I_{n-1} \end{pmatrix} \prod_{j=n+1}^{2n} \rho_{1j}(*).$$

(ii)

$$\prod_{j=1}^{n} \rho_{n+1,j}(*) \begin{pmatrix} 1 & & & \\ & I_{n-1} & & \chi \\ & & 1 & \\ & & & I_{n-1} \end{pmatrix} = \prod_{j=2}^{n} \rho_{j1}(*) \begin{pmatrix} 1 & & & \\ & I_{n-1} & & \chi \\ & & 1 & \\ & & & I_{n-1} \end{pmatrix} \prod_{j=1}^{n} \rho_{n+1,j}(*).$$

PROOF OF PROPOSITION 2. (i) Let $v = (a_1, \dots, a_n, b_1, \dots, b_n)^t$ be the first column of $\theta$. Since $\operatorname{sr}(R) \le \operatorname{asr}(R) \le n-1$, we can find a matrix $\delta = \operatorname{diag}(\eta_1, \eta_1^{*^{-1}}) \in B^+$ such that $a_1, \dots, a_{n-1}, b_1, \dots, b_n$ in $\delta v$ form a unimodular vector.

Suppose that $c_n' b_n + \sum_{i=1}^{n-1}(c_i a_i + c_i' b_i) = 1$. Multiplying the equation by $1-a_n$ on the left and replacing $v$ by $\prod_{i=1}^{n-1} \rho_{n,\sigma i}((1-a_n)c_i') \prod_{i=1}^{n-1} \rho_{ni}((1-a_n)c_i)v$, we get $a_n = 1 + xb_n$ for some $x \in R$.

Since $\operatorname{asr}(R) \le n-2$, by Lemma 7 there exist $t_i \in R$ such that $R(1 + hb_1) + \sum_{i=2}^{n-1} R(a_i + t_i b_1) = R$. Replacing $v$ by $\prod_{i=2}^{n-1} \rho_{i,n+1}(t_i)v$, we have $R(1 + hb_1) + \sum_{i=2}^{n-1} Ra_i = R$ for every $h$ in $R$.

Since $a_n = 1 + xb_n$, $Ra_n + Rb_n = R$, there exist $y_1, y_2 \in R$ such that $y_1 a_n + y_2 b_n = -a_1 + 1 + hb_1$. Replacing $v$ by $\rho_{1,2n}(y_2)v$, we get $\sum_{i=1}^{n} Ra_i = R$.

There exists a $\eta_2 \in GL_n R$ such that $\operatorname{diag}(\eta_2, \eta_2^{*^{-1}})v = (1, 0, \dots, 0, *, \dots, *)^t$. Then multiplying $v$ by $\prod_{i=n+1}^{2n} \rho_{i,1}(*)$ , we get $v = (1, 0, \dots, 0)^t$.

Summarizing this procedure, we have

https://doi.org/10.1017/S1446788700037253 Published online by Cambridge University Press

$$\prod_{i=n+1}^{2n} \rho_{i1}(*) \operatorname{diag}(\eta_2, \eta_2^{*^{-1}}) \rho_{1,2n}(y_2) \prod_{i=2}^{n-1} \rho_{i,n+1}(*) \prod_{i=1}^{n-1} \rho_{n,\sigma i}(*) \operatorname{diag}(\eta_3, \eta_3^{*^{-1}}) \theta \psi_4$$

$$= \begin{pmatrix} 1 & & & \\ & \alpha & & \beta \\ & & 1 & \\ & \gamma & & \delta \end{pmatrix},$$

where $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in O_{2(n-1)}R$, and hence we can write $\theta$ as

$$\theta = \begin{pmatrix} \eta_4 & 0 \\ 0 & \eta_4^{*^{-1}} \end{pmatrix} \psi_3 \prod_{i=n+1}^{2n} \rho_{i1}(*) \begin{pmatrix} 1 & & & \\ & \alpha & & \beta \\ & & 1 & \\ & \gamma & & \delta \end{pmatrix} \psi_4^{-1},$$

