Canad. Math. Bull. Vol. 62 (3), 2019 pp. 479–489
http://dx.doi.org/10.4153/S0008439518000097


# The Quotient Problem for Entire Functions

Ji Guo

*Abstract.* Let $\{\mathbf{F}(n)\}_{n\in\mathbb{N}}$ and $\{\mathbf{G}(n)\}_{n\in\mathbb{N}}$ be linear recurrence sequences. It is a well-known Diophantine problem to determine the finiteness of the set $\mathcal{N}$ of natural numbers such that their ratio $\mathbf{F}(n)/\mathbf{G}(n)$ is an integer. In this paper we study an analogue of such a divisibility problem in the complex situation. Namely, we are concerned with the divisibility problem (in the sense of complex entire functions) for two sequences $F(n) = a_0 + a_1 f_1^n + \cdots + a_l f_l^n$ and $G(n) = b_0 + b_1 g_1^n + \cdots + b_m g_m^n$, where the $f_i$ and $g_j$ are nonconstant entire functions and the $a_i$ and $b_j$ are non-zero constants except that $a_0$ can be zero. We will show that the set $\mathcal{N}$ of natural numbers such that $F(n)/G(n)$ is an entire function is finite under the assumption that $f_1^{i_1}\cdots f_l^{i_l} g_1^{j_1}\cdots g_m^{j_m}$ is not constant for any non-trivial index set $(i_1,\ldots,i_l,j_1,\ldots,j_m)\in\mathbb{Z}^{l+m}$.

## 1 Introduction

A sequence of complex numbers $\{\mathbf{G}(n)\}_{n\in\mathbb{N}}$ is called a *linear recurrence* if there exist complex numbers $c_0,\ldots,c_{k-1}$ $(k \geq 1)$ such that $\mathbf{G}(n+k) = c_0\mathbf{G}(n) + \cdots + c_{k-1}\mathbf{G}(n+k-1)$ for all $n \in \mathbb{N}$. This is equivalent to a unique expression

$$\mathbf{G}(n) = \sum_{i=1}^{r} g_i(n)\alpha_i^n, \quad \text{for all } n \in \mathbb{N},$$

with nonzero polynomials $g_i \in \mathbb{C}[X]$ and distinct nonzero $\alpha_i \in \mathbb{C}^*$. The recurrence is called *simple* when all the $g_i(n)$ are constant. The "Hadamard-quotient theorem", a conjecture of Pisot, was solved by van der Poorten. (See [7,9] for a detailed argument and see [2,12] for an overview of the existing improvements.) We now state a simple version of the theorem: *if* $\mathbf{F}(n)$ *and* $\mathbf{G}(n)$ *are linear recurrences such that their ratio* $\mathbf{F}(n)/\mathbf{G}(n)$ *is an integer for all large* $n \in \mathbb{N}$, *then* $\mathbf{F}(n)/\mathbf{G}(n)$ *is itself a linear recurrence.* In particular, it implies that: *given integers* $a, b > 1$, *if* $a^n - 1$ *divides* $b^n - 1$ *for all large positive integers* $n$, *then* $b$ *is a power of* $a$. The following recent result can be viewed as an analogue in the complex situation, while a non-Archimedean analogue was established in [6].

***Theorem 1.1*** ([4]) *Let* $f$ *and* $g$ *be entire functions on* $\mathbb{C}$. *Then* $T_f(r) \asymp T_g(r)$, *and there exists an infinite set* $\mathcal{N}$ *of positive integers such that* $g^n - 1 | f^n - 1$ *for each* $n \in \mathcal{N}$ *if and only if* $f = \xi \cdot g^\ell$, *where* $\ell$ *is a positive integer and* $\xi$ *is a* $d$*-th root of unity with* $d = \gcd\{n : n \geq 2 \text{ and } n \in \mathcal{N}\}$.

Here, $T_f(r)$ denotes the Nevanlinna characteristic function (see Section 2). The notation $T_f(r) \asymp T_g(r)$ means that there exist positive numbers $a, b$ such that $aT_f(r) < T_g(r) < bT_f(r)$ for $r$ sufficiently large.

Received by the editors May 8, 2018; revised October 8, 2018.
Published online on Cambridge Core April 17, 2019.
AMS subject classification: **30D30,** 32H30, 11J97.
Keywords: quotient problem, entire functions, second main theorem, linear recurrence.

Our purpose is to show a multi-variable version of the above theorem as follows.

***Theorem 1.2*** *Let $l, m \geq 1$ be two positive integers. Let $f_1, \dots, f_l$ and $g_1, \dots, g_m$ be nonconstant entire functions such that*

$$\max_{i=1,\dots,l} T_{f_i}(r) \asymp \max_{j=1,\dots,m} T_{g_j}(r).$$

*Let*

$$F(n) = a_0 + a_1 f_1^n + \cdots + a_l f_l^n \quad \text{and} \quad G(n) = b_0 + b_1 g_1^n + \cdots + b_m g_m^n,$$

*where $a_0 \in \mathbb{C}$ and $a_1, \dots, a_l, b_0, \dots, b_m \in \mathbb{C}^*$.*

(i) *If the ratio $F(n)/G(n)$ is an entire function for infinitely many $n \in \mathbb{Z}^+$, or*

(ii) *$f_1, \dots, f_l$ and $g_1, \dots, g_m$ are all units,* i.e.*, entire functions without zero, and if the ratio $F(1)/G(1)$ is an entire function,*

*then $f_1^{i_1} \cdots f_l^{i_l} g_1^{j_1} \dots g_m^{j_m} \in \mathbb{C}$ for some $(i_1, \dots, i_l, j_1, \dots, j_m) \neq (0, \dots, 0) \in \mathbb{Z}^{l+m}$.*

***Remark*** This growth condition is essential as, for part (ii), there are examples satisfying $g - 1 | f - 1$, such as $g(z) = \exp(2\pi\sqrt{-1}z)$ and $f(z) = \exp(2\pi\sqrt{-1}p(z))$, where $p(z)$ is a polynomial of degree at least 2 with coefficients in $\mathbb{Z}$, while $\lim_{r\to\infty} T_f(r)/T_g(r) = \infty$.

