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ON THE STRUCTURE OF A REAL CROSSED GROUP ALGEBRA

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The main result of this paper is that there exist non-principal left ideals in a certain twisted group algebra A of the infinite dihedral group $< a, b \mid b^{-1}ab = a^{-1}, b^2 = 1 >$ over the field $\mathbf{R}$ of real numbers: namely in the A defined by $b^{-1}ab = a^{-1}$, $b^2 = -1$, and $\lambda a = a\lambda$, $\lambda b = b\lambda$ *for all real* λ.

The motivation comes from the study (in a series of papers by Berman and the author) of finitely generated torsion-free $\mathbf{R}G$-modules for groups G which have an infinite cyclic subgroup of finite index. In a sense, this amounts to studying modules over (full matrix algebras over) a finite set of $\mathbf{R}$-algebras [namely, for the groups in question, these algebras take on the role played by $\mathbf{R}$, $\mathbf{C}$ and $\mathbf{H}$ (the real quaternions) in the theory of real representations of finite groups]. For all but two algebras in that finite set, satisfying results have been obtained by exploiting the fact that each of them is either a ring with zero divisors or a principal left ideal ring. The other two are known to have no zero divisors. One of them is the present A. The point of the main result is that new ideas will be needed for understanding A-modules.

A number of subsidiary results are concerned with convenient generating sets for left ideals in A.

0. INTRODUCTION

Let G be an arbitrary group containing an infinite cyclic subgroup of finite index. Berman and the author showed (see [**1**]) that G contains a normal subgroup H such that $(G : H) = 2^{\alpha}(\alpha = 0, 1)$ and $H = F.(a)$, where F is a finite normal subgroup in H and (a) the infinite cyclic group. Let K be an arbitrary field with Char $K \nmid |F|$. It was proved in [**1**] that the investigation of finitely generated KG-modules can be reduced to the study of finitely generated modules over algebras of so-called type E over K.

Berman and Buzási described in [**2**] all the algebras of type E over the real field $\mathbf{R}$ and discussed the structure of finitely generated modules over them. It was shown that the algebras

$$A = (\mathbf{R}, a, b),\ \lambda a = a\lambda,\ \lambda b = b\lambda,\ b^{-1}ab = a^{-1},\ b^2 = -1(\lambda \in \mathbf{R}),$$
$$B = (\mathbf{C}, a, b),\ \lambda a = a\lambda,\ \lambda b = b\overline{\lambda},\ b^{-1}ab = a^{-1},\ b^2 = -1(\lambda \in \mathbf{C}),$$

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contain no zero divisors. All the other algebras of type E over $\mathbf{R}$ are either principal left ideal rings or contain zero divisors, so the investigation of the structure of finitely generated torsion-free modules over such algebras can be considered completed by aplyiong classical results and the results of [**2**]. If the algebra A or B is not a principal left ideal ring, then the structure of finitely generated torsion-free A-modules needs additional investigation.

It will be shown in this paper that the algebra A defined above is not a principal left ideal ring.

1. The structure of left ideals

Throughout this paper A denotes the algebra defined above. It was shown in [**2**] that the group algebra $\mathbf{R}(a) \subset A$ of the infinite cyclic group (a) is a Euclidean ring with respect to the norm:

$$|f(a)| = |\lambda_n a^n + \cdots + \lambda_m a^m| = n - m \qquad (\lambda_i \in \mathbf{R}; n \geqslant m; n, m \in \mathbf{Z}).$$

It is easy to see that when $f(a)$, $g(a) \in \mathbf{R}(a)$ and $g(a) \neq 0$, there exist $h(a)$ and $r(a) \in \mathbf{R}(a)$ such that

$$f(a) = g(a).h(a) + r(a),$$

where $r(a) = 0$ or $|g(a)| > |r(a)|$; however the elements $h(a)$ and $r(a)$ are not uniquely determined.