where

$$\psi_3 = \operatorname{diag}(\eta_2, \eta_2^{*^{-1}}) \prod_{i=1}^{n-1} \rho_{n,\sigma i}(*) \prod_{i=2}^{n-1} \rho_{i,n+1}(*) \rho_{1,2n}(-y_2) \operatorname{diag}(\eta_2^{-1}, \eta_2^*) \in B^+,$$

$$\psi_4 = \prod_{i=1}^{n} \rho_{1i}(*) \prod_{i=1}^{n} \rho_{1,\sigma i}(*) \in B^+,$$

and $\begin{pmatrix} \eta_4 & 0 \\ 0 & \eta_4^{*^{-1}} \end{pmatrix} = \operatorname{diag}(\eta_3^{-1}, \eta_3^*) \operatorname{diag}(\eta_2^{-1}, \eta_2^*)$.

Applying induction on $n$, we may assume that

$$\begin{pmatrix} 1 & & & \\ & \alpha & & \beta \\ & & 1 & \\ & \gamma & & \delta \end{pmatrix} = \psi_3' \lambda_1' \theta' \psi_4',$$

where $\psi_3'$, $\psi_4'$ are in $B^+$, $\lambda_1'$ is in $B^-$, and $\theta' \in I_{2(n-k)} \oplus O_{2k}R$, where $k = \operatorname{asr}(R)+1$.

Writing $\psi_3'$ as

$$\begin{pmatrix} 1 & & & \\ & I_{n-1} & & \chi_1 \\ & & 1 & \\ & & & I_{n-1} \end{pmatrix} \begin{pmatrix} 1 & & & \\ & \eta_5 & & \\ & & 1 & \\ & & & \eta_5^{*^{-1}} \end{pmatrix}$$

and applying Lemma 8, we can write $\theta$ as $\begin{pmatrix} \eta & 0 \\ 0 & \eta^{*^{-1}} \end{pmatrix} \psi_5 \lambda_3 \theta_1 \psi_6$, where $\psi_5 = \begin{pmatrix} I & \zeta_1 \\ 0 & I \end{pmatrix}$, $\lambda_3 = \begin{pmatrix} I & 0 \\ \zeta_2 & I \end{pmatrix}$ and $\psi_6$ is in $B^+$.

By decomposing $\eta$ as $\psi_7\lambda_4\eta_6\psi_8$, where the $\psi_i(\lambda_i)$ are upper (lower) triangular matrices, $\eta_6 \in I_{n-k} \oplus GL_k R$, and rearranging these matrices, we obtain $\theta = \psi_1\lambda\theta_2\psi_2$, where $\psi_i$ are in $B^+$, $\lambda$ is in $B^-$, and $\theta_2 \in I_{2(n-k)} \oplus O_{2k}R$ with $k = \operatorname{asr}(R) + 1$.

In the case where $R$ is commutative and $*$ is the trivial involution, the result can be improved by [5, p. 539].

(ii) Follows easily from Lemma 9 of [4] and (i).

REMARKS. (i) If necessary, we can make $\psi_1 \in U^+$ and $\lambda_1 \in U^-$ by including the diagonal entries in $\psi_2$,

(ii) If $R$ is commutative and $\operatorname{asr}(R) \leq 1$, we can write $\theta \in O_{2n}R$ as $\theta_1\psi$, where $\theta_1 \in O_2R$ and $\psi \in EO_{2n}R$.

PROOF OF COROLLARY 3. (i) By Proposition 2, it suffices to show the result for $EO_4R$. From [7], we know that any matrix in $E_2R$ can be factored as a product of an upper triangular matrix, a lower triangular matrix, and an upper triangular matrix. The exact sequence

$$1 \to \left\{\left(\begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}, \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}\right) : z^2 = 1\right\} \to E_2R \times E_2R \to EO_4R \to 1$$

allows us to obtain the conclusion.

(ii) Follows easily from Lemma 9 of [4] and (i).

## 3. Proof of the main results

To prove Theorems 4 and 5, we need the following three lemmas.

