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ON OPTIMAL TERMINAL WEALTH PROBLEMS WITH RANDOM TRADING TIMES AND DRAWDOWN CONSTRAINTS

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Abstract

We consider an investment problem where observing and trading are only possible at random times. In addition, we introduce drawdown constraints which require that the investor's wealth does not fall under a prior fixed percentage of its running maximum. The financial market consists of a riskless bond and a stock which is driven by a Lévy process. Moreover, a general utility function is assumed. In this setting we solve the investment problem using a related limsup Markov decision process. We show that the value function can be characterized as the unique fixed point of the Bellman equation and verify the existence of an optimal stationary policy. Under some mild assumptions the value function can be approximated by the value function of a contracting Markov decision process. We are able to use Howard's policy improvement algorithm for computing the value function as well as an optimal policy. These results are illustrated in a numerical example.

Keywords: Portfolio optimization; illiquid market; random trading time; drawdown constraint; limsup Markov decision process; Howard's policy improvement algorithm; Lévy process

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1. Introduction

We consider a terminal wealth problem with a finite horizon T and an underlying financial market in which the assets are driven by Lévy processes. Moreover, we take the liquidity risks of the financial assets into our considerations and therefore assume that investors cannot observe and trade the assets at any time. An example for such an illiquid market is an over-the-counter (OTC) market in which missing counter parties yield high-liquidity risks. In the literature, there are several approaches to take liquidity risks into account. In this paper, we are interested in the approach of [7], [11], [12], and [13] in which observing and trading are only possible at discrete random times. More precisely, we assume that these random times are given by the jump times of an inhomogeneous Poisson process whose intensity process rises to infinity when tending to the finite horizon T. Furthermore, we introduce drawdown constraints which require that the investor's wealth does not fall under a prior fixed percentage of its running maximum. Portfolio problems with such drawdown constraints are, among others, discussed

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in [4], [5], and [6]. One essential result of this paper is the reduction of the terminal wealth problem to a limsup Markov decision process (MDP). Such an MDP can be solved by applying a *structure theorem* which is introduced and proven in [14] (see also Appendix A). Then we are able to show the existence of an optimal stationary policy and, moreover, the value function can be characterized as the unique fixed point of the Bellman equation. If we consider a terminal wealth problem with a shortened horizon, the number of random observations and trading times is finite. This crucial property enables us to solve the terminal wealth problem by a contracting MDP. Howard's policy improvement algorithm can be used to compute an optimal stationary policy as well as the value function. It turns out that the solution of the limsup MDP is close to the solution of a contracting MDP. Under a mild assumption we approximate the value function of the terminal wealth problem by the value function of the contracting MDP. Such an approximation is also valid for optimal policies. These results are illustrated in a numerical example and are compared with the solution of the classical Merton problem.

The outline of this paper is as follows. Section 2 is concerned with the introduction of the terminal wealth problem. In Section 3 the terminal wealth problem is solved by a limsup MDP. Section 4 presents the solution of the terminal wealth problem with a finite number of random observations and trading times. The approximation results are presented in Section 5. In the last section, we illustrate the results in a numerical example and compare them with the solution of the classical Merton problem.

2. Financial market and terminal wealth problem

Let us fix the investor's finite horizon at $T > 0$ and a complete stochastic basis $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P})$ in the sense of [8]. All stochastic processes are defined on that complete stochastic basis.

In the following we consider a financial market consisting of a stock, a bond, and exogenous random times. The stock price $S = (S_t)_{0\le t\le T}$ is given by $S_t = \mathcal{E}(L)_t$ where $\mathcal{E}$ is the stochastic exponential operator and $L = (L_t)_{0\le t\le T}$ is an adapted Lévy process with characteristics (b, c, F) satisfying $c > 0$, $F\big((-\infty, -1]\big) = 0$, and

$$\int_{|x|\ge 1} e^x F(\mathrm{d}x) < \infty.$$

Moreover, we suppose that the price of the riskless bond $B = (B_t)_{0\le t\le T}$ is given by $B_t \equiv 1$ and that the exogenous random times $\tau = (\tau_n)_{n\in\mathbb{N}}$ are described by the jump times of an adapted inhomogeneous Poisson process $N = (N_t)_{0\le t<T}$. This process is assumed to be independent of the stock price S and to have a deterministic intensity process $\lambda = (\lambda_t)_{0\le t<T}$ which satisfies $\lambda\colon [0, T) \to (0, \infty)$, λ is bounded on $[0, t]$ for each $t \in [0, T)$, and

$$\int_0^T \lambda_t \,\mathrm{d}t = \infty.$$

Remark 1. Since an exponential of a Lévy process can be represented by the stochastic exponential of another Lévy process (cf. [1, Section 5.1]), the exponential-Lévy model is included in the introduced financial market.

Remark 2. Let $\hat{N}$ be a homogeneous Poisson process with intensity 1 and $\lambda = (\lambda_t)_{0\le t<T}$ be an intensity process as above. Then $\hat{N}_{\int_0^t \lambda_s \,\mathrm{d}s}$ is an inhomogeneous Poisson process with intensity process λ (cf. [9, Section 8.31]) and the exogenous random times $\tau = (\tau_n)_{n\in\mathbb{N}}$ converge to the finite horizon T.

On the given financial market we consider an investor who starts to invest his initial capital $x_0 > 0$ at time $\tau_0 = 0$. We assume that he observes and trades his assets only at the exogenous random times $\tau = (\tau_n)_{n\in\mathbb{N}}$. Hence, the investor's information is given by the filtration $\mathbb{G} = (\mathcal{G}_n)_{n\in\mathbb{N}_0}$ with

$$\mathcal{G}_0 = \{\varnothing, \Omega\} \quad \text{and} \quad \mathcal{G}_n = \sigma\{(\tau_k, S_{\tau_k})\colon 1 \le k \le n\}, \qquad n \ge 1.$$

A policy is defined as a real-valued $\mathbb{G}$-adapted process

$$\pi = (a_n)_{n\in\mathbb{N}_0},$$

where a_n denotes the amount invested in the stock over the period $(\tau_n, \tau_{n+1}]$ after observing the stock price at time τ_n. Now let

$$\tilde{\pi}_t := \sum_{n=0}^{\infty} \frac{a_n}{S_{\tau_n}} \mathbf{1}_{\{\tau_n < t \le \tau_{n+1}\}}, \qquad 0 \le t < T,$$

be the number of stocks which the investor owns at time t. Since we assume a self-financing portfolio, the investor's wealth process $X^\pi = (X_t^\pi)_{0\le t<T}$, with respect to the policy π, is given by

$$X_t^\pi = x_0 + \int_0^t \tilde{\pi}_s \, \mathrm{d}S_s.$$

Remark 3. Since there exists an equivalent local martingale measure $\mathbb{Q}$ of the Lévy process L it follows that the wealth process X^π is a $\mathbb{Q}$-supermartingale. Hence, $\lim_{t\to T} X_t^\pi$ exists $\mathbb{P}$-almost surely and $\lim_{t\to T} X_t^\pi = \lim_{n\to\infty} X_{\tau_n}^\pi$. Since the asset prices do not jump at time T almost surely, we may define $X_T^\pi := \lim_{n\to\infty} X_{\tau_n}^\pi$.

For $0 \le s < t \le T$ we introduce the return of the stock by

$$Z_{s,t} := \frac{S_t - S_s}{S_s} = S_{t-s} - 1,$$

and denote the distribution of $Z_{s,t}$ by $p(t - s, \mathrm{d}z)$. Thus, we obtain a more convenient representation of the investor's wealth at the exogenous random times τ which is given by

$$X_{\tau_n}^\pi = x_0 + \sum_{k=0}^{n-1} a_k Z_{\tau_k,\tau_{k+1}}, \qquad n \ge 1.$$

Let $\beta \in [0, 1)$ be the prior fixed parameter determining the percentage of wealth which will be guaranteed by the drawdown constraints. To establish these drawdown constraints we introduce the process $M^\pi = (M_t^\pi)_{0\le t<T}$ which is given by $M_0^\pi = m_0$ and

$$M_t^\pi := \max\{m_0, X_{\tau_1}^\pi, \ldots, X_{\tau_n}^\pi\}, \quad \text{if } \tau_n \le t < \tau_{n+1},$$

where $m_0 \in (0, x_0/\beta)$ is fixed. In the following the process M is called the *running maximum*. A policy $\pi = (a_0, a_1, a_2, \ldots)$ is called *admissible* if the following two conditions hold.