We write $f(a^{-1}) = \overline{f(a)}$. The element $f(a) \in \mathbf{R}(a)$ is called symmetric if $\overline{f(a)} = \mu a^{-m} f(a)$, for some $m \in \mathbf{Z}$ and $\mu \in \mathbf{R}$. The units of $\mathbf{R}(a)$ are exactly the elements μa^m.

Lemma 1.1. *Let $f(a)$ and $g(a) \in \mathbf{R}(a)$ with $f(a) \neq 0$ and $|f(a)| \geqslant |g(a)|$. Then there exist elements $h(a)$ and $r(a) \in \mathbf{R}(a)$ such that*

$$f(a) = g(a).h(a) + r(a),$$

where $r(a) = 0$ or $|r(a)| < |g(a)|$, and $|f(a)| = |g(a).h(a)|$.

Proof: First let $f(a) = \alpha_n a^n + \cdots + \alpha_0$ and $g(a) = \beta_m a^m + \cdots + \beta_0$, where $\alpha_i, \beta_j \in \mathbf{R}$, $\alpha_0 \neq 0$ and $\beta_0 \neq 0$. Then there exist elements $h_1(a) = \gamma_k a^k + \cdots + \gamma_0$ and $r_1(a) = \lambda_s a^s + \cdots + \lambda_0$ in $\mathbf{R}(a)$ such that

$$f(a) = g(a).h_1(a) + r_1(a), \tag{1.1}$$

where $r_1(a) = 0$ or $r_1^0(a) < g^0(a)$. Here $\psi^0(a)$ denotes the degree of the polynomial $\psi(a) \in \mathbf{R}(a)$. (1.1) implies

$$\alpha_0 = \beta.\gamma_0 + \lambda_0. \tag{1.2}$$

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If $\gamma_0 \neq 0$, then $|h_1(a)| = h_1^0(a)$, and since $n \geqslant m$,

$$|f(a)| = f^0(a) = (g(a).h_1(a))^0 = |g(a).h_1(a)|.$$

If $\gamma_0 = 0$, then (1.2) implies $\lambda_0 \neq 0$. Consider the element

$$h(a) = h_1(a) - \frac{\lambda_0}{\beta_0}.$$

Then the equation

$$f(a) = g(a).h(a) + [r_1(a) - \frac{\lambda_0}{\beta_0}.g(a)]$$

holds. It is clear that the element

$$r(a) = r_1(a) - \frac{\lambda_0}{\beta_0}.g(a)$$

has no constant term and so $|r(a)| < r^0(a)$. Consequently, as $r_1^0(a) < g^0(a)$ it follows that $r^0(a) = g^0(a)$ and

$$|r(a)| < r^0(a) = g^0(a) = |g(a)|.$$

So we obtain the equation

$$f(a) = g(a).h(a) + r(a), \tag{1.3}$$

where $|r(a)| < |g(a)|$ and $|f(a)| = f^0(a) = (g(a).h(a))^0 = |g(a).h(a)|$.

Now let $f_1(a) = \alpha'_{n_1} a^{n_1} + \cdots + \alpha'_{n_2} a^{n_2}$ and $g_1(a) = \beta'_{m_1} a^{m_1} + \cdots + \beta'_{m_2} a^{m_2}$ with α'_i, $\beta'_j \in \mathbf{R}; n_i$, $m_j \in \mathbf{Z}; n_1 \geqslant n_2$ and $m_1 \geqslant m_2$. Then $f_1(a) = a^{n_2}.f(a)$ and $g_1(a)a^{m_2}.g(a)$, where

$$f(a) = \alpha_n a^n + \cdots + \alpha_0, \qquad g(a) = \beta_m a^m + \cdots + \beta_0,$$
$$\alpha'_{n_1} = \alpha_n, \ldots, \alpha'_{n_2} = \alpha_0, \quad n = n_1 - n_2,$$
$$\beta'_{m_1} = \beta_m, \ldots, \beta'_{m_2} = \beta_0, \quad m = m_1 - m_2.$$