LEMMA 9. *Let $R$ be a commutative ring with* 1 *and $n \geq 1$. Suppose that $\theta$ is in $U^+$ and $\pi = \begin{pmatrix} 0 & \epsilon^* \\ I_{2n-1} & 0 \end{pmatrix} \in O_{2n}R$. Then there is a $\kappa$ in $U^+$ such that $\kappa^{-1}\theta\pi\kappa$ is of the form*

$$Uc(a_1, \dots, a_n) = \begin{pmatrix} 0 & & & & & & & \epsilon^* \\ 1 & 0 & & & & & & a_1 \\ & \ddots & \ddots & & & & & a_2 \\ & & 1 & 0 & & & & \vdots \\ & & & 1 & -\epsilon^* a_1^* & -\epsilon^* a_2^* & \cdots & a_n \\ & & & & 1 & 0 & & \\ & & & & & \ddots & \ddots & \\ & & & & & & 1 & 0 \end{pmatrix} \quad (a_n^* \in \Lambda).$$

https://doi.org/10.1017/S1446788700037253 Published online by Cambridge University Press

PROOF. Write $\theta$ in the form $\begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{*^{-1}} \end{pmatrix}$.

Let $\{e_1, \dots, e_n\}$ be the standard basis of $R^n$ and let $\eta = (e_1, \omega e_1, \dots, \omega^{n-1} e_1)$, where $\omega = \alpha\mu$, $\mu = \begin{pmatrix} & 0 \\ I_{n-1} & \end{pmatrix}$.

Then the matrix $\begin{pmatrix} \eta & 0 \\ 0 & \eta^{*^{-1}} \end{pmatrix}$ is in $U^+$ and $\begin{pmatrix} \eta^{-1} & 0 \\ 0 & \eta^* \end{pmatrix} \theta\pi \begin{pmatrix} \eta & 0 \\ 0 & \eta^{*^{-1}} \end{pmatrix} = \theta_1$ has the form

$$\theta_1 = \begin{pmatrix} 0 & & & c_{11} & c_{12} & \cdots & c_{1n} & \epsilon^* \\ 1 & \ddots & & c_{21} & c_{22} & \cdots & c_{2n} & 0 \\ & \ddots & 0 & \vdots & \vdots & & \vdots & \vdots \\ & & 1 & c_{n1} & c_{n2} & \cdots & c_{nn} & 0 \\ & & & 1 & 0 & 0 & \cdots & 0 \\ & & & & 1 & 0 & & \\ & & & & & \ddots & \ddots & \\ & & & & & & 1 & 0 \end{pmatrix},$$

where the matrix

$$\gamma = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{pmatrix} \in \Lambda_n^*.$$

Then we can write $\theta_1$ as $\begin{pmatrix} I & \gamma \\ 0 & I \end{pmatrix} \pi$ which is similar to $\pi \begin{pmatrix} I & \gamma \\ 0 & I \end{pmatrix} = \theta'$, where

$$\theta' = \begin{pmatrix} 0 & & & & 0 & \cdots & 0 & \epsilon^* \\ 1 & 0 & & & c_{11} & c_{12} & \cdots & c_{1n} \\ & \ddots & \ddots & & c_{21} & c_{22} & \cdots & c_{2n} \\ & & 1 & 0 & \vdots & \vdots & & \vdots \\ & & & 1 & c_{n1} & c_{n2} & \cdots & c_{nn} \\ & & & & 1 & 0 & & \\ & & & & & \ddots & \ddots & \\ & & & & & & 1 & 0 \end{pmatrix}.$$

Let

$$\nu = \prod_{i<\sigma j, 2\le i\le n-1, n+3\le j\le 2n} \rho_{ij} \left( \sum_{k=1}^{i-1} c_{k,k+\sigma j-i} \right) \prod_{\sigma j=2}^{n} \rho_{\sigma j, j} \left( \sum_{l=2}^{\sigma j} c_{l-1,l-1} \right).$$

https://doi.org/10.1017/S1446788700037253 Published online by Cambridge University Press

Then $\nu^{-1}\theta'\nu = Uc(a_1, \dots, a_n)$, and $\kappa = \begin{pmatrix} \eta & 0 \\ 0 & \eta^{*^{-1}} \end{pmatrix} \begin{pmatrix} I & \gamma \\ 0 & I \end{pmatrix} \nu$ is in $U^+$.