One can also view this as a complex analogue of the Hadamard quotient theorem. Indeed, our proof is inspired by the article [2], where Corvaja and Zannier proved a stronger version of the Hadamard quotient theorem with a sophisticated application of Schmidt's subspace theorem. However, their methods applied to the complex case via Vojta's dictionary of Diophantine geometry and Nevanlinna theory ([11] or [8]) can only cover the case where the $f_i$ and $g_j$ are units, *i.e.*, they are entire functions with no zeros. Moreover, a stronger statement, with the aid of Borel's lemma, can be formulated in this situation as the second part of Theorem 1.2. To deal with the more general situation; *i.e.*, allowing the entire functions $f_i$ and $g_j$ having zeros, we need to use a general version of the Navanlinna second main theorem (see Theorem 2.5) with a ramification term to derive an estimate with truncated counting function. This part of the argument only works for the complex case, since the corresponding result in the number field situation is a special case of the yet to be proven Vojta's conjecture ([11]).

## 2 Preliminaries

Now let us recall some notation, definitions, and basic results in Nevanlinna theory. Refer to [5] or [8] for details.

Let $f$ be a meromorphic function, and let $z \in \mathbb{C}$ be a complex number. Denote

$$v_z(f) := \mathrm{ord}_z(f), \quad v_z^+(f) := \max\{0, v_z(f)\}, \quad v_z^-(f) := -\min\{0, v_z(f)\}.$$

Let $n_f(\infty, r)$ denote the number of poles of $f$ in $\{z : |z| \leq r\}$, counting multiplicity. The *counting function* of $f$ at $\infty$ is defined by

$$N_f(\infty,r) := \int_0^r \frac{n_f(\infty,t)-n_f(\infty,0)}{t}\,dt + n_f(\infty,0)\log r$$
$$= \sum_{0<|z|\le r} \nu_z^-(f)\log\left|\frac{r}{z}\right| + \nu_0^-(f)\log r.$$

Then the *counting function* $N_f(a,r)$ for $a \in \mathbb{C}$ is defined as

$$N_f(a,r) := N_{1/(f-a)}(\infty,r).$$

The *proximity function* $m_f(\infty,r)$ is defined by

$$m_f(\infty,r) := \int_0^{2\pi} \log^+|f(re^{i\theta})|\frac{d\theta}{2\pi},$$

where $\log^+ x = \max\{0,\log x\}$ for $x \ge 0$. For any $a \in \mathbb{C}$, the *proximity function* $m_f(a,r)$ is defined by

$$m_f(a,r) := m_{1/(f-a)}(\infty,r).$$

The *characteristic function* is defined by

$$T_f(r) := m_f(\infty,r) + N_f(\infty,r).$$

It satisfies the inequalities $T_{fg}(r) \le T_f(r) + T_g(r) + O(1)$ and $T_{f+g}(r) \le T_f(r) + T_g(r) + O(1)$ for any entire functions $f$ and $g$. It also satisfies the First Main Theorem as follows.

***Theorem 2.1*** *Let $f$ be a non-constant meromorphic function on $\mathbb{C}$. Then for every $a \in \mathbb{C}$ and for any positive real number $r$,*

$$m_f(a,r) + N_f(a,r) = T_f(r) + O(1),$$

*where $O(1)$ is independent of $r$.*

The above theorem can be deduced from the following version of Jensen's formula.

***Theorem 2.2*** *Let $f$ be a meromorphic function on $\{z : |z| \le r\}$ that is not the zero function. Then*

$$\int_0^{2\pi} \log|f(re^{i\theta})|\frac{d\theta}{2\pi} = N_f(r,0) - N_f(r,\infty) + \log|c_f|,$$

*where $c_f$ is the leading coefficient of $f$ expanded as Laurent series in $z$,* i.e., $f = c_f z^m + \cdots$ *with $c_f \ne 0$.*

For a holomorphic map $\mathbf{f} : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$, we take a reduced form of $\mathbf{f} = (f_0,\dots,f_n)$; *i.e.*, $f_0,\dots,f_n$ are entire functions on $\mathbb{C}$ without common zeros. The Nevanlinna–Cartan *characteristic function* $T_{\mathbf{f}}(r)$ is defined by

$$T_{\mathbf{f}}(r) = \int_0^{2\pi} \log\|\mathbf{f}(re^{i\theta})\|\frac{d\theta}{2\pi},$$

where $\|\mathbf{f}(z)\| = \max\{|f_0(z)|,\dots,|f_n(z)|\}$. This definition is independent, up to an additive constant, of the choice of the reduced representation of $\mathbf{f}$. From the definition of the characteristic functions, we can derive the following proposition.

https://doi.org/10.4153/S0008439518000097 Published online by Cambridge University Press