(i) The process $(X_{\tau_n}^\pi)_{n\in\mathbb{N}}$ satisfies $X_{\tau_n}^\pi \ge \beta M_{\tau_n}^\pi$ for all $n \in \mathbb{N}$.

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(ii) Let $E := \{(t, x, m) \colon t \in [0, T), x \in (0, \infty), m \in (0, x/\beta)\}$. For each $n \in \mathbb{N}_0$ there exists a measurable function $f_n \colon E^{n+1} \to \mathbb{R}$ such that

$$a_n = f_n((\tau_0, X^\pi_{\tau_0}, M^\pi_{\tau_0}), \ldots, (\tau_n, X^\pi_{\tau_n}, M^\pi_{\tau_n})).$$

The investor aims to achieve

$$V(y) := \sup_{\pi \in \mathcal{A}(y)} \mathbb{E}_y[U(X^\pi_T)] = \sup_{\pi \in \mathcal{A}(y)} \mathbb{E}_y\Big[\lim_{n\to\infty} U(X^\pi_{\tau_n})\Big], \qquad y = (t, x, m) \in E, \tag{P1}$$

where, as usual, $\mathbb{E}_y[\,\cdot\,]$ denotes the conditional expectation $\mathbb{E}[\,\cdot \mid X_t = x,\ M_t = m]$ and $\mathcal{A}(y)$ is the set of admissible policies in the state $y \in E$.

Moreover, $U \colon (0, \infty) \to \mathbb{R}$ denotes the investor's utility function which is strictly increasing, strictly concave, and continuously differentiable. The Fenchel–Legendre transform of U has domain $(0, \infty)$ and we have the following conditions.

(i) There exist $p \in (0, 1)$ and $C_U > 0$ such that $U^+(x) \le C_U(1 + x^p)$ for all $x > 0$.

(ii) If $U(0) := U(0+) = -\infty$, then there exist $p' < 0$ and $C'_U > 0$ such that $U^-(x) \le C'_U(1 + x^{p'})$ for all $x > 0$.

(iii) Moreover, if $U(0) := U(0+) = -\infty$, then we assume that, for the underlying Lévy process L, there exists $\delta > 0$ and $\xi < p'$ such that

$$\int_{(-1,-1+\delta)} ((1 + y)^\xi - 1 - \xi y) F(\mathrm{d}y) < \infty. \tag{1}$$

These conditions stand for the rest of this paper. In particular, the above assumptions (i) and (ii) are satisfied for power utility functions $U(x) = \alpha^{-1}x^\alpha$ with $\alpha < 1$ and $\alpha \ne 0$, the logarithmic utility function $U(x) = \log(x)$, and exponential utility functions of the form $U(x) = -\alpha^{-1}e^{-\alpha x}$ with $\alpha > 0$.

3. Solution via a limsup MDP

In what follows we introduce a limsup MDP in order to compute the value function V as well as an optimal policy of the terminal wealth problem (P1).

limsup MDP

- The state space $E := \{(t, x, m) \colon t \in [0, T), x \in (0, \infty), m \in (0, x/\beta)\}$ is endowed with the Borel σ-algebra $\mathcal{B}(E)$, where t is the time, x the current wealth, and m the current value of the running maximum. The state process is denoted by $Y_n = (\tau_n, X_{\tau_n}, M_{\tau_n})$.
- The action space $A := [0, \infty)$ is endowed with the Borel σ-algebra $\mathcal{B}(A)$.
- The admissible actions are given by $D(t, x, m) := [0, x - \beta m]$.
- The stochastic transition kernel Q is given by

$$Q(B \mid (y, a)) := \int_t^T \int_{(-1,\infty)} \mathbf{1}_B(u, x + az, \max\{m, x + az\})\lambda_u \times \exp\bigg(-\int_t^u \lambda_s \,\mathrm{d}s\bigg) p(u - t, \mathrm{d}z)\,\mathrm{d}u,$$

where $y = (t, x, m) \in E$, $a \in D(t, x, m)$, and $B \in \mathcal{B}(E)$.

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- The terminal reward function $g\colon E \to \mathbb{R}$ is given by $g(t, x, m) := U(x)$.

As usual, a decision rule is a measurable mapping $f\colon E \to A$ such that we have $f(t, x, m) \in D(t, x, m)$ for all $(t, x, m) \in E$. Moreover, we define a Markovian policy π as a sequence of decision rules, i.e. $\pi := (f_0, f_1, f_2, f_3, \dots)$ where f_k is the decision rule at time τ_k. The set of all Markovian policies is denoted by Π and the gain, with respect to a Markovian policy π, is defined by

$$V_{1,\pi}(y) := \mathbb{E}_y^\pi\Big[\lim_{n\to\infty} U(X_{\tau_n})\Big], \qquad y \in E.$$

Thereby $\mathbb{E}_y^\pi$ denotes the expectation with respect to the probability measure $\mathbb{P}_y^\pi$ induced by a policy $\pi \in \Pi$ and initial state $y \in E$. Furthermore, we define the value function of the limsup MDP by

$$V_1(y) := \sup_{\pi\in\Pi} V_{1,\pi}(y), \qquad y \in E.$$

Theorem 1.

(a) *For $\pi \in \Pi$, the following holds*

$$\mathbb{E}_y\Big[\lim_{n\to\infty} U(X_{\tau_n}^\pi)\Big] = V_{1,\pi}(y), \qquad y \in E.$$

(b) *We have $V = V_1$.*

Proof. (a) Let π be a Markovian policy. Then $\pi \in \mathcal{A}(y)$ and we obviously have

$$\mathbb{E}_y\Big[\lim_{n\to\infty} U(X_{\tau_n}^\pi)\Big] = V_{1,\pi}(y).$$

(b) Now let $\pi \in \mathcal{A}(y)$. Then, there exist measurable functions $f_k\colon E^{k+1} \to \mathbb{R}$ such that

$$a_k = f_k((\tau_0, X_{\tau_0}^\pi, M_{\tau_0}^\pi), \dots, (\tau_k, X_{\tau_k}^\pi, M_{\tau_k}^\pi)), \qquad k \in \mathbb{N}_0.$$

Since the state process of the limsup MDP is Markovian, the maximal expected gain cannot be improved by history-dependent policies. Therefore, we obtain

$$V_1(y) = \sup_{\pi\in\Pi} \mathbb{E}_y\Big[\lim_{n\to\infty} U(X_{\tau_n}^\pi)\Big] = V(y).$$

Due to Theorem 1 it is sufficient to solve the limsup MDP. For that we introduce the function $h\colon [0, T] \times (0, \infty) \to \mathbb{R}$ defined by

$$h(t, x) := \inf_{z\in(0,\infty)} \{\mathbb{E}[\tilde{U}(zY_{t,T})] + xz\},$$

where $Y_{t,T} := \exp(-(b'/\sqrt{c})W_{T-t} - \frac{1}{2}((b')^2/c)(T-t))$, $W = (W_t)_{0\le t\le T}$ is a standard Brownian motion, $b' := b + \int_{\{x>1\}} x F(\mathrm{d}x)$, (b, c, F) are the characteristics of the Lévy process L, and $\tilde{U}$ is the Fenchel–Legendre transform of the utility function U.

Proposition 1.

(a) *Let $b(t, x) := e^{\gamma(T-t)}(1 + x + x^{p'})$ for some $\gamma > 0$ (if $U(0) > -\infty$ then $p' := 0$). Then there exist $C > 0$ and $\gamma > 0$ such that*

$$U(x) \le h(t, x) \le Cb(t, x) \quad \textit{for all } (t, x) \in [0, T] \times (0, \infty).$$

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(b) *We have* $h(t,\cdot)$ *concave on* $(0,\infty)$ *for fixed* t.

(c) *For* $(t,x,m)\in E$, *it holds*

$$\sup_{a\in[0,x-\beta m]}\int_t^T\int_{(-1,\infty)} h(u,x+az)\lambda_u \exp\biggl(-\int_t^u \lambda_s\,\mathrm{d}s\biggr)p(u-t,\mathrm{d}z)\,\mathrm{d}u \le h(t,x).$$

(d) *We have* $\lim_{t\nearrow T,\,x'\to x} h(t,x') = U(x)$.