LEMMA 10. *Let $R$ be a commutative ring with $1$, $*$ the trivial involution, and $\zeta\Lambda = \Lambda$ for any unit $\zeta \in R$. Suppose that $\theta \in O_{2n}R$ for $n \geq 1$ is in $B^+$ with diagonal entries $d_1, \dots, d_n, d_1^{-1}, \dots, d_n^{-1} \in GL_1R$, $\pi = \begin{pmatrix} 0 & \epsilon \\ I_{2n-1} & 0 \end{pmatrix} \in O_{2n}R$. Then there is a matrix $\mu = \begin{pmatrix} I & 0 \\ 0 & zI \end{pmatrix} \psi \in GO_{2n}R$, where $\psi$ is in $B^+$, such that $\mu^{-1}\theta\pi\mu = Uc(a_1, \dots, a_n)$.*

PROOF. $\theta\pi$ has the form

$$\theta\pi = \begin{pmatrix}
* & & \cdots & * & * & & * & \epsilon d_1 \\
d_2 & * & & & & & * & 0 \\
0 & \ddots & \ddots & \vdots & \vdots & & \vdots & \vdots \\
 & \ddots & d_n & * & * & \cdots & * & 0 \\
 & & 0 & d_1^{-1} & 0 & \cdots & 0 & 0 \\
 & & 0 & * & d_2^{-1} & 0 & & \\
 & & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
 & & 0 & * & \cdots & * & d_n^{-1} & 0
\end{pmatrix}.$$

Let $\beta = \operatorname{diag}(d_1, d_1d_2, \dots, d_1d_2\cdots d_n, d_1^{-1}, d_2^{-1}d_1^{-1}, \dots, d_n^{-1}\cdots d_2^{-1}d_1^{-1})$, and take $z = d_1 \cdots d_n$. Then by Lemma 9, $\mu^{-1}\beta^{-1}\theta\pi\beta\mu$ is similar to the matrix $Uc(a_1, \dots, a_n)$, where $\mu = \begin{pmatrix} I & 0 \\ 0 & zI \end{pmatrix}$.

LEMMA 11. *Let $R$ be a commutative ring with $1$. Then*

(i) *$Uc(b_1, \dots, b_n)^{-1}Uc(a_1, \dots, a_n) = \prod_{i=1}^{n} \rho_{i,2n}(a_i - b_i)$ where $a_n, b_n \in \Lambda^*$,*

(ii) *When $n \geq 3$, then $\prod_{i=1}^{n-1} \rho_{i,2n}(a_i)\rho_{n,2n}(a_n)$, where $a_n \in \Lambda^*$, can be written as a product of two commutators, and when $a_n = 0$, it is a commutator,*

(iii) *When $n = 2$ and $1$ is the sum of two units in $R$, then $\prod_{i=1}^{n-1} \rho_{i,2n}(a_i)\rho_{n,2n}(a_n)$, where $a_n \in \Lambda^*$, can be written as a product of two commutators,*

(iv) *For any $\alpha \in O_{2n}R$, $\alpha^{-1}$ is similar to $\alpha^*$.*

PROOF. (i) is a direct calculation.

(ii) By the identity $\rho_{n,2n}(a_n) = \rho_{n,n+1}(-a_n)[\rho_{n1}(1), \rho_{1,n+1}(a_n)]$, we can show

$$\prod_{i=1}^{n-1} \rho_{i,2n}(a_i)\rho_{n,2n}(a_n) = \prod_{i=1}^{n-1} \rho_{i,2n}(a_i)\rho_{1,2n}(\epsilon^* a_n^*)c,$$