***Proposition 2.3*** ([8, Theorem A3.1.2]) *Let* $\mathbf{f} = (f_0, \ldots, f_n) : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ *be a linearly non-degenerate holomorphic curve (*i.e., *the image of* $\mathbf{f}$ *is not contained in any proper linear subspaces) of a reduced form. Then*

$$T_{f_j/f_i}(r) + O(1) \leq T_{\mathbf{f}}(r) \leq \sum_{j=0}^{n} T_{f_j/f_0}(r) + O(1).$$

Let $H$ be a hyperplane in $\mathbb{P}^n(\mathbb{C})(n > 0)$ and let $a_0X_0 + \cdots + a_nX_n$ be a linear form defining it. Let $P = [x_0 : \cdots : x_n] \in \mathbb{P}^n(\mathbb{C}) \setminus H$ be a point. The Weil function $\lambda_H : \mathbb{P}^n(\mathbb{C}) \setminus H \to \mathbb{R}$ is defined as

$$\lambda_H(P) = -\log \frac{|a_0x_0 + \cdots + a_nx_n|}{\max\{|x_0|, \ldots, |x_n|\}}.$$

This definition depends on $a_0, \ldots, a_n$, but only up to an additive constant and is independent of the choice of homogeneous coordinates for $P$. The *proximity function* of $\mathbf{f}$ with respect to $H$ is defined by

$$m_{\mathbf{f}}(H, r) = \int_0^{2\pi} \lambda_H\big(\mathbf{f}(re^{i\theta})\big)\frac{d\theta}{2\pi}.$$

Let $\mathbf{n}_{\mathbf{f}}(H, r)$ (resp. $\mathbf{n}_{\mathbf{f}}^{(Q)}(H, r)$) be the number of zeros of $a_0f_0 + \cdots + a_nf_n$ in the disk $|z| \leq r$, counting multiplicity (resp. ignoring multiplicity bigger than $Q \in \mathbb{N}$). The integrated counting function with respect to $H$ is defined by

$$N_{\mathbf{f}}(H, r) = \int_0^r \frac{\mathbf{n}_{\mathbf{f}}(H, t) - \mathbf{n}_{\mathbf{f}}(H, 0)}{t}dt + \mathbf{n}_{\mathbf{f}}(H, 0)\log r,$$

and the $Q$-truncated counting function with respect to $H$ is defined by

$$N_{\mathbf{f}}^{(Q)}(H, r) = \int_0^r \frac{\mathbf{n}_{\mathbf{f}}^{(Q)}(H, t) - \mathbf{n}_{\mathbf{f}}^{(Q)}(H, 0)}{t}dt + \mathbf{n}_{\mathbf{f}}^{(Q)}(H, 0)\log r.$$

The following general second main theorem with ramification term is due to Vojta ([10, Theorem 1]).

***Theorem 2.4*** *Let* $\mathbf{f} : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ *be a holomorphic curve whose image is not contained in any proper subspaces and let* $(f_0, \ldots, f_n)$ *be a reduced form of* $\mathbf{f}$*. Let* $H_1, \ldots, H_q$ *be arbitrary hyperplanes in* $\mathbb{P}^n(\mathbb{C})$*. Denote by* $W(\mathbf{f})$ *the Wronskian of* $f_0, \ldots, f_n$*. Then for any* $\varepsilon > 0$*, we have*

$$\int_0^{2\pi} \max_K \sum_{k \in K} \lambda_{H_k}\big(\mathbf{f}(re^{i\theta})\big)\frac{d\theta}{2\pi} + N_{W(\mathbf{f})}(0, r) \leq_{\mathrm{exc}} (n + 1 + \varepsilon)T_{\mathbf{f}}(r),$$

*where the maximum is taken over all subsets* $K$ *of* $\{1, \ldots, q\}$ *such that* $H_k$ $(k \in K)$ *are in general position and* $\leq_{\mathrm{exc}}$ *means the estimate holds except for* $r$ *in a set of finite Lebesgue measure.*

We also need the following inequality with truncated counting functions.

***Lemma 2.5*** ([8, Lemma A3.2.1]) *Let* $\mathbf{f}:\mathbb{C}\to\mathbb{P}^n(\mathbb{C})$ *be a holomorphic curve whose image is not contained in any proper subspaces and let* $(f_0,\dots,f_n)$ *be a reduced form of* $\mathbf{f}$. *Let* $H_1,\dots,H_q$ *be the hyperplanes in* $\mathbb{P}^n$ *in general position. Then*

$$\sum_{j=1}^{q} N_{\mathbf{f}}(H_j,r)-N_{W(\mathbf{f})}(0,r)\le\sum_{j=1}^{q}N_{\mathbf{f}}^{(n)}(H_j,r).$$

Finally, we recall the following results of Green [3] (or see [5, Chapter VII, Theorem 4.1]) and Borel [1] (or see [5, Chapter VII, Theorem 1.1]).

***Lemma 2.6*** *Let* $f_0,\dots,f_n$ *be entire functions with no common zeros satisfying*

$$f_0^k+\cdots+f_n^k=0.$$

*Suppose none of the* $f_i$ *is* 0. *Define an equivalence relation:* $i\sim j$ *iff* $f_i/f_j$ *is constant. If* $k\ge n^2$, *then for each equivalence class* $S$, *we have*

$$\sum_{i\in S}f_i^k=0.$$

***Lemma 2.7*** *Let* $f_0,\dots,f_n$ *be units satisfying*

$$f_0+\cdots+f_n=0.$$

*Define an equivalence relation:* $i\sim j$ *iff* $f_i/f_j$ *is constant. Then for each equivalence class* $S$, *we have*

$$\sum_{i\in S}f_i=0.$$

## 3 Proof of the Main Theorem

Our proof is based on the method used to show [2, Proposition 2.1]. As mentioned in the introduction, we need extra work in order to cover the non-units case.