Proof. We prove only part (a) since the other parts follow from [7, Lemma 3.3]. Using $\mathbb{E}[Y_{t,T}]=1$ and Jensen's inequality we obtain

$$h(t,x)\ge \inf_{z>0}\{\tilde U(z\mathbb{E}[Y_{t,T}])+xz\} = \inf_{z>0}\{\tilde U(z)+xz\} = U(x).$$

For $z>0$ it holds that $\tilde U(z)\le \sup_{x>0}\{C_U(1+x^p)-xz\}\le C'(1+z^{-p/(1-p)})$ for some large $C'>0$. Hence,

$$\mathbb{E}[\tilde U(Y_{t,T})]\le C'\biggl(1+\exp\biggl(\frac{p}{2(1-p)}\biggl(1+\frac{p}{1-p}\biggr)\frac{(b')^2}{c}(T-t)\biggr)\biggr)$$

and so $h(t,x)\le \mathbb{E}[\tilde U(Y_{t,T})]+x\le 2C'e^{\gamma(T-t)}(1+x+x^{p'})$ for γ large enough.

Now we introduce the important set of functions

$$\mathbb{B}_b(E) := \{v\colon E\to\mathbb{R}\mid v \text{ is measurable and there exists } C=C(v)>0 \text{ such that } |v(t,x,m)|\le Cb(t,x) \text{ for all } (t,x,m)\in E\}.$$

Moreover, we define the following operators for $v\in\mathbb{B}_b(E)$:

$$L_1 v(t,x,m,a) := \int_t^T\int_{(-1,\infty)} v(u,x+az,\max\{m,x+az\})\lambda_u \times\exp\biggl(-\int_t^u\lambda_s\,\mathrm{d}s\biggr)p(u-t,\mathrm{d}z)\,\mathrm{d}u;$$

$$(\mathcal{T}_1 v)(t,x,m) := \sup_{a\in[0,x-\beta m]} L_1 v(t,x,m,a),\qquad (t,x,m)\in E.$$

The next theorem contains the main results of this section. The value function is an element of the set $\mathbb{M}_1\subset\mathbb{B}_b(E)$. For each $v\in\mathbb{M}_1$, the following hold:

(1) $U(x)\le v(t,x,m)\le h(t,x)$, $(t,x,m)\in E$;

(2) $v(t,\cdot,\cdot)$ is concave on $\mathcal{D}:=\{(x,m)\colon x\in(0,\infty), m\in(0,x/\beta)\}$ for fixed t;

(3) $v(t,x,\cdot)$ is decreasing on $(0,x/\beta)$ for fixed (t,x);

(4) $v(t,\cdot,m)$ is increasing on $(\beta m,\infty)$ for fixed (t,m);

(5) the function $s\to v(t,x+s,m+s)$ is increasing on $[0,\infty)$ for fixed $(t,x,m)\in E$.

Theorem 2.

(a) *We have* $V=V_1\in\mathbb{M}_1$ *and* V_1 *is the unique fixed point of* $\mathcal{T}_1$ *in* $\mathbb{M}_1$ *which, for all* $\pi\in\Pi$ *and* $y\in E$, *satisfies:*

(i) $\lim_{n\to\infty} V_1(Y_n) = \lim_{n\to\infty} U(X_{\tau_n})$, $\mathbb{P}_y^\pi$*-almost surely;*

(ii) $(V_1(Y_n))_{n\geq 0}$ *is uniformly* $\mathbb{P}_y^\pi$*-integrable.*

(b) *We have* $V_1 = \lim_{n\to\infty} \mathcal{T}_1^n g$ *for all* $g \in \mathbb{M}_1$.

(c) *There exists a maximizer* f^* *of* V_1 *and each maximizer of* V_1 *defines an optimal stationary policy* $(f^*, f^*, f^*, \dots)$ *for (P1).*

Proof. The proof will apply the *structure theorem* for limsup MDPs (see Theorem 7 of Appendix A). Therefore, we have to check the conditions (i)–(iv) of Theorem 7.

(i) This is obvious.

(iii) Let $v \in \mathbb{M}_1$ and $(c_n)_{n\in\mathbb{N}}$ be a convergent sequence in $[0, x - \beta m]$ with limit c_0. Since $v \leq h \leq b$, we can apply Fatou's lemma and obtain

$$\limsup_{n\to\infty} L_1 v(t, x, m, c_n) \leq L_1 v(t, x, m, c_0).$$

Therefore, it follows that $L_1 v$ is an upper semicontinuous function on $[0, x - \beta m]$ for fixed $(t, x, m) \in E$. Hence, we can find a decision rule f^* which is a maximizer of v.

(ii) Let $v \in \mathbb{M}_1$ and $(t, x, m) \in E$. Here $\mathcal{T}_1 v$ is measurable on E, since $\mathcal{T}_1 v(t, x, m) = L_1 v(t, x, m, f^*(t, x, m))$ and $L_1 v$ is a measurable function. From Proposition 1 it follows that $\mathcal{T}_1 v(t, x, m) \leq h(t, x)$ and $\mathcal{T}_1 v(t, x, m) \geq U(x)$. Let us now fix $t \in [0, T)$ and define the convex set

$$G := \{(x, m, a)\colon x \in (0, \infty), m \in (0, x/\beta), a \in [0, x - \beta m]\}.$$

Using the definition of concavity yields $v(u, x + az, \max\{m, x + az\})$ is a concave function on G. Hence, $L_1 v$ is concave on G and, consequently, $\mathcal{T}_1 v(t, \cdot, \cdot)$ is a concave function on $\mathcal{D}$. By standard arguments we can show the remaining properties of $\mathcal{T}_1 v$.

(iv) *Step 1.* Let $(v_m)_{m\in\mathbb{N}_0}$ and $(\tilde{v}_m)_{m\in\mathbb{N}_0}$ be recursively defined by

$$v_0 := U, \qquad v_{m+1} := \mathcal{T}_1 v_m \quad \text{for all } m \geq 0,$$
$$\tilde{v}_0 := h, \qquad \tilde{v}_{m+1} := \mathcal{T}_1 \tilde{v}_m \quad \text{for all } m \geq 0.$$

By induction we can easily show that $v_m \leq v_{m+1} \leq h$ and $\tilde{v}_m \geq \tilde{v}_{m+1} \geq U$ for $m \geq 0$ holds.

Step 2. Owing to the monotonicity of v_n and since $v_n \leq h$, $(v_n)_{n\geq 0}$ converges pointwise to a function $v_\infty\colon E \to \mathbb{R}$ and $v_\infty \leq h$. Analogously $(\tilde{v}_n)_{n\geq 0}$ converges pointwise to a function $\tilde{v}_\infty\colon E \to \mathbb{R}$ and $\tilde{v}_\infty \geq U$.

Step 3. The function $v_\infty = \lim_{n\to\infty} v_n$ is measurable and $U \leq v_\infty \leq h$. Moreover, $v_\infty(t, \cdot, \cdot)$ is concave on $\mathcal{D}$ and, for $m \leq m'$,

$$v_\infty(t, x, m) = \lim_{n\to\infty} v_n(t, x, m) \geq \lim_{n\to\infty} v_n(t, x, m') = v_\infty(t, x, m'),$$

i.e. v_∞ is decreasing in m. By the same arguments the remaining properties of v_∞ can be shown. Thus, $v_\infty \in \mathbb{M}_1$. Analogously it follows that $\tilde{v}_\infty \in \mathbb{M}_1$.

Step 4. By monotone convergence, v_∞ and $\tilde{v}_\infty$ are fixed points of $\mathcal{T}_1$.

Step 5. The inequalities $U(x) \le v_\infty(t,x,m) \le h(t,x)$ imply that $\lim_{n\to\infty} v_\infty(Y_n) = \lim_{n\to\infty} g(Y_n)$ $\mathbb{P}_y^\pi$-almost surely. The same holds for $\lim_{n\to\infty} \tilde{v}_\infty(Y_n)$.