where $c$ is a commutator. But $\rho_{1,2n}(a_1+\epsilon^* a_n^*)\prod_{i=2}^{n-1}\rho_{i,2n}(a_i)$ is similar to $\begin{pmatrix}\eta & 0\\ 0 & \eta^{*^{-1}}\end{pmatrix}$ where $\eta=\begin{pmatrix} I & v\\ 0 & 1\end{pmatrix}$, and $v=(a_1+\epsilon^* a_n^*, a_2, \dots, a_{n-1})^t$. When $n \geq 3$, we can find an invertible matrix $\kappa_{n-1}\in E_{n-1}R$ such that $\kappa_{n-1}-I\in E_{n-1}R$ (see [7]). So

$$\eta=\left[\begin{pmatrix}\kappa_{n-1} & 0\\ 0 & 1\end{pmatrix},\begin{pmatrix} I & u\\ 0 & 1\end{pmatrix}\right],$$

where $u=(\kappa_{n-1}-I)^{-1}v$. Then $\rho_{1,2n}(a_1+\epsilon^* a_n^*)\prod_{i=2}^{n-1}\rho_{i,2n}(a_i)$ is a commutator.

(iii) Proceed as in (ii). It suffices to show that $\eta=\begin{pmatrix}1 & (a_1+\epsilon^* a_2^*)\\ 0 & 1\end{pmatrix}$ is a commutator. Write $1=u_1+u_2$, $b=a_1+\epsilon^* a_2^*$. Then as in [2],

$$\eta=\left[\begin{pmatrix}u_1 & 0\\ 0 & 1\end{pmatrix},\begin{pmatrix}1 & -u_2^{-1}b\\ 0 & 1\end{pmatrix}\right].$$

(iv) This follows directly from the observation that $\alpha^{-1}=\phi_n\alpha^*\phi_n^{-1}$, where $\phi_n=\begin{pmatrix}0 & I_n\\ \epsilon I_n & 0\end{pmatrix}$.

Proof of Theorem 4. (i) By Proposition 1, in this case every matrix $\theta\in EO_{2n}R=[EO_{2n}R, EO_{2n}R]=[O_{2n}R, O_{2n}R]$ can be written as $\psi_1\lambda_1\psi_2\lambda_2$ where $\psi_i$ are in $U^+$ and $\lambda_i$ are in $U^-$. Moreover,

$$\theta=\psi_1\lambda_1\psi_2\lambda_2=\psi_3c_1\lambda_3=c_2\psi_3\lambda_3=c_2(\psi_3\pi)(\pi^{-1}\lambda_3)=c_2\psi\lambda$$

where $c_1=\psi_2^{-1}\lambda_1\psi_2\lambda_1^{-1}$, $\psi=\psi_3\pi$, $\lambda=\pi^{-1}\lambda_3$, and $\pi$ is defined as before.

By Lemmas 9 and 11, we have $\tau^{-1}\psi\tau=Uc(a_1,\dots,a_n)$, $\omega^{-1}\lambda^{-1}\omega=Uc(b_1,\dots,b_n)$ for some $\tau,\omega\in EO_{2n}R$, where $a_n, b_n\in\Lambda^*$. By Lemma 11, there is some $\zeta=\prod_{i=1}^{n}\rho_{i,2n}(a_i-b_i)$ such that $\tau^{-1}\psi\tau=\omega^{-1}\lambda^{-1}\omega\zeta$, so

$$\psi=\tau\omega^{-1}\lambda^{-1}\omega\zeta\tau^{-1}$$

and

$$\psi\lambda=\tau\omega^{-1}\lambda^{-1}\omega\zeta\tau^{-1}\lambda=\tau\zeta\tau^{-1}\tau\zeta^{-1}\omega^{-1}\lambda^{-1}\omega\zeta\tau^{-1}\lambda=\tau\zeta\tau^{-1}\left[\tau\zeta^{-1}\omega^{-1},\lambda^{-1}\right].$$

Since $\zeta$ can be written as a product of two commutators by Lemma 11, we see that $\theta=c_2c_3c_4c_5$. So $c(EO_{2n}R)\le 4$ and $c(O_{2n}R)\le 4$.