As was shown above, the equation (1.3) holds for the elements $f(a)$ and $g(a)$. Then, applying (1.3), we obtain

$$\begin{aligned} f_1(a) &= a^{n_2}.f(a) \\ &= a^{n_2}[g(a).h(a) + r(a)] \\ &= a^{m_2}.g(a).a^{n_2-m_2}.h(a) + a^{n_2}.r(a) \\ &= g_1(a).a^{n_2-m_2}.h(a) + a^{n_2}.r(a). \end{aligned}$$

Since $|f_1(a)| = |f(a)|$, $|g_1(a)| = |g(a)| = g(a)$ and $|r(a)| = |a^{n_2}.r(a)|$, the inequality $|g_1(a)| > |a^{n_2}.r(a)|$ follows from $|g(a)| > |r(a)|$; consequently we obtain

$$|f_1(a)| = |f(a)| = |g(a).h(a)| = |g_1(a).a^{n_2-m_2}.h(a)|.$$

∎

LEMMA 1.2. *Let $I \subseteq A$ be a left ideal generated by elements p and $1+qb$, where $p, q \in \mathsf{R}(a)$ and p is the generator of the ideal $I \cap \mathsf{R}(a)$. Then p is a symmetric element.*

PROOF: Since $p.(1+qb) = p + pqb \in I$, we have $pqb \in I$. This implies that $b\overline{pq} \in I$ and so $\overline{p}.\overline{q} \in I \cap \mathsf{R}(a)$. The element p generates the ideal $I \cap \mathsf{R}(a)$, so it follows that

$$\overline{p}.\overline{q} \equiv 0 \pmod p. \tag{1.4}$$

But

$$(1-qb)(1+qb) = 1 + q.\overline{q} \in I \cap \mathsf{R}(a),$$

and hence

$$1 + q.\overline{q} \equiv 0 \pmod p. \tag{1.5}$$

Thus the congruence (1.4) implies that $\overline{p} \equiv 0 (\mathrm{mod}\ p)$. Then $\overline{p} = s.p$, for an element $s \in \mathsf{R}(a)$. Under the action of the automorphism $f \to \overline{f}$, this equation implies

$$p = \overline{s}.\overline{p} = \overline{s}.s.p.$$

As the ring $\mathsf{R}(a)$ has no zero divisors, it follows from this equation that s is a unit in $\mathsf{R}a$, and p is a symmetric element. ∎

LEMMA 1.3. *Every left ideal I of the algebra A can be generated by elements p and $s_0 + s_1 b$, where p, s_0, $s_1 \in \mathsf{R}(a)$; here p is a symmetric element and generates the ideal $I \cap \mathsf{R}(a)$.*

PROOF: Every element of A can be expressed by the form $\alpha + \beta b$ with $\alpha, \beta \in \mathsf{R}(a)$. Consider the elements $x = t_0 + t_1 b$ of the left ideal I. As x runs over I, the corresponding elements t_1 run over an ideal L_1 in $\mathsf{R}(a)$. Since $\mathsf{R}(a)$ is a principal ideal ring, $L_1 = (s_1)$ for some element $s_1 \in \mathsf{R}(a)$. Consider a fixed element

$$x_0 = s_0 + s_1 b \in I$$

and let $x = \lambda_0 + \lambda_1 b$ be an arbitrary element of I. Here $\lambda_1 \in L_1$ and so $\lambda_1 = ts_1$ for some $t \in \mathsf{R}(a)$. Consider the element

$$p_0 = x - tx_0 = (\lambda_0 + ts_1 b) - t(s_0 + s_1 b) = \lambda_0 - ts_0 \in I.$$

As the element x runs over the I, the element p_0 runs over an ideal L_0 of $\mathsf{R}(a)$. Let $L_0 = (p_1)$. Then $p_0 = t_0 p_1$ for some $t_0 \in \mathsf{R}(a)$ and the element of x can be expressed in the form

$$x = t_0 p_1 + tx_0,$$

that is the elements p_1, $x_0 + s_1 b$ generate the left ideal I. If $I \cap \mathsf{R}(a) = (p)$, then $p_1 = t_1 p$ for some $t_1 \in \mathsf{R}(a)$, and consequently I can be generated by elements p, $s_0 + s_1 b$. By Lemma 1.2 the element p is symmetric. ∎