Step 6. Let $y = (t,x,m) \in E$ and $\pi \in \mathcal{A}(y)$. The stochastic logarithm yields

$$X_t^\pi = x_0 + \int_0^t \tilde{\pi}_s \, \mathrm{d}S_s = x_0 + \int_0^t \pi_s \frac{X_{s-}^\pi}{S_{s-}} \, \mathrm{d}S_s = x_0 + \int_0^t \pi_s X_{s-}^\pi \, \mathrm{d}L_s \quad \text{for all } t \in [0,T),$$

where $\pi_s := \tilde{\pi} S_{s-}/X_{s-}$ is an adapted càglàd process with values in $[0,1]$. Applying Itô's formula, Gronwall's inequality, and Fatou's lemma, as in [14, Proof of Proposition 2.14], yields $\mathbb{E}[(X_\tau^\pi)^2] \le x^2 C_2 < \infty$ for a stopping time τ with values in $[0,T)$. Hence, $\mathbb{E}_y^\pi[(X_{\tau_n})^2] \le x^2 C_2 < \infty$ for $n \ge 0$ and $\{(X_{\tau_n})_{n\ge0}\}$ is uniformly $\mathbb{P}_y^\pi$-integrable. Moreover, it follows that $h^+(\tau_n, X_{\tau_n})$ is uniformly $\mathbb{P}_y^\pi$-integrable, since by [7, Lemma 3.3] $0 \le h^+(\tau_n, X_{\tau_n}) \le C(1 + X_{\tau_n})$. If $U(0) > -\infty$, then $\{U^-(X_{\tau_n})\}$ is bounded and so uniformly $\mathbb{P}_y^\pi$-integrable. On the other hand, if $U(0) = -\infty$, then, by using the assumption (1), $\mathbb{E}[(X_\tau^\pi)^\xi] \le x^\xi C_\xi < \infty$. It follows that $\{(X_{\tau_n}^{p'})_{n\ge0}\}$ is uniformly $\mathbb{P}_y^\pi$-integrable and, since $0 \le U^-(X_{\tau_n}) \le C'_U(1 + X_{\tau_n}^{p'})$, it also follows that $U^-(X_{\tau_n})$ is uniformly $\mathbb{P}_y^\pi$-integrable. Owing to

$$0 \le |v_\infty| \le v_\infty^+ + v_\infty^- \le h^+ + U^- \quad \text{and} \quad 0 \le |\tilde{v}_\infty| \le \tilde{v}_\infty^+ + \tilde{v}_\infty^- \le h^+ + U^-,$$

$v_\infty(Y_n)$ and $\tilde{v}_\infty(Y_n)$ are uniformly $\mathbb{P}_y^\pi$-integrable.

Since $\mathcal{T}_1^n U \le \mathcal{T}_1^n g \le \mathcal{T}_1^n h$ and $g \in \mathbb{M}_1$, the statements of Theorem 2 follow from the *structure theorem* (see Appendix A).

In the case of a constant relative risk aversion (CRRA) utility function we obtain the following representation of the value function.

Proposition 2. *In the case of a power utility function (i.e.* $U(x) = \alpha^{-1}x^\alpha$, $\alpha < 1, \alpha \ne 0$*), there exists a function* $F\colon [0,T] \times (0, \beta^{-1}) \to \mathbb{R}$ *such that* $V_1(t,x,m) = U(x)F(t, m/x)$ *for* $(t,x,m) \in E$.

Proof. It can be shown by induction that $\mathcal{T}_1^n U(t,x,m) = U(x)F_n(t, m/x)$ for some function $F_n\colon [0,T] \times (0,\beta^{-1}) \to \mathbb{R}$. Since $\mathcal{T}_1^n U$ converges to V_1, it follows that F_n converges to a function F such that

$$V_1(t,x,m) = U(x)F(t, m/x), \qquad (t,x,m) \in E.$$

Proposition 3. *In the case of a logarithmic utility function (i.e.* $U(x) = \log(x)$*), there exists a function* $F\colon [0,T] \times (0,\beta^{-1}) \to \mathbb{R}$ *such that* $V_1(t,x,m) = U(x) + F(t, m/x)$ *for* $(t,x,m) \in E$.

Note that for general CRRA utility functions each maximizer f^* of V_1 has the form

$$f^*(t,x,m) = \bar{f}\left(t, \frac{m}{x}\right)x.$$

Here $\bar{f}$ indicates the *fraction* of wealth which is invested in the stock.

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4. Terminal wealth problem with a finite number of random trading times

The goal of this section is to present a terminal wealth problem which can be used to approximate the value function as well as an optimal policy of the given optimization problem (P1). We consider the following terminal wealth problem

$$\tilde{V}(y) := \sup_{\pi \in \mathcal{A}(y)} \mathbb{E}_y[U(X^{\pi}_{\tilde{T}})], \qquad y \in \tilde{E}, \tag{P2}$$

where $0 < \tilde{T} < T$ and $\tilde{E} := \{(t, x, m) : t \in [0, \tilde{T}], x \in (0, \infty), m \in (0, x/\beta)\}$. We *assume* that the investor can observe and trade the assets at time $\tilde{T}$.

This optimization problem (P2) can be solved by the following contracting MDP. A similar contracting MDP was used in [2] to solve terminal wealth problems in pure jump markets.

Contracting MDP

- The state space $\tilde{E}$ is endowed with the Borel σ-algebra $\mathcal{B}(\tilde{E})$. The state process is denoted by $Y_n = (\tau_n, X_{\tau_n}, M_{\tau_n})$ and there is a cemetery state (i.e. a state which will never be left once it is reached and where we obtain no reward) $\Delta \notin \tilde{E}$ such that Y_n equals Δ if $\tau_n > \tilde{T}$.
- The action space $A := [0, \infty)$ is endowed with the Borel σ-algebra $\mathcal{B}(A)$.
- The admissible actions are given by $D(t, x, m) := [0, x - \beta m]$, $D(\Delta) = \{0\}$.
- The stochastic transition kernel Q is given by

$$\begin{aligned} Q(B \mid (y, a)) := \int_t^{\tilde{T}} \int_{(-1,\infty)} &\mathbf{1}_B(u, x + az, \max\{m, x + az\})\lambda_u \\ &\times \exp\Big(-\int_t^u \lambda_s \,\mathrm{d}s\Big) p(u - t, \mathrm{d}z)\,\mathrm{d}u, \\ Q(\Delta \mid (y, a)) := 1 - Q(\tilde{E} \mid (y, a)), &\qquad Q(\Delta \mid (\Delta, 0)) = 1, \end{aligned}$$

  where $y = (t, x, m) \in \tilde{E}$, $a \in D(t, x, m)$, and $B \in \mathcal{B}(\tilde{E})$.
- The one-stage reward r is given by

$$r(t, x, m, a) := \exp\Big(-\int_t^{\tilde{T}} \lambda_u \,\mathrm{d}u\Big) \int_{(-1,\infty)} U(x + az) p(\tilde{T} - t, \mathrm{d}z), \qquad r(\Delta, 0) := 0,$$

  where $(t, x, m) \in E$ and $a \in D(t, x, m)$.

The gain with respect to a Markovian policy $\pi = (f_0, f_1, \ldots)$ is defined by

$$V_{2,\pi}(y) := \mathbb{E}^{\pi}_y\Big[\sum_{k=0}^{\infty} r(Y_k, f_k(Y_k))\Big], \qquad y \in \tilde{E},$$

and the value function of the MDP by

$$V_2(y) := \sup_{\pi \in \Pi} V_{2,\pi}(y), \qquad y \in \tilde{E}.$$

The following proposition shows that the MDP is contracting (cf. [3, Section 7.3]).

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Proposition 4. *Let* $b(t,x) := e^{\gamma(T-t)}(1+x+x^{p'})$ *for some* $\gamma > 0$ *(if* $U(0) > -\infty$ *then* $p' := 0$*). Then* $b(t,x)$ *is a bounding function for the contracting MDP, i.e. there exist* $C_r > 0$ *and* $C_\gamma > 0$ *such that, for* $(t,x,m) \in \tilde{E}$ *and* $a \in D(t,x,m)$*, the following holds:*

- $|r(t,x,m,a)| \le C_r b(t,x)$;
- $\int_t^T \int_{(-1,\infty)} b(u, x+az)\lambda_u \exp(-\int_t^u \lambda_s \,\mathrm{d}s) p(t,u,\mathrm{d}z)\,\mathrm{d}u \le C_\gamma b(t,x)$.