(ii) By Proposition 1, in this case every $\theta\in EO_{2n}R=[GO_{2n}R, GO_{2n}R]$ is similar to the product $\psi_1\lambda_1$, where $\psi_1$ is in $B^+$ and $\lambda_1$ is in $U^-$. Then $\psi_1\lambda_1=(\psi_1\pi)(\pi^{-1}\lambda_1)=\psi\lambda$, where $\pi$ is defined as before. Then by Lemma 10, there exists

$\tau \in GO_{2n}R$ such that $\tau\psi\tau^{-1} = Uc(a_1, \dots, a_n)$, and there exists $\omega \in O_{2n}R$ such that $\omega\lambda\omega^{-1} = Uc(b_1, \dots, b_n)$.

Continuing as in the proof of part (i), we obtain $\theta = c_1c_2c_3$, where the $c_i$ are commutators. Hence $c(GO_{2n}R) \leq 3$.

PROPOSITION 12. *Let $R$ be a commutative ring with* 1 *and* $n \geq \max\{\mathrm{asr}(R) + 1, 3\}$. *Then*

(i) $c(O_{2n}R) \leq 4 + c(O_{2k}R)$, *where* $k = \mathrm{asr}(R)$,

(ii) *when* $*$ *is the trivial involution and* $\Lambda = R^{\epsilon}$, *then* $c(GO_{2n}R) \leq 3 + c(GO_{2k}R)$, *where* $k = \mathrm{asr}(R)$.

PROOF. Similar to the proof of Theorem 4 after Proposition 2 is applied to the decomposition of $\theta \in O_{2n}R$.

PROOF OF THEOREM 5. (i) Note that $EO_{2n}R = [EO_{2n}R, EO_{2n}R] = [O_{2n}R, O_{2n}R] = [GO_{2n}R, GO_{2n}R]$ for $n \geq 2$ in this case (see [6] and Remark (ii)). Applying Corollary 3 to the decomposition of $\theta \in EO_{2n}R$, we can write $\theta$ as $\psi_1\lambda_1\psi_2\lambda_2$, where $\psi_i$ are in $U^+$ and $\lambda_i$ are in $U^-$. In this case, $\Lambda = 0$, hence $a_n = 0$ in the companion matrix $Uc(a_1, \dots, a_n)$. Then by Lemma 11, $\zeta$ in the proof of Theorem 4 is a commutator. Thus we have $\theta = c_1c_2c_3$ where the $c_i$ are commutators. So $c(EO_{2n}R) \leq 3$ and $c(O_{2n}R) \leq 3$.

(ii) Similar to the proof of (i) and Theorem 4(ii).

## References


[1] A. Bak, *K-theory of forms* (Princeton University Press, Princeton, 1981).

[2] H. Bass, '*K*-theory and stable algebra', *Publ. Math. IHES* **22** (1964), 5–60.

[3] ———, 'Unitary algebraic *K*-theory', in: *Lecture Notes in Math.*, 343 (Springer, Berlin, 1973) pp. 57–265.

[4] R. K. Dennis and L. N. Vaserstein, 'On a question of M. Newman on the number of commutators', *J. Algebra* **118** (1988), 150–161.

[5] B. A. Magurn, W. van der Kallen and L. N. Vaserstein, 'Absolute stable rank and Witt cancellation for noncommutative rings', *Inv. Math.* **91** (1988), 525–542.

[6] L. N. Vaserstein, 'On normal subgroups of Chevalley groups over commutative rings', *Tohôku Math. J.* **38** (1986), 219–230.

[7] L. N. Vaserstein and E. Wheland, 'Commutators and companion matrices over rings of stable rank 1', *Linear Algebra and its Applications* **142** (1990), 263–277.

[8] H. You, 'Commutators and unipotents in symplectic groups', *Acta Math. Sinica* **10** (1994), 173–179.

Department of Mathematics
Youngstown State University
Youngstown, Ohio 44455
USA

Department of Mathematics
The Pennsylvania State University
University Park, Pennsylvania 16802
USA
e-mail: vstein@math.psu.edu

Department of Mathematics
Northeast Normal University
Changchun 130024
People's Republic of China