LEMMA 1.4. *Let the left ideal $I \subseteq A$ be generated by elements p and $s_0 + s_1 b$, where p, $s_0, s_1 \in \mathsf{R}(a)$ and $(p) = I \cap \mathsf{R}(a)$. If either $(p, s_0) = 1$, or $(p, \overline{s}_1) = 1$, then there exists an element $q \in \mathsf{R}(a)$ such that the left ideal I can be generated by elements p and $1 + qb$.*

PROOF: Let $(s_0, p) = 1$. It can be assumed that $|p| > |s_0|$. Indeed, if $|p| \leqslant |s_0|$ then $s_0 = ph + r$ for some $h, r \in \mathsf{R}(a)$, where $r = 0$ or $|r| < |p|$. We set

$$(s_0 + s_1 b) - hp = ph + r + s_1 b - hp = r + s_1 b,$$

and the elements p, $r + s_1 b$ generate the left ideal I, where $|p| > |r|$.

Applying the Euclidean algorithm to the elements p and s_0, we obtain

$$\begin{aligned}
p &= s_0 h_0 + r_0, && |r_1| < |s_0|, \\
s_0 &= r_0 h_1 + r_1, && |r_1| < |r_0|, \\
r_0 &= r_1 h_2 + r_2, && |r_2| < |r_{k-1}|, \\
&\vdots \\
r_{k-2} &= r_{k-1} h_k + r_k, && |r_k| < |r_{k-1}|, \\
r_{k-1} &= r_k h_{k+1}.
\end{aligned}$$

We have $r_k = (p, s_0) = 1$. We use this algorithm in the following way:
As the first step we form an element

$$p - h_0(s_0 + s_1 b) = r_0 - h_0 s_1 b = r_0 + m_0 b \qquad (m_0 \in \mathsf{R}(a))$$

and change the generator elements of I to the generators

$$r_0 + m_0 b,\ s_0 + s_1 b.$$

At the second step we form the element

$$(s_0 + s_1 b) - h_1(r_0 + m_0 b) = r_1 + m_1 b \qquad (m_1 \in \mathsf{R}(a))$$

and change the generators to

$$r_0 + m_0 b,\ r_1 + m_1 b,$$

and so on. At the last step we get the generators

$$m_k b,\ 1 + m_{k+1} b \qquad (m_k,\ m_{k+1} \in \mathsf{R}(a)).$$

Since b is an invertible element and $m_k \in I \cap \mathsf{R}(a) = (p)$, we obtain the generators p and $1 + qb$, where $m_{k+1} = q$.

If $(p, \overline{s}_1) = 1$, then the element

$$b(s_0 + s_1 b) = -\overline{s}_1 + \overline{s}_0 b$$

is also a generator, and we have the case considered above. ∎

Theorem 1.1. *Every left ideal $I \subseteq A$ can be expressed in the form*

$$I = I_1.d,$$

where I_1 is a left ideal generated by elements p and $1 + qb$; here $p, q \in \mathsf{R}(a)$, $(p) = I_1 \cap \mathsf{R}(a)$ and $d \in \mathsf{R}(a)$.

Proof: By Lemma 1.3 the left ideal I can be expressed as $I = (p_1, s_0 + s_1 b)$, where $p_1, s_0, s_1 \in \mathsf{R}(a)$ and $(p_1) = I \cap \mathsf{R}(a)$. If $(s_0, p_1) = 1$ or $(p_1, \overline{s}_1) = 1$, then by Lemma 1.4 we obtain the theorem with $d = 1$.