Moreover, $C_\gamma < 1$ *for large* γ.

Proof. Let $(t,x,m) \in \tilde{E}$ and $a \in D(t,x,m)$. Since

$$\int_{(-1,\infty)} U^+(x+az)p(\tilde{T}-t,dz) \le \tilde{C}_U(1+x)\int_{(-1,\infty)}\left(1+\frac{a}{1+x}|z|\right)p(\tilde{T}-t,\mathrm{d}z)$$
$$\le \tilde{C}_U b(t,x)$$

for some $\tilde{C}_U > 0$, we obtain

$$|r(t,x,m,a)| \le \tilde{C}_U b(t,x) + \int_{(-1,\infty)} U^-(x+az)p(\tilde{T}-t,\mathrm{d}z).$$

If $U(0) > -\infty$, then $U^-(x)$ is bounded and the claim follows. If $U(0) = -\infty$, then

$$\int_{(-1,\infty)} U^-(x+az)p(\tilde{T}-t,\mathrm{d}z) \le C'_U(1+\mathbb{E}[(x+aZ_{t,\tilde{T}})^{p'}]).$$

Defining the process $Y^s = (Y_t^s)_{0\le t\le \tilde{T}}$ by $Y_t^s := x + a(S_t - S_s)/S_s$ for $s < t$ yields

$$x + aZ_{t,\tilde{T}} = Y_{\tilde{T}}^t = x + \int_t^{\tilde{T}} \pi_u Y_{u-}^t \frac{1}{S_{u-}}\,\mathrm{d}S_u = x + \int_0^{\tilde{T}} \mathbf{1}_{\{u>t\}}\pi_u Y_{u-}^t\,\mathrm{d}L_u,$$

where $\pi_u := aS_{u-}/S_t Y_{u-}^t$ is an adapted càglàd process with values in [0, 1]. As in the proof of Theorem 2 we have $\mathbb{E}[(Y_{\tilde{T}}^t)^{p'}] \le x^{p'} C_{p'}$ for some $C_{p'} > 0$. Hence, there exists $C_r > 0$ such that $|r(t,x,m,a)| \le C_r b(t,x)$.

Let C_λ be an upper bound of the intensity process λ on $[0,\tilde{T}]$. Then we have

$$\begin{aligned}
&\int_t^{\tilde{T}} \int_{(-1,\infty)} b(u,x+az)\lambda_u \exp\left(-\int_t^u \lambda_s\,\mathrm{d}s\right)p(u-t,\mathrm{d}z)\,\mathrm{d}u\\
&\quad\le \int_t^{\tilde{T}} \mathrm{e}^{\gamma(\tilde{T}-u)}(1+x)\int_{(-1,\infty)}\left[1+\frac{a}{1+x}|z|\right]\lambda_u \exp\left(-\int_t^u \lambda_s\,\mathrm{d}s\right)p(u-t,\mathrm{d}z)\,\mathrm{d}u\\
&\qquad + \int_t^{\tilde{T}} \mathrm{e}^{\gamma(\tilde{T}-u)}\int_{(-1,\infty)}(x+az)^{p'}\lambda_u\exp\left(-\int_t^u\lambda_s\,\mathrm{d}s\right)p(u-t,\mathrm{d}z)\,\mathrm{d}u\\
&\quad\le C_\lambda \int_t^{\tilde{T}} \mathrm{e}^{\gamma(\tilde{T}-u)}(1+x)(2+\mathrm{e}^{C(u-t)})\,\mathrm{d}u + x^{p'}C_{p'}C_\lambda\frac{1}{\gamma}\mathrm{e}^{\gamma(\tilde{T}-t)}\\
&\quad\le \left[\frac{3C_\lambda}{\gamma - C} + \frac{C_{p'}C_\lambda}{\gamma}\right]b(t,x) =: C_\gamma b(t,x),
\end{aligned}$$

where $C_{p'} > 0$ and $C > 0$. Hence, b is a bounding function and we have $C_\gamma \in (0,1)$ when γ is chosen large enough.

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Theorem 3.

(a) *For* $\pi \in \Pi$ *it holds*

$$\mathbb{E}_y[U(X^\pi_{\tilde{T}})] = V_{2,\pi}(y), \qquad y \in \tilde{E}.$$

(b) *We have* $\tilde{V} = V_2$.

Proof. (a) Let π be a Markovian policy and $\tilde{\mu}$ be the one point measure in $\tilde{T}$. To avoid heavy notation we define $\tilde{Y}_k := (\tau_k, X^\pi_{\tau_k}, M^\pi_{\tau_k})$. Then $\pi \in \mathcal{A}(y)$ and

$$\mathbb{E}_y[U(X^\pi_{\tilde{T}})] = \sum_{k=0}^{\infty} \mathbb{E}_y\biggl[\int_{[\tau_k,\tau_{k+1})} U(X^\pi_u)\tilde{\mu}(\mathrm{d}u)\biggr],$$

where $\tau_0 = t$ and τ_k is the kth exogenous random time after time t. Moreover, we obtain

$$\begin{aligned}
&\sum_{k=0}^{\infty} \mathbb{E}_y\biggl[\int_{[\tau_k,\tau_{k+1})} U(X^\pi_u)\tilde{\mu}(\mathrm{d}u)\biggr] \\
&\quad = \sum_{k=0}^{\infty} \mathbb{E}_y\biggl[\mathbb{E}_y\biggl[\mathbb{E}_y\biggl[\int_{[\tau_k,\tau_{k+1})} U\biggl(X^\pi_{\tau_k} + f_k(\tilde{Y}_k)\frac{S_u - S_{\tau_k}}{S_{\tau_k}}\biggr)\tilde{\mu}(\mathrm{d}u) \biggm| \mathcal{G}_k \vee \sigma(\tau_{k+1})\biggr] \biggm| \mathcal{G}_k\biggr]\biggr] \\
&\quad = \sum_{k=0}^{\infty} \mathbb{E}_y\biggl[\int_{\tau_k}^{T} \int_{(-1,\infty)} \mathbf{1}_{\{\tau_k \le \tilde{T} < u\}} U(X^\pi_{\tau_k} + f_k(\tilde{Y}_k)z)\lambda_u \\
&\qquad\qquad \times \exp\biggl(-\int_{\tau_k}^{u} \lambda_s \,\mathrm{d}s\biggr) p(\tilde{T} - \tau_k, \mathrm{d}z)\,\mathrm{d}u\biggr] \\
&\quad = \sum_{k=0}^{\infty} \mathbb{E}_y\biggl[\exp\biggl(-\int_{\tau_k}^{\tilde{T}} \lambda_u \,\mathrm{d}u\biggr) \int_{(-1,\infty)} \mathbf{1}_{\{\tau_k \le \tilde{T}\}} U(X^\pi_{\tau_k} + f_k(\tilde{Y}_k)z) p(\tilde{T} - \tau_k, \mathrm{d}z)\biggr] \\
&\quad = \sum_{k=0}^{\infty} \mathbb{E}_y[r(\tilde{Y}_k, f_k(\tilde{Y}_k))] = \sum_{k=0}^{\infty} \mathbb{E}^\pi_y[r(Y_k, f_k(Y_k))] = \mathbb{E}^\pi_y\biggl[\sum_{k=0}^{\infty} r(Y_k, f_k(Y_k))\biggr].
\end{aligned}$$

(b) Let $\pi \in \mathcal{A}(y)$. Using the same arguments as in the proof of Theorem 1(b) yields

$$V_2(y) = \sup_{\pi \in \Pi} V_{2,\pi}(y) = \tilde{V}(y).$$

Consider $\mathbb{B}_b(\tilde{E})$ and let d be the metric on $\mathbb{B}_b(\tilde{E})$ defined by

$$d(v, w) := \sup_{(t,x,m)\in\tilde{E}} \frac{|v(t,x,m) - w(t,x,m)|}{b(t,x)}, \qquad v, w \in \mathbb{B}_b(\tilde{E}).$$

Note that $(\mathbb{B}_b(\tilde{E}), d)$ is a complete metric space. Moreover, we define the following operators