**Proof of Theorem 1.2** Assume that $f_1,\dots,f_l,g_1,\dots,g_m$ are entire functions such that $f_1^{i_1}\cdots f_l^{i_l}g_1^{j_1}\cdots g_m^{j_m}$ is not constant for any non-trivial index set $(i_1,\dots,i_l,j_1,\dots,j_m)\in\mathbb{Z}^{l+m}$. Suppose that

$$q(n):=\frac{F(n)}{G(n)}=\frac{a_0+a_1f_1^n+\cdots+a_lf_l^n}{b_0+b_1g_1^n+\cdots+b_mg_m^n}$$

is an entire function for infinitely many $n$. Since

$$\max_{1\le i\le l}T_{f_i}(r)\asymp\max_{1\le j\le m}T_{g_j}(r),$$

there exist two positive constants $a,b$ such that

$$a\max_{1\le j\le m}T_{g_j}(r)\ge\max_{1\le i\le l}T_{f_i}(r)\ge b\max_{1\le j\le m}T_{g_j}(r).$$

By the pigeonhole principle, there exists a subset $R$ of $\mathbb{R}^+$, of infinite Lebesgue measure, such that $\max_{1\le j\le m}T_{g_j}(r)=T_{g_k}(r)$ for $r\in R$ and for some $k\in\{1,\dots,m\}$. By rearranging the indices, we can assume that $k=1$. Thus,

$$T_{f_i}(r)\le\max_{1\le i\le l}T_{f_i}(r)\le a\max_{1\le j\le m}T_{g_j}(r)=aT_{g_1}(r)$$

https://doi.org/10.4153/S0008439518000097 Published online by Cambridge University Press

for $1 \le i \le l$ and $r \in R$. Without loss of generality, we can assume that $a > 1$. Then for $r \in R$,

$$T_{f_i}(r) \le aT_{g_1}(r),\ 1 \le i \le l,\ \text{and } T_{g_j}(r) \le aT_{g_1}(r),\ 1 \le j \le m. \tag{3.1}$$

Fix two positive integers $s, t$ that will be determined later. Let

$$G_1(n) = G(n) - b_1 g_1^n.$$

Then

$$G_1(n)^s q(n) = F(n)\Big(\sum_{k=0}^{s-1}\binom{s}{k}G(n)^{s-1-k}(-b_1 g_1^n)^k\Big) + (-b_1 g_1^n)^s q(n). \tag{3.2}$$

We will use the following notation throughout the proof. Denote

$$\mathbf{c} := (c_2,\dots,c_m) \in (\mathbb{Z}_{\ge 0})^{m-1} \quad \text{and} \quad \mathbf{d} := (d_1,\dots,d_m) \in (\mathbb{Z}_{\ge 0})^m.$$

Let $|\mathbf{c}| := c_2 + \cdots + c_m$ and $|\mathbf{d}| = d_1 + \cdots + d_m$. We use the graded lexicographic order to arrange the index sets $\mathbf{c} \in (\mathbb{Z}_{\ge 0})^{m-1}$ and $\mathbf{d} \in (\mathbb{Z}_{\ge 0})^m$; *i.e.*, $\mathbf{c}_i > \mathbf{c}_j$ if and only if $|\mathbf{c}_i| > |\mathbf{c}_j|$ or $|\mathbf{c}_i| = |\mathbf{c}_j|$ and the left-most nonzero entry of $\mathbf{c}_i - \mathbf{c}_j$ is positive. Let $g_2^{nc_2}\cdots g_m^{nc_m}$ be abbreviated to $\underline{\alpha}(n)^{\mathbf{c}}$ and $g_1^{nd_1}\cdots g_m^{nd_m}$ to $\underline{\beta}(n)^{\mathbf{d}}$. For each $\mathbf{c}_i$ with $|\mathbf{c}_i| \le t$, we define

$$\varphi_{\mathbf{c}_i} := \bigg(G_1(n)^s q(n) - F(n)\Big(\sum_{k=0}^{s-1} G(n)^{s-1-k}(-b_1 g_1^n)^k\Big)\bigg)\underline{\alpha}(n)^{\mathbf{c}_i}. \tag{3.3}$$

Note that the number of such $\varphi_{\mathbf{c}}$ is

$$M = \binom{m-1+t}{m-1}.$$

Observe that every $\varphi_{\mathbf{c}_i}$ is a linear combination of $\underline{\alpha}(n)^{\mathbf{c}}q(n)$ where $|\mathbf{c}| \le t+s$ and of the forms $\underline{\beta}(n)^{\mathbf{d}} f_i^n$ with $|\mathbf{d}| \le s+t$ and $0 \le i \le l$ (letting $f_0 = 1$). Thus the number of such forms $\underline{\alpha}(n)^{\mathbf{c}}q(n)$ is

$$N_1 := \binom{m-1+t+s}{m-1}.$$

Suppose that the number of $\mathbf{d}$ with $|\mathbf{d}| \le t+s-1$ is $N_2$. Denote $N := N_1 + (l+1)N_2$.