Let us consider the set of all elements

$$x = \mu_0 + \mu_1 b \qquad (\mu_0, \mu_1 \in \mathsf{R}(a))$$

of I. As the element x runs over I, the corresponding elements μ_i form ideals L_i in $\mathsf{R}(a)(i = 0, 1)$. Let $L_0 = (d)$. Then $L_1 = (\overline{d})$. Indeed, $bx = -\overline{\mu}_1 + \overline{\mu}_0 b \in I$, consequently if $\mu_0 \in L_0$ then $\overline{\mu}_0 \in L_1$. Since $p_1 \in L_0$, so $p_1 = p.d$ for some $p \in \mathsf{R}(a)$. Since $L_0 = (d)$, applying the Euclidean algorithm, as in the proof of Lemma 1.4, we can replace the generator $s_0 + s_1 b$ by an element $d + s_1' b$. As $s_1' \in L_1$, it follows that $s_1' = q.\overline{d}$ for some $q \in \mathsf{R}a$. Consequently, every element $y \in I$ can be expressed in the form

$$\begin{aligned} y &= (\lambda_0 + \lambda_1 b)p_1 + (\lambda_0' + \lambda_1' b)(d + s_1' b) \\ &= (\lambda_0 + \lambda_1 b)p.d + (\lambda_0' + \lambda_1' b)\big(d + q\overline{d}b\big) \\ &= [(\lambda_0 + \lambda_1 b)p + (\lambda_0' + \lambda_1' b)(1 + qb)]d, \end{aligned}$$

where $\lambda_0 + \lambda_1 b$ and $\lambda_0' + \lambda_1' b \in A$. The elements

$$(\lambda_0 + \lambda_1 b)p + (\lambda_0' + \lambda_1' b)(1 + qb)$$

form a left ideal I_1 generated by elements p and $1 + qb$. ∎

Lemma 1.5. *Let the left ideal $I \subseteq A$ be generated by two pairs of elements $p, 1+qb$ and $p, 1 + q_1 b$, where $(p) = I \cap \mathsf{R}(a)$ and $q, q_1 \in \mathsf{R}(a)$. Then $q \equiv q_1 (\mathrm{mod}\ p)$.*

Proof: Clearly

$$(1 + qb) - (1 + q_1 b) = (q - q_1)b \in I,$$

which implies $\overline{q} - \overline{q}_1 \in I$. But $\overline{q} - \overline{q}_1 \in \mathsf{R}(a)$ and hence $\overline{q} \equiv \overline{q}_1 (\mathrm{mod}\ q)$. Since p is symmetric, the lemma follows from this congruence. ∎

2. Construction of a left ideal which is not a principal left ideal

For the element $x = \alpha + \beta b(\alpha, \beta \in \mathsf{R}(a))$ of A we define a norm $N(x)$ by the formula

$$N(x) = (\overline{\alpha} - \beta b)(\alpha + \beta b) = \alpha, \overline{\alpha} + \beta.\overline{\beta}.$$

It is easy to see that $N(x.y) = N(x).N(y)$ for all $x, y \in A$ and $N(x) \in I \cap \mathsf{R}(a)$ for all $x \in I$ where I is a left ideal of A.

Lemma 2.1. *Let $I \subseteq A$ be a principal left ideal generated by the element $s_0 + s_1 b$ with $s_0, s_1 \in \mathsf{R}(a)$. If $(p) = I \cap \mathsf{R}(a)$, then the elements $d.p$ and $N(s_0 + s_1 b)$ are associates, where $(\overline{s}_0, s_1) = d$.*

Proof: First let $(\overline{s}_0, s_1) = 1$. We have that

(2.1) $$p = (\lambda_0 + \lambda_1 b)(s_0 + s_1 b)$$

for some $\lambda_0 + \lambda_1 b \in A$. This implies

(2.2) $$p = \lambda_0.s_0 - \lambda_1, \overline{s}_1 \qquad \text{and} \qquad 0 = \lambda_0.s_1 + \lambda_1.\overline{s}_0.$$