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for $v \in \mathbb{B}_b(\tilde{E})$:

$$\begin{aligned} L_2 v(t,x,m,a) &:= \exp\biggl(-\int_t^{\tilde{T}} \lambda_u \,\mathrm{d}u\biggr) \int_{(-1,\infty)} U(x+az) p(\tilde{T}-t, \mathrm{d}z) \\ &\quad + \int_t^{\tilde{T}} \int_{(-1,\infty)} v(u, x+az, \max\{m, x+az\}) \lambda_u \\ &\qquad \times \exp\biggl(-\int_t^u \lambda_s \,\mathrm{d}s\biggr) p(u-t, \mathrm{d}z) \,\mathrm{d}u; \\ (\mathcal{T}_2 v)(t,x,m) &:= \sup_{a\in[0,x-\beta m]} L_2 v(t,x,m,a), \qquad (t,x,m) \in \tilde{E}. \end{aligned}$$

The value function of the contracting MDP belongs to $\mathbb{M}_2 \subset \mathbb{B}_b(\tilde{E})$. This result should be compared with Theorem 2. For each $v \in \mathbb{M}_2$ the following hold:

(1) $U(x) \le v(t,x,m)$, $(t,x,m) \in \tilde{E}$;

(2) $v(t,\cdot,\cdot)$ is concave on $\mathcal{D} := \{(x,m) \colon x \in (0,\infty), m \in (0, x/\beta)\}$ for fixed t;

(3) $v(t,x,\cdot)$ is decreasing on $(0, x/\beta)$ for fixed (t,x);

(4) $v(t,\cdot,m)$ is increasing on $(\beta m, \infty)$ for fixed (t,m);

(5) the function $s \to v(t, x+s, m+s)$ is increasing on $[0,\infty)$ for fixed $(t,x,m) \in \tilde{E}$.

Theorem 4.

(a) *We have* $\tilde{V} = V_2 \in \mathbb{M}_2$ *and* V_2 *is the unique fixed point of* $\mathcal{T}_2$ *in* $\mathbb{M}_2$.

(b) *Let* $g \in \mathbb{M}_2$. *Then it holds*

$$d(V_2, \mathcal{T}_2^n g) \le \frac{C_\gamma^n}{1-C_\gamma} d(\mathcal{T}_2 g, g), \qquad n \in \mathbb{N}.$$

(c) *There exists a maximizer* f^* *of* V_2 *and each maximizer of* V_2 *defines an optimal stationary policy* $(f^*, f^*, f^*, \ldots)$ *for (P2).*

Proof. We are going to prove the statements by applying the *structure theorem* for contracting MDPs (see [3, Theorem 7.3.5]). Since it is not guaranteed that $0 \in \mathbb{M}_2$, we first have to enlarge the set $\mathbb{M}_2$ by canceling condition (1). This enlargement is denoted by $\tilde{\mathbb{M}}$. Then, analogous to the proof of Theorem 2, $\tilde{\mathbb{M}}$ fulfils the conditions of the *structure theorem*. This yields $V_2 = \lim_{n\to\infty} \mathcal{T}_2^n g$ for all $g \in \tilde{\mathbb{M}}$. Since $|U(x)| \le C_U(1+x^p) + C_U'(1+x^{p'}) \le Cb(t,x)$ for some $C > 0$, we have $U \in \tilde{\mathbb{M}}$. Moreover, $\mathcal{T}_2 g \ge \mathcal{T}_2 U \ge U$ for $g \in \mathbb{M}_2$. This yields $\mathcal{T}_2^n g \ge U$, which implies that $V_2 \in \mathbb{M}_2$. The other statements follow directly from the *structure theorem* for contracting MDPs.

In the case of a CRRA utility function we obtain the following representation of the value function. The proof follows by induction.

Proposition 5. *In the case of a power utility function (i.e.* $U(x) = \alpha^{-1} x^\alpha$, $\alpha < 1, \alpha \neq 0$*), there exists a function* $F \colon [0,\tilde{T}] \times (0, \beta^{-1}) \to \mathbb{R}$ *such that* $V_2(t,x,m) = U(x) F(t, m/x)$ *for* $(t,x,m) \in \tilde{E}$.

Proposition 6. *In the case of a logarithmic utility function (i.e.* $U(x) = \log(x)$*), there exists a function* $F \colon [0,\tilde{T}] \times (0,\beta^{-1}) \to \mathbb{R}$ *such that* $V_2(t,x,m) = U(x) + F(t, m/x)$ *for* $(t,x,m) \in \tilde{E}$.

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5. Approximation results

In this section we want to approximate the terminal wealth problem (P1) by the terminal wealth problem (P2) which has only a finite number of random observation and trading times. The following mild *assumption* is needed.

The utility function U and the financial market are given in such a way that the map

$$u \to \int_{(-1,\infty)} U(x+az)p(u-t,\mathrm{d}z) \tag{2}$$

is non-decreasing on $[t,T)$ for fixed $t \in [0,T)$, $x \in (0,\infty)$, and $a \in [0,x]$.

Assumption (2) ensures that the investor invests his money in a profitable financial market.

Remark 4. In the case of a power utility function $U(x) = \alpha^{-1}x^{\alpha}$, $\alpha < 1, \alpha \neq 0$, we may apply Itô's formula which shows that assumption (2) is fulfilled if

$$2b' \geq (1-\alpha)c,$$

where $b' := b + \int_{\{x>1\}} xF(\mathrm{d}x)$ and (b,c,F) are the characteristics of the Lévy process L.

In the following we write $V_{2,\tilde{T}}$ instead of V_2 to indicate explicitly the underlying horizon of the terminal wealth problem. Moreover, let $\{\tilde{T}_n \in (0,T), n \geq 0\}$ be an increasing sequence which converges to T.

Theorem 5. *We have* $\lim_{n\to\infty} V_{2,\tilde{T}_n}(y) = V_1(y),\ y \in E$.

Proof. Let $y = (t,x,m) \in E$ and $\pi = (a_0,a_1,a_2,\dots) \in \mathcal{A}(y)$ be such that $a_k = 0$ if $\tau_k \geq \tilde{T}_n$ and n is large such that $t < \tilde{T}_n$. As in the proof of Theorem 3, we are able to show that

$$\begin{aligned}\mathbb{E}_y[U(X_T^{\pi})] = \sum_{k=0}^{\infty} \mathbb{E}_y \int_{\tau_k}^{T} \int_{(-1,\infty)} \mathbf{1}_{\{\tau_k \leq \tilde{T}_n < u\}} U(X_{\tau_k}^{\pi} + a_k z)\lambda_u \\ \times \exp\biggl(-\int_{\tau_k}^{u} \lambda_s \,\mathrm{d}s\biggr) p(u-\tau_k,\mathrm{d}z)\,\mathrm{d}u,\end{aligned}$$

where $\tau_0 = t$ and τ_k is the kth exogenous random time after time t. Moreover, we obtain

$$\begin{aligned}V_1(y) &\geq \sum_{k=0}^{\infty} \mathbb{E}_y \int_{\tau_k}^{T} \int_{(-1,\infty)} \mathbf{1}_{\{\tau_k \leq \tilde{T}_n < u\}} U(X_{\tau_k}^{\pi} + a_k z)\lambda_u \exp\biggl(-\int_{\tau_k}^{u} \lambda_s \,\mathrm{d}s\biggr) p(u-\tau_k,\mathrm{d}z)\,\mathrm{d}u \\ &= \sum_{k=0}^{\infty} \mathbb{E}_y \int_{\tilde{T}_n}^{T} \int_{(-1,\infty)} \mathbf{1}_{\{\tau_k \leq \tilde{T}_n\}} U(X_{\tau_k}^{\pi} + a_k z)\lambda_u \exp\biggl(-\int_{\tau_k}^{u} \lambda_s \,\mathrm{d}s\biggr) p(u-\tau_k,\mathrm{d}z)\,\mathrm{d}u \\ &\geq \sum_{k=0}^{\infty} \mathbb{E}_y \int_{\tilde{T}_n}^{T} \int_{(-1,\infty)} \mathbf{1}_{\{\tau_k \leq \tilde{T}_n\}} U(X_{\tau_k}^{\pi} + a_k z)\lambda_u \exp\biggl(-\int_{\tau_k}^{u} \lambda_s \,\mathrm{d}s\biggr) p(\tilde{T}_n-\tau_k,\mathrm{d}z)\,\mathrm{d}u \\ &= \sum_{k=0}^{\infty} \mathbb{E}_y \exp\biggl(-\int_{\tau_k}^{\tilde{T}_n} \lambda_s \,\mathrm{d}s\biggr) \int_{(-1,\infty)} \mathbf{1}_{\{\tau_k \leq \tilde{T}_n\}} U(X_{\tau_k}^{\pi} + a_k z) p(\tilde{T}_n-\tau_k,\mathrm{d}z),\end{aligned}$$

where (2) is used for the second inequality. From the proof of Theorem 3 we obtain

$$V_{2,\tilde{T}_n}(y) = \sup_{\pi \in \mathcal{A}(y)} \sum_{k=0}^{\infty} \mathbb{E}_y \exp\biggl(-\int_{\tau_k}^{\tilde{T}_n} \lambda_s \,\mathrm{d}s\biggr) \int_{(-1,\infty)} \mathbf{1}_{\{\tau_k \leq \tilde{T}_n\}} U(X_{\tau_k}^{\pi} + a_k z) p(\tilde{T}_n-\tau_k,\mathrm{d}z).$$