Define $x_i(n) := \underline{\alpha}(n)^{\mathbf{c}_i}q(n)$ for $i = 1,\dots,N_1$ and $x_{N_1+iN_2+j}(n) := f_i^n \underline{\beta}(n)^{\mathbf{d}_j}$ for $i = 0,\dots,l$, and $j = 1,\dots,N_2$. Since $G(n)$ has a non-zero constant term, the graded lexicographic order imposed on $\mathbf{d}$ implies that $\mathbf{d}_1 = (0,\dots,0) \in \mathbb{Z}^m$. Then the $x_i(n)$ can be expressed as

$$\begin{aligned}\mathbf{x}(n) &:= \big(x_1(n),\dots,x_N(n)\big) \\ &= \big(\underline{\alpha}(n)^{\mathbf{c}_1}q(n),\dots,\underline{\alpha}(n)^{\mathbf{c}_{N_1}}q(n), 1, \underline{\beta}(n)^{\mathbf{d}_2},\dots,\underline{\beta}(n)^{\mathbf{d}_{N_2}}, \\ &\qquad f_1^n, f_1^n\underline{\beta}(n)^{\mathbf{d}_2},\dots, f_l^n\underline{\beta}(n)^{\mathbf{d}_{N_2}}\big).\end{aligned} \tag{3.4}$$

We note that $\mathbf{x}(n)$ is a holomorphic map from $\mathbb{C}$ to $\mathbb{P}^{N-1}$ and $(x_1(n),\dots,x_N(n))$ is a reduced form, since $x_1(n),\dots,x_N(n)$ are entire functions and $x_{N_1+1}(n) = 1$. Moreover, we claim that this map is not contained in any proper linear subspace if $n$ is sufficiently large. If the claim does not hold for a large enough $n$, there exist constants $u_1, \dots, u_{N_1}, v_{0,1}, v_{0,2}, \dots, v_{l,N_2}$ in $\mathbb{C}$ that are not all zero such that

$$\sum_{i=1}^{N_1} u_i \underline{\alpha}(n)^{\mathbf{c}_i} q(n) + \sum_{i=0}^{l}\sum_{j=1}^{N_2} v_{i,j}\underline{\beta}(n)^{\mathbf{d}_j} f_i^n = 0,$$

and hence

$$\sum_{i=1}^{N_1} u_i \underline{\alpha}(n)^{\mathbf{c}_i}(a_0 + a_1 f_1^n + \cdots + a_l f_l^n) + \Big(\sum_{i=0}^{l}\sum_{j=1}^{N_2} v_{i,j}\underline{\beta}(n)^{\mathbf{d}_j} f_i^n\Big)(b_0 + b_1 g_1^n + \cdots + b_m g_m^n) = 0. \tag{3.5}$$

If $v_{0,1}, \dots, v_{l,N_2}$ are all zero, then by Lemma 2.6, for $n \geq (l+1)^2 N_1^2$, there exist two distinct terms $f_i^n \underline{\alpha}(n)^{\mathbf{c}_j}$ and $f_{i'}^n \underline{\alpha}(n)^{\mathbf{c}_{j'}}$ such that their quotient

$$\frac{f_i^n \underline{\alpha}(n)^{\mathbf{c}_j}}{f_{i'}^n \underline{\alpha}(n)^{\mathbf{c}_{j'}}} = f_i^n f_{i'}^{-n} \underline{\alpha}(n)^{\mathbf{c}_j - \mathbf{c}_{j'}}$$

is a constant, which contradicts the assumption that $f_1^{i_1}\cdots f_l^{i_l} g_1^{j_1}\cdots g_m^{j_m}$ is not constant for any non-trivial index set $(i_1, \dots, i_l, j_1, \dots, j_m) \in \mathbb{Z}^{l+m}$. Therefore, the set $\{\mathbf{d} : v_{i,j} \neq 0 \text{ for some } 0 \leq i \leq l\}$ is non-empty and it contains a maximal element with respect to the graded lexicographic order, denoted by $\mathbf{d}_k$. Then $v_{i,k} \neq 0$ for some $0 \leq i \leq l$. Expanding (3.5), we find the coefficient of $f_i^n \underline{\beta}(n)^{\mathbf{d}_k} g_1^n$ is $v_{i,k} \neq 0$. By Lemma 2.6 again, there exists another term with nonzero coefficient in (3.5), say $f_{i'}^n \underline{\beta}(n)^{\mathbf{d}_{k'}} g_{j'}^n$ or $f_{i'}^n \underline{\alpha}(n)^{\mathbf{c}_{k'}}$, such that

$$\frac{f_i^n \underline{\beta}(n)^{\mathbf{d}_k} g_1^n}{f_{i'}^n \underline{\beta}(n)^{\mathbf{d}_{k'}} g_{j'}^n} \quad \text{or} \quad \frac{f_i^n \underline{\beta}(n)^{\mathbf{d}_k} g_1^n}{f_{i'}^n \underline{\alpha}(n)^{\mathbf{c}_{k'}}}$$

is a constant for $n \geq n_1 := (l+1)^2(N_1 + N_2(m+1))^2$. However, the first quotient is not constant, since the graded lexicographic order associated with the index set of $\beta(n)^{\mathbf{d}_k} g_1^n$ is bigger than the one with $\beta(n)^{\mathbf{d}_{k'}} g_{j'}^n$; the second quotient is not constant either, since $\underline{\alpha}(n)^{\mathbf{c}_{k'}}$ is a product of powers of $g_2, \dots, g_m$.