Because $(\overline{s}_0, s_1) = 1$, it follows from the second equality of (2.2) that $\lambda_0 = t.\overline{s}_0$ and $\lambda_1 = -t.s_1 (t \in \mathsf{R}(a))$. Then (2.2) implies that $p = t(s_0\overline{s}_0 + s_1\overline{s}_1)$, that is, $p \equiv 0(\text{mod}\, N(s_0 + s_1 b))$. On the other hand, $N(s_0 + s_1 b) \in I \cap \mathsf{R}(a)$ and so $N(s_0 + s_1 b) \equiv 0(\text{mod}\, p)$, that is, p and $N(s_0 + s_1 b)$ are associates. Now let

(2.3) $$(\overline{s}_0, s_1) = d \neq 1 \text{ with } \overline{s}_0 = \overline{h}_0 d \text{ and } s_1 = h_1.d \quad (h_0, h_1 \in \mathsf{R}(a)),$$

where $(h_0, h_1) = 1$. In this case $s_0 + s_1 b = (h_0 + h_1 b)\overline{d}$, that is, $I = I_1.\overline{d}$, where I_1 is a principal left ideal generated by $h_0 + h_1 b$. Here $(h_0, h_1) = 1$, and if $(p_1) = I_1 \cap \mathsf{R}(a)$, then p_1 and $N(h_0 + h_1 b)$ are associates. it follows, at the same time, that $p = p_1\overline{d}$, and so $p.d = p_1\overline{d}.d$ and

$$N(s_0 + s_1 b) = s_0\overline{s}_0 + s_1\overline{s}_1 = (h_0\overline{h}_0 + h_1\overline{h}_1)d.\overline{d} = N(h_0 + h_1 b)d.\overline{d}$$

are associates too. ∎

Lemma 2.2. *Let $I \subseteq A$ be a left ideal generated by elements p and $1 + qb$, where $(p) = I \cap \mathsf{R}(a)$, $q \in \mathsf{R}(a)$. Then every element of I can be expressed in the form $(xb)p + y(1 + qb)$, where $x, y \in \mathsf{R}(a)$.*

Proof: Let $s_0 + s_1 b \in I$ be an arbitrary element of I. Then

(2.4) $$s_0 + s_1 b - s_0(1 + qb) = (s_1 - s_0 q)b \in I,$$

that is

$$b(s_1 - s_0 q)b = -\overline{s}_1 + \overline{s}_0\overline{q} \in I \cap \mathsf{R}(a).$$

Since $(p) = I \cap \mathsf{R}(a)$, so $\overline{s}_1 - \overline{s}_0\overline{q} = p.\overline{x}$ for some $\overline{x} \in \mathsf{R}(a)$. This implies that $s_1 - s_0 q = p.x(x \in \mathsf{R}(a))$, because by Lemma 1.2, the element p is symmetric. By 2.4 we have $(xb)p + s_0(1 + qb) = s_0 + s_1 b$, which proves the lemma. ∎

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THEOREM 2.1. *Algebra A is not a principal left ideal ring.*

PROOF: We shall construct a left ideal I generated by certain elements p, $1+qb$ which is not a principal left ideal.

Let $q = a^3+1$. Since $(p) = I \cap \mathsf{R}(a)$, the element p divides the element $N(1+qb) = 1+q.\overline{q}$. The element $1+q.\overline{q}$ is expressed as a product of prime elements as follows:

$$1+q.\overline{q} = (a-\alpha)(a^{-1}-\alpha)(a^2-\alpha a+\alpha^2)(a^{-2}-\alpha a^{-1}+\alpha^2),$$

where α is a real value of

$$\alpha = {}^3\sqrt{\frac{-3+5}{2}}.$$

Let $p = (a^2-\alpha a+\alpha^2)(a^{-2}-\alpha a^{-1}+\alpha^2)$. For p and q there exist elements h and r such that

(2.5) $$p = q.h + r.$$

Here

(2.6) $$h = \alpha^2 a^{-1} - (\alpha+\alpha^3)a^{-2},$$
$$r = (1+\alpha^2+\alpha^4) - (\alpha+\alpha^2+\alpha^3)a^{-1} + (\alpha+\alpha^2+\alpha^3)a^{-2}.$$

It is true in (2.5) that $|r| = 2 < |q|$ and $|h| = 1$. We construct an element

(2.7) $$u = bp - h(1+qb) = (qh+r)b - h(1+qb) = -h + rb \in I.$$

It follows that $|N(u)| = |h.\overline{h} + r.\overline{r}| = 4 = |p|$.