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Hence, $V_1(y) \geq V_{2,\tilde{T}_n}(y)$ and $\limsup_{n\to\infty} V_{2,\tilde{T}_n}(y) \leq V_1(y)$.

On the other hand $\mathbb{E}_y[U(X^\pi_{\tilde{T}_n})] \leq V_{2,\tilde{T}_n}(y)$ for $\pi \in \mathcal{A}(y)$. Because $\{U(X^\pi_{\tilde{T}_n}), n \geq 0\}$ is uniformly $\mathbb{P}_y$-integrable, we obtain

$$\mathbb{E}_y[U(X^\pi_T)] = \lim_{n\to\infty} \mathbb{E}_y[U(X^\pi_{\tilde{T}_n})] \leq \liminf_{n\to\infty} V_{2,\tilde{T}_n}(y).$$

Hence, $\liminf_{n\to\infty} V_{2,\tilde{T}_n} \geq V_1$ which yields the statement.

Next we consider the sequence of maximizers $f_{\tilde{T}_n}$ of $V_{2,\tilde{T}_n}$. Since there exists a convergent subsequence of $\{f_{\tilde{T}_n}(y), n \geq 0\}$ for each fixed $y \in E$, we define

$$f_1^*(y) := \limsup_{n\to\infty} f_{T_n}(y) \quad \text{and} \quad f_2^*(y) := \liminf_{n\to\infty} f_{T_n}(y) \quad \text{for } y \in E.$$

Theorem 6. *Let $V_1 \geq 0$. Then the stationary policies $(f_1^*, f_1^*, \ldots)$ and $(f_2^*, f_2^*, \ldots)$ are optimal for the terminal wealth problem (P1).*

Proof. We have $V_{2,\tilde{T}_n} = \mathcal{T}_{2,\tilde{T}_n} V_{2,\tilde{T}_n}$, where the operator $\mathcal{T}_{2,\tilde{T}_n}$ is the maximal reward operator of the terminal wealth problem (P2) with horizon $\tilde{T}_n$. Let $f_{\tilde{T}_n}$ be a maximizer of $V_{2,\tilde{T}_n}$. Since $V_1 = \lim_{n\to\infty} V_{2,\tilde{T}_n}$, it follows, for $y \in E$,

$$\begin{aligned} V_1(y) = \lim_{n\to\infty} \int_t^{\tilde{T}_n} \int_{(-1,\infty)} & V_{2,\tilde{T}_n}(u, x + f_{\tilde{T}_n}(y)z, \max\{m, x + f_{\tilde{T}_n}(y)z\})\lambda_u \\ & \times \exp\left(-\int_t^u \lambda_s \,\mathrm{d}s\right) p(u-t, \mathrm{d}z)\,\mathrm{d}u. \end{aligned}$$

Let $\{f_{\tilde{T}_{n_k}}(y), k \geq 0\}$ be a subsequence of $\{f_{\tilde{T}_n}(y), n \geq 0\}$ such that

$$\lim_{k\to\infty} f_{\tilde{T}_{n_k}}(y) = \limsup_{n\to\infty} f_{\tilde{T}_n}(y).$$

From $V_{2,\tilde{T}_n} \leq V_1$ and Fatou's lemma we obtain

$$\begin{aligned} V_1(y) &\leq \int_t^T \int_{(-1,\infty)} \limsup_{k\to\infty} V_1(u, x + f_{\tilde{T}_{n_k}}(y)z, \max\{m, x + f_{\tilde{T}_{n_k}}(y)z\})\lambda_u \\ &\qquad \times \exp\left(-\int_t^u \lambda_s \,\mathrm{d}s\right) p(u-t, \mathrm{d}z)\,\mathrm{d}u \\ &= \int_t^T \int_{(-1,\infty)} V_1\left(u, x + \lim_{k\to\infty} f_{\tilde{T}_{n_k}}(y)z, \max\left\{m, x + \lim_{k\to\infty} f_{\tilde{T}_{n_k}}(y)z\right\}\right)\lambda_u \\ &\qquad \times \exp\left(-\int_t^u \lambda_s \,\mathrm{d}s\right) p(u-t, \mathrm{d}z)\,\mathrm{d}u \\ &\leq V_1(y). \end{aligned}$$

Hence, $\limsup_{n\to\infty} f_{\tilde{T}_n}(y)$ is a maximizer of V_1 and the stationary policy $(f_1^*, f_1^*, f_1^*, \ldots)$ is optimal for the terminal wealth problem (P1). Similar arguments show the optimality of $(f_2^*, f_2^*, f_2^*, \ldots)$.

6. Numerical analysis for the power utility function

In this section we present a numerical study of the terminal wealth problems (P1) and (P2). We fix the following data.

(i) The finite horizon equals 0.99, i.e. $T = 0.99$.

(ii) We consider variance gamma stock price dynamics as in [10, Section 3], i.e.

$$S_t = S_0 \exp(\mu t + L(t; \sigma, \nu, \theta) + wt), \tag{3}$$

where $w = \nu^{-1} \log(1 - \theta\nu - \sigma^2\nu/2)$ and $L(t; \sigma, \nu, \theta)$ is a variance gamma process with parameters $\mu = 0.045$, $\sigma = 0.4$, $\nu = 0.004$, and $\theta = -0.2$. These parameters were estimated from the German Stock Index (DAX Index) via a moment based estimation.

(iii) We assume a *power utility* function $U(x) = \alpha^{-1}x^{\alpha}$ with parameter $\alpha = \frac{1}{3}$.

(iv) The intensity process $\lambda = (\lambda_t)_{0 \le t < T}$ is given by $\lambda_t := 1/(1-t)$.

(v) One half of the wealth is guaranteed by the drawdown constraints, i.e. $\beta = \frac{1}{2}$.

Since the intensity process λ is bounded on $[0, T]$ we are facing optimization problem (P2). We want to emphasize that Theorem 4 is still true if c equals 0. This means that there is no Brownian motion in the driving Lévy process needed to solve optimization problem (P2). Because of that and the *power utility* function, we may write

$$\tilde{V}(t, x, m) = U(x)F(t, m/x), \qquad (t, x, m) \in E,$$

for some function $F\colon [0, T] \times (0, \beta^{-1}) \to \mathbb{R}$ (see Proposition 5).

The initial policy, for Howard's policy improvement algorithm, is chosen to be the so-called *generalized Merton ratio* f_0 with

$$f_0(t, x, m) := \min\left\{\pi^*, 1 - \beta\frac{m}{x}\right\}x,$$

where $\pi^* := 0.422$ denotes the solution of the classical Merton problem in the market from above. Furthermore, the *fraction* $f_0(t, x, m)/x$ is independent of the time t and depends on (x, m) only through $v := m/x$.

Figure 1 shows the computed approximation of the optimal fraction f^*. It turns out that Howard's policy improvement algorithm works very well and that the first improvement already yields a very good approximation.

To obtain more insight into the micro-structure of the fraction, the slice planes $t \to f^*(t, \frac{1}{2})$ and $v \to f^*(\frac{1}{2}, v)$ are shown in Figure 2.