We will now construct a set of hyperplanes in order to apply Theorem 2.4. We first let

$$H_i := \{X_{i-1} = 0\} \quad \text{for } i = 1, \dots, N$$

be the coordinate hyperplanes in $\mathbb{P}^{N-1}$. Next we observe that, as $G_1(n) = b_0 + b_2 g_2^n + b_3 g_3^n + \cdots + b_m g_m^n$ with $b_0 \neq 0$, the graded lexicographic order imposed on the $\mathbf{c}$ and the choice of the $x_i(n)$ give the following expression of $\phi_{\mathbf{c}_i}$ for $1 \leq i \leq M$:

$$\varphi_{\mathbf{c}_i} = b_0^s x_i(n) + A_{i,i+1}x_{i+1}(n) + \cdots + A_{i,N}x_N(n) \tag{3.6}$$

for some $A_{1,2}, \dots, A_{M,N} \in \mathbb{C}$. Then we let

$$H_{N+i} : b_0^s X_{i-1} + A_{i,i+1}X_i + \cdots + A_{i,N}X_{N-1} = 0, \text{ for } i = 1, \dots, M \tag{3.7}$$

https://doi.org/10.4153/S0008439518000097 Published online by Cambridge University Press

be hyperplanes according to the expression (3.6) of $\phi_{\mathbf{c}_i}$. It is clear that the hyperplanes $H_{M+1},\dots,H_{N+M}$ in $\mathbb{P}^{N-1}$ are in general position. In addition, (3.2) and (3.3) implies that

$$\varphi_{\mathbf{c}_i}=(-b_1g_1^n)^s q(n)\underline{\alpha}(n)^{\mathbf{c}_i}=(-b_1)^s x_i(n)g_1^{sn} \tag{3.8}$$

for $i=1,\dots,M$.

Now we can apply Theorem 2.4, the general second main theorem, to the linearly non-degenerate map $\mathbf{x}(n)$ with the hyperplanes $H_1,\dots,H_{N+M}$. Then for any $\varepsilon>0$,

$$\int_0^{2\pi}\max_J\sum_{j\in J}\lambda_{H_j}\big(\mathbf{x}(n)(re^{\sqrt{-1}\theta})\big)\frac{d\theta}{2\pi}+N_W(0,r)\leq_{\text{exc}}(N+\varepsilon)T_{\mathbf{x}(n)}(r), \tag{3.9}$$

where $J$ runs over the subsets of $\{1,\dots,N+M\}$ such that the hyperplanes $H_j(j\in J)$ are in general position and $W$ is the Wronskian of the reduced form of $\mathbf{x}(n)$ in (3.4).

We now proceed to derive a lower bound for the left-hand side of (3.9). For any meromorphic function $\xi$, denote

$$|\xi|_{r,\theta}:=|\xi(re^{\sqrt{-1}\theta})|.$$

For the holomorphic map $\mathbf{x}(n)$ and the hyperplanes $H_1,\dots,H_{N+M}$, we claim that the following inequality holds:

$$\max_J\sum_{j\in J}\lambda_{H_j}\big(\mathbf{x}(n)(re^{\sqrt{-1}\theta})\big)\geq M\log^+\frac{1}{|g_1|_{r,\theta}^{sn}}+\sum_{i=1}^N\log\frac{1}{|x_i(n)|_{r,\theta}} \\ +N\log\|\mathbf{x}(n)\|_{r,\theta}+O(1), \tag{3.10}$$

where $J$ runs over the subsets of $\{1,\dots,N+M\}$ such that the hyperplanes $H_j(j\in J)$ are in general position and

$$\|\mathbf{x}(n)\|_{r,\theta}:=\max_{1\leq i\leq N}\{|x_i(n)|_{r,\theta}\}.$$

For $\theta\in S_r^+:=\{\theta:|g_1|_{r,\theta}\geq 1\}$, we consider

$$\sum_{i=1}^N\lambda_{H_i}\big(\mathbf{x}(n)(re^{\sqrt{-1}\theta})\big)=\sum_{i=1}^N\log\frac{1}{|x_i(n)|_{r,\theta}}+N\log\|\mathbf{x}(n)\|_{r,\theta}.$$

Since $H_1,\dots,H_N$ are in general position and $\log^+(1/|g_1|_{r,\theta})=0$ for $\theta\in S_r^+$, it implies (3.10). For $\theta\in S_r^-:=\{\theta:|g_1|_{r,\theta}<1\}$, we consider

$$\begin{aligned}\sum_{i=1}^N\lambda_{H_{M+i}}&\big(\mathbf{x}(n)(re^{\sqrt{-1}\theta})\big)\\ &=\sum_{i=1}^M\log\frac{\|\mathbf{x}(n)\|_{r,\theta}}{|\varphi_{\mathbf{c}_i}|_{r,\theta}}+\sum_{i=M+1}^N\log\frac{\|\mathbf{x}(n)\|_{r,\theta}}{|x_i(n)|_{r,\theta}}\\ &=\sum_{i=1}^M\log\frac{|x_i(n)|_{r,\theta}}{|\varphi_{\mathbf{c}_i}|_{r,\theta}}+\sum_{i=1}^N\log\frac{1}{|x_i(n)|_{r,\theta}}+N\log\|\mathbf{x}(n)\|_{r,\theta}\\ &=M\log^+\frac{1}{|g_1|_{r,\theta}^{sn}}+\sum_{i=1}^N\log\frac{1}{|x_i(n)|_{r,\theta}}+N\log\|\mathbf{x}(n)\|_{r,\theta}+O(1),\end{aligned} \tag{3.11}$$