We show that the element u does not generate the left ideal I. Indeed, assume that $I = (u)$. (2.7) implies

(2.8) $$u - bp = -h(1+qb).$$

Because $N(u) \in I \cap \mathsf{R}(a)$, it follows that $n(u) \equiv 0 (\mathrm{mod}\ p)$. However, $|N(u)| = |p|$, so the elements $N(u)$ and p are associates. This means that $p = \delta \cdot N(u)$, and it is easy to calculate that $\delta = \alpha \cdot (1+\alpha+2\alpha^2+\alpha^3+2\alpha^4+\alpha^5+\alpha^6)^{-1}$. Consequently $p = \delta(-\overline{h} - rb)(-h+rb)$, so (2.8) implies

(2.9) $$u - \delta b(-\overline{h} - rb)u = -h(1+qb).$$

Since $I = (u)$, there exists an element $\mu_0 + \mu_1 b \in A$ such that

$$1 + qb = (\mu_0 + \mu_1 b)u.$$

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Then (2.9) can be expressed in the form

$$[1 - \delta b(-\overline{h} - rb)]u = -h(\mu_0 + \mu_1 b)u.$$

But the algebra A contains no zero divisors, so we have

$$1 - \delta\overline{r} + \delta hb = -h(\mu_0 + \mu_1 b),$$

or $1 - \delta\overline{r} = -h\mu_0$, that is

$$1 - \delta\overline{r} \equiv 0 (\mathrm{mod}\ h). \tag{2.10}$$

We show that congruence (2.10) gives rise to a contradiction. Indeed, applying (2.6) we have

$$f(a) = 1 - \delta\overline{r} = -\delta(\alpha + \alpha^2 + \alpha^3)a^2 + \delta(\alpha + \alpha^2 + \alpha^3)a - \delta(1 + \alpha^2 + \alpha^4) + 1.$$

We set

$$g(a) = a^2.h = \alpha^2 a - (\alpha + \alpha^3).$$

It is clear that h divides the element $1 - \delta\overline{r}$ if and only if $g(a)$ divides $f(a)$. Since $g(a) = \alpha^2[a - \alpha^{-1}(1 + \alpha^2)]$, the congruence $f(a) \equiv 0(\mathrm{mod}\ g(a))$ is true if and only if $f[\alpha^{-1}(1 + \alpha^2)] = 0$. It is easy to calculate that this is not true for the real value of α mentioned above. This proves that the element $u = -h + rb$ does not generate the left ideal I.

Now let us assume that I is a principal left ideal generated by an element $z = s_0 + s_1 b$. Then it follows that

$$u = -h + rb = (\lambda_0 + \lambda_1 b)(s_0 + s_1 b) \tag{2.11}$$

for some element $\lambda_0 + \lambda_1 b \in A$. (2.11) implies the equation

$$N(u) = N(\lambda_0 + \lambda_1 b).N(z), \tag{2.12}$$

that is, the element $N(z)$ divides $N(u)$. On the other hand, $N(z) \in I \cap \mathbf{R}(a)$, that is, $N(z) = m.p$ for some $m \in \mathbf{R}(a)$. Then it follows from (2.12) that $N(u) = N(\lambda_0 + \lambda_1 b).m.p$. Since $\delta N(u) = p$, we obtain that $\lambda_0 + \lambda_1 b$ is an invertible element.

By (2.11) this implies that the elements u and z are associates. This is in contradiction with the fact that element u does not generate the left ideal I. ∎

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