In Figure 2 we see that the fraction f^* is not constant in time and that its value is very close to the optimal fraction π^* of the classical Merton problem for small values of v. Clearly, the time dependence arises from the rising liquidity. Furthermore, we see that the fraction f^* is constant as a function of the ratio $v = m/x$ for small values thereof and that it then decreases linearly to 0 with slope $-\beta$.

We obtain that the terminal wealth V_{f_0} under the policy f_0 is very close to the value function $\tilde{V}$. This shows that the policy f_0 is a very good approximation of an optimal policy.

Finally, we present in Figure 3 an approximation of the function F which is denoted by F^*.

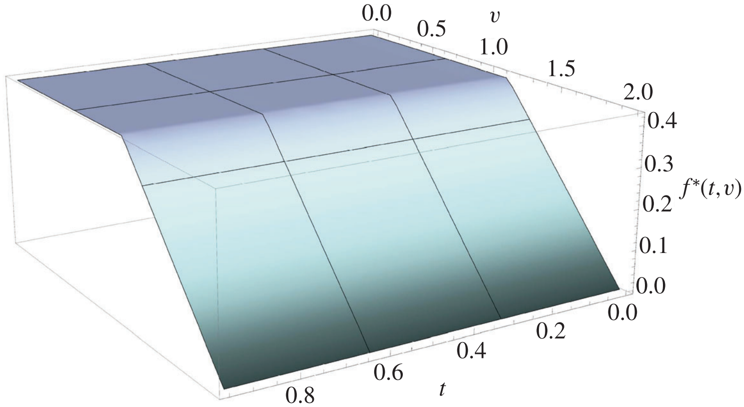

FIGURE 1: Approximation f^* of the optimal fraction.

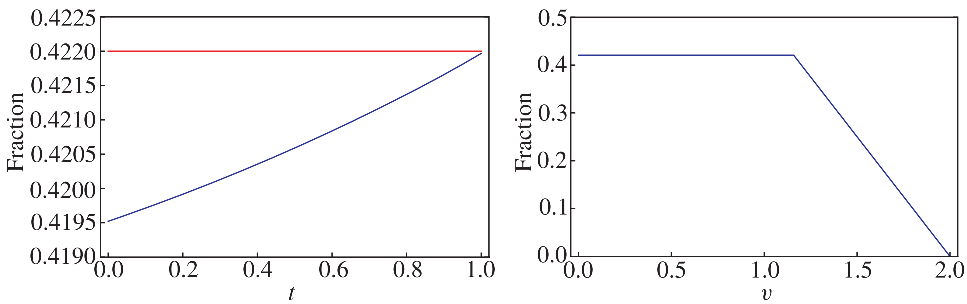

FIGURE 2: Slice planes for insight into the micro-structure. The left-hand side shows $f^*(t, 1/2)$ and the right-hand side shows $f^*(1/2, v)$.

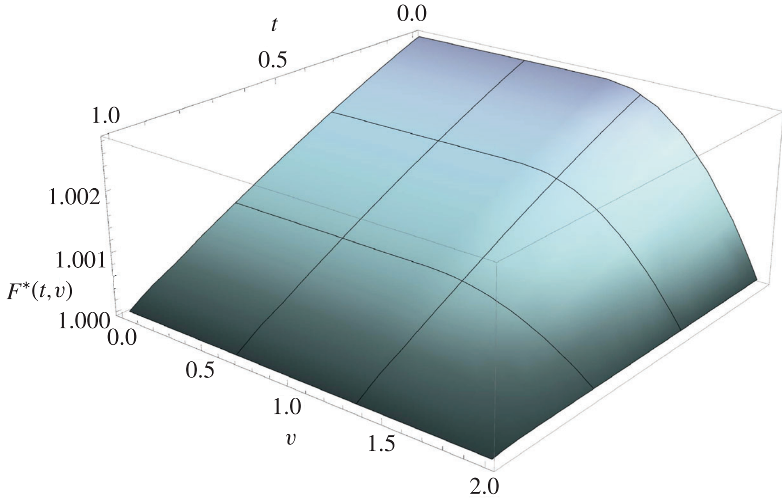

FIGURE 3: Approximation F^* of the function F.

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If we add a Brownian motion with a tiny variance to the variance gamma process in (3) and choose $T = 1$, then we are dealing with optimization problem (P1). Again, we may write

$$V(t, x, m) = U(x)F\left(t, \frac{m}{x}\right), \qquad (t, x, m) \in E,$$

due to Proposition 2. From a numerical point of view, the distribution of the stock returns does not change when choosing Brownian motion with very small variance. In this case the solution of optimization problem (P2) is an approximation of the solution of optimization problem (P1).

Appendix A. Structure theorem for limsup MDPs

In this appendix we formulate a *structure theorem* for general limsup MDPs. For this, consider an MDP (E, A, D, Q, g) with the following meaning (cf. [3]):

- E is the state space, endowed with a σ-algebra $\mathfrak{E}$;
- A is the action space, endowed with a σ-algebra $\mathfrak{A}$;
- D is a measurable subset of $E \times A$ and denotes the admissible state-action combinations; it is assumed that D contains the graph of a measurable mapping; moreover, the set $D(x) := \{a \in A \mid (x, a) \in D\}$ is the set of admissible actions;
- Q is a stochastic transition kernel from D to E;
- $g: E \to \mathbb{R}$ is a measurable function, the so-called terminal reward function.

A Markovian policy $\pi := (f_0, f_1, f_2, \ldots)$ is given by a sequence of decision rules f_n, where $f_n: E \to A$ is a measurable mapping such that $f_n(x) \in D(x)$ for all $x \in E$. The set of all Markovian policies is denoted by Π and we *assume* that

$$\sup_{\pi \in \Pi} \mathbb{E}_x^{\pi}\Big[\limsup_{n\to\infty} g^+(X_n)\Big] < \infty, \qquad x \in E.$$

The gain corresponding to a Markovian policy π is defined by

$$V_\pi(x) := \mathbb{E}_x^{\pi}\Big[\limsup_{n\to\infty} g(X_n)\Big], \qquad x \in E,$$

and we aim to maximize the gain over all Markovian policies, i.e. we are interested in

$$V(x) := \sup_{\pi \in \Pi} V_\pi(x), \qquad x \in E. \tag{P}$$

Let $\mathbb{M}(E) := \{v: E \to \mathbb{R} \mid v \text{ is measurable}\}$. Then we define, for $v \in \mathbb{M}(E)$, the operators

$$Lv(x, a) := \int_E v(x')Q(x'|x, a), \qquad (x, a) \in D,$$

whenever the integral exists, and

$$(\mathcal{T}v)(x) := \sup_{a \in D(x)} Lv(x, a), \qquad x \in E.$$

The following *structure theorem* contains the solution of the limsup MDP. A proof can be found in [14, Appendix A].

Theorem 7. (Structure theorem.) *Let $\mathbb{M}$ be a subset of $\mathbb{M}(E)$ such that the following conditions are satisfied:*

(i) $g \in \mathbb{M}$;

(ii) $\mathcal{T} v$ *is well defined for each* $v \in \mathbb{M}$ *and* $\mathcal{T} v \in \mathbb{M}$;

(iii) *for each* $v \in \mathbb{M}$ *there exists a maximizer of* v;

(iv) *for* $x \in E$ *there exists* $v_\infty(x) := \lim_{n\to\infty} \mathcal{T}^n g(x)$ *and* $v_\infty \in \mathbb{M}$; *furthermore* v_∞ *has the following three properties:*

- $v_\infty = \mathcal{T} v_\infty$,
- $\lim_{n\to\infty} v_\infty(X_n) = \limsup_{n\to\infty} g(X_n)$ $\mathbb{P}_x^\pi$*-almost surely, and*
- $(v_\infty(X_n))_{n\geq 0}$ *is uniformly* $\mathbb{P}_x^\pi$*-integrable for all* $\pi \in \Pi$ *and* $x \in E$.

Then we have:

(a) $V \in \mathbb{M}$ *and* V *is the unique fixed point of* $\mathcal{T}$ *in* $\mathbb{M}$ *with the three properties of condition* (iv); *moreover,* $V = \lim_{n\to\infty} \mathcal{T}^n g$;

(b) *there exists a maximizer* f^* *of* V *and each maximizer of* V *defines an optimal stationary policy* $(f^*, f^*, f^*, \ldots)$ *for (P).*

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