https://doi.org/10.4153/S0008439518000097 Published online by Cambridge University Press

where the last equality follows from (3.8). Since the hyperplanes $H_{M+1}, \dots, H_{N+M}$ are in general position, (3.11) implies the inequality (3.10) as well. Integrating (3.10) over $d\theta$ from 0 to $2\pi$, we derive from Theorem 2.2 and the definition of the proximity and characteristic functions that

$$\begin{aligned}(3.12)\qquad &\int_0^{2\pi} \max_J \sum_{j\in J} \lambda_{H_j}(\mathbf{x}(n))\frac{d\theta}{2\pi}\\ &\geq M m_{g_1^{sn}}(0,r) - \sum_{i=1}^{N} N_{x_i(n)}(0,r) + N T_{\mathbf{x}(n)}(r) + O(1)\\ &= M T_{g_1^{sn}}(r) - M N_{g_1^{sn}}(0,r) - \sum_{i=1}^{N} N_{x_i(n)}(0,r) + N T_{\mathbf{x}(n)}(r) + O(1)\\ &= M T_{g_1^{sn}}(r) - \sum_{i=1}^{N} N_{\mathbf{x}(n)}(H_{M+i},r) + N T_{\mathbf{x}(n)}(r) + O(1),\end{aligned}$$

where the second equation follows by Theorem 2.1, and the last one is due to the identification

$$N_{g_1^{sn}}(0,r) + N_{x_i(n)}(0,r) = N_{x_i(n)g_1^{sn}}(0,r) = N_{\mathbf{x}(n)}(H_{N+i},r)$$

by (3.6), (3.7), and (3.8) for $i = 1, \dots, M$. We now use Lemma 2.5 to obtain the following inequality:

$$(3.13)\qquad \sum_{i=1}^{N} N_{\mathbf{x}(n)}(H_{M+i},r) - N_W(0,r) + O(1) \leq \sum_{i=1}^{N} N^{(N-1)}_{\mathbf{x}(n)}(H_{M+i},r).$$

Putting together (3.9), (3.12), and (3.13), we conclude that

$$(3.14)\qquad M T_{g_1^{sn}}(r) - \sum_{i=1}^{N} N^{(N-1)}_{\mathbf{x}(n)}(H_{M+i},r) \leq_{\text{exc}} \varepsilon T_{\mathbf{x}(n)}(r) + O(1).$$

Since the inequality holds except for $r \in \mathbb{R}^+$ in a set of finite Lebesgue measure, we can assume that it holds for all $r \in R$ by shrinking $R$. By the property of characteristic function, it is easy to obtain

$$\begin{aligned}&T_{x_i(n)}(r) \leq a(s+t)T_{g_1^n}(r) + T_{q(n)}(r) &&\text{for } 1 \leq i \leq N_1,\\ &T_{x_i(n)}(r) \leq a(s+t+1)T_{g_1^n}(r) &&\text{for } N_1+1 \leq i \leq N.\end{aligned}$$

Then by Proposition 2.3, (3.1), and (3.4), for $r \in R$, we have

$$\begin{aligned}(3.15)\qquad T_{\mathbf{x}(n)}(r) &\leq \sum_{j=1}^{N} T_{x_j(n)}(r)\\ &\leq N(s+t+1)aT_{g_1^n}(r) + N_1(T_{F(n)} + T_{G(n)}) + O(1)\\ &\leq N(s+t+1)aT_{g_1^n}(r) + N_1 a(l+m)T_{g_1^n}(r) + O(1)\\ &= a(N(s+t+1) + N_1(l+m))T_{g_1^n}(r) + O(1).\end{aligned}$$

On the other hand, for $r \in R$, we have

$$
\begin{aligned}
(3.16)\qquad \sum_{i=1}^{N} N^{(N-1)}_{\mathbf{x}(n)}(H_{M+i}, r)
&\le N(N-1)\Big(\sum_{i=1}^{l} N_{f_i}(0,r) + \sum_{j=1}^{m} N_{g_j}(0,r)\Big) + N_1 N_{q(n)}(0,r) + O(1)\\
&\le N(N-1)\Big(\sum_{i=1}^{l} T_{f_i}(r) + \sum_{j=1}^{m} T_{g_j}(r)\Big) + N_1 N_{F(n)}(0,r) + O(1)\\
&\le N(N-1)a(l+m)T_{g_1}(r) + N_1 T_{F(n)}(r) + O(1)\\
&\le \frac{N(N-1)(l+m)a}{n} T_{g_1^n}(r) + N_1(T_{f_1^n}(r) + \cdots + T_{f_l^n}(r)) + O(1)\\
&\le \Big(aN_1 l + \frac{N(N-1)(l+m)a}{n}\Big) T_{g_1^n}(r) + O(1).
\end{aligned}
$$

Combining (3.14), (3.15), and (3.16), for $r \in R$, we have

$$
\begin{aligned}
(3.17)\qquad \Big(Ms - N_1 al - \frac{N(N-1)(l+m)a}{n}\Big) T_{g_1^n}(r) &\le_{\text{exc}}\\
&\varepsilon a(N(s+t+1) + N_1(l+m)) T_{g_1^n}(r) + O(1).
\end{aligned}
$$

Finally, we will choose our $s$, $t$, and $\varepsilon$ to derive a contradiction from the above inequality. First, we fix $s > al$. Since

$$
Ms = s\binom{m-1+t}{m-1} = \frac{s}{(m-1)!} t^{m-1} + o(t^{m-1})
$$

and

$$
N_1 al = al\binom{m-1+t+s}{m-1} = \frac{al}{(m-1)!} t^{m-1} + o(t^{m-1})
$$

can be regarded as polynomials in $t$, both with degrees $m-1$, and in which the leading coefficient of $Ms$ is larger than the one for $aN_1 l$, there exists a sufficiently large integer $t$ such that $Ms > N_1 al$. Then we can choose $\varepsilon$ satisfying

$$
0 < \varepsilon < \frac{Ms - aN_1 l}{a(N(s+t+1) + N_1(l+m))}.
$$

Consequently, since $g_1$ is nonconstant, $T_{g_1}(r)$ is not bounded, and we can deduce from (3.17) that

$$
n \le n_0 := \frac{N(N-1)(l+m)a}{Ms - aN_1 l - \varepsilon a(N(s+t+1) + N_1(l+m))}.
$$

In conclusion, if

$$
f_1^{i_1} \cdots f_l^{i_l} g_1^{j_1} \cdots g_m^{j_m}
$$

is not constant for any non-trivial index set $(i_1, \dots, i_l, j_1, \dots, j_m) \in \mathbb{Z}^{l+m}$, then the ratio $F(n)/G(n)$ is not an entire function for $n > \max\{n_0, n_1\}$, where $n_1 = (l+1)^2 (N_1 + N_2(m+1))^2$ is the number to assure that $\mathbf{x}(n)$ is linearly non-degenerate for $n \ge n_1$. ∎

For the second part of Theorem 1.2, we first notice that the expression for $x(1)$ in (3.4) is not contained in any proper linear subspace, by Borel's lemma (Lemma 2.7) if $f_1, \dots, f_l, g_1, \dots, g_m$ are units. Next, the condition that $f_1, \dots, f_l, g_1, \dots, g_m$ are units implies that the counting function in (3.16) is just zero. In this case, (3.17) becomes

$$MsT_{g_1}(r) \leq_{\text{exc}} \varepsilon a(N(s+t+1) + N_1(l+m))T_{g_1}(r) + O(1). \tag{3.18}$$

Finally, let $\varepsilon$ satisfy

$$0 < \varepsilon < \frac{Ms}{a(N(s+t+1) + N_1(l+m))}.$$

Since $g_1$ is nonconstant, for all $r$ large enough, we have

$$MsT_{g_1}(r) \geq \varepsilon a\Big(N(s+t+1) + N_1(l+m)\Big)T_{g_1}(r) + O(1),$$

which contradicts (3.18).

**Acknowledgments** This work was done as part of my PhD thesis, and I want to express my gratitude to my advisor Julie Tzu-Yueh Wang for her careful guidance and suggestions. I would also like to thank the referee for helpful comments on the first version of this article, and in particular, for pointing out a mistake in an example that I had previously constructed.

## References

[1] E. Borel, *Sur les zéros des fonctions entières*. Acta math. 20(1897), no. 1, 357–396. https://doi.org/10.1007/BF02418037.

[2] P. Corvaja and U. Zannier, *Finiteness of integral values for the ratio of two linear recurrences*. Invent. math. 149(2002), no. 2, 431–451. https://doi.org/10.1007/s002220200221.

[3] M. Green, *Some Picard theorems for holomorphic maps to algebraic varieties*. Amer. J. Math. 97(1975), no. 1, 43–75. https://doi.org/10.2307/2373660.

[4] J. Guo and J.T.-Y. Wang, *Asymptotic gcd and divisible sequences for entire functions*. Trans. Amer. Math. Soc., to appear.

[5] S. Lang, *Introduction to complex hyperbolic spaces*. Springer-Verlag, New York, 1987. https://doi.org/10.1007/978-1-4757-1945-1.

[6] H. Pasten and J.T.-Y. Wang, *GCD Bounds for analytic functions*. Int. Math. Res. Not. IMRN (2017), no. 1, 47–95. https://doi.org/10.1093/imrn/rnw028.

[7] A. J. van der Poorten, *Solution de la conjecture de Pisot sur le quotient de Hadamard de deux fractions rationnelles*. C. R. Acad. Sci. Paris 302(1988), 97–102.

[8] M. Ru, *Nevanlinna theory and its relation to Diophantine approximation*. World Scientific, Publishing Co., Inc., River Edge, NJ, 2001. https://doi.org/10.1142/9789812810519.

[9] R. Rumely, *Notes on van der Poorten's proof of the Hadamard quotient theorem*. In: *Séminaire de Théorie des Nombres, Paris 1986–87*, Progr. Math., 75, Birkhäuser Boston, Boston, MA, 1988, pp. 349–409.

[10] P. Vojta, *On Cartan's theorem and Cartan's conjecture*. Amer. J. Math. 119(1997), no. 1, 1–17.

[11] P. Vojta, *Diophantine approximation and Nevanlinna theory*. In: *Arithmetic geometry*, Lecture Notes in Mathematics, 2009, Springer-Verlag, Berlin, 2011, pp. 111–224. https://doi.org/10.1007/978-3-642-15945-9_3.

[12] U. Zannier, *Diophantine equations with linear recurrences. An overview of some recent progress*. J. Théor. Nombres Bordeaux 17(2005), 423–435.

*Department of Mathematics, National Tsing Hua University, No. 101, Sec. 2, Kuang-Fu Road, Hsinchu, 30013, Taiwan*
*e-mail*: s104021881@m104.nthu.edu.tw