COMPOSITIO MATHEMATICA

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Compositio Math. **147** (2011), 1003–1021.

doi:10.1112/S0010437X11005379

FOUNDATION COMPOSITIO MATHEMATICA

The London Mathematical Society

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Abstract

For the dual pair $\mathrm{Sp}(n) \times O(m)$ with $m \leqslant n$, we prove an identity between a special value of a certain Eisenstein series and the regularized integral of a theta function. The proof uses the functional equation of the Eisenstein series and the regularized Siegel–Weil formula for $\mathrm{Sp}(n) \times O(2n+2-m)$. Analogous results for unitary and orthogonal groups are included.

Introduction

The classical Siegel–Weil formula, which was first proven by Siegel [Sie35, Sie51] and then generalized by Weil [Wei65], identifies a special value of a certain Eisenstein series with an integral of a theta function under the assumption that the Eisenstein series is absolutely convergent. Kudla and Rallis [KR88a, KR88b, KR94] extended this result for the dual pair $\mathrm{Sp}(n) \times O(V)$ beyond the convergent range. When $\dim V > n+1$, Kudla and Rallis [KR88b] proved that the Siegel–Weil formula is valid under Weil's condition for the absolute convergence of the theta integrals, where they showed that the Eisenstein series is holomorphic at the point in question, which lies within the right half-plane determined by the functional equation. When $\dim V \leqslant n+1$, Kudla and Rallis [KR94] constructed a natural extension of the divergent theta integral, the so-called regularized theta integral. The regularized Siegel–Weil formula identifies the regularized theta integral with the value of the Eisenstein series at its center of symmetry if $\dim V = n+1$, and with a residue of another Eisenstein series in the right half-plane if $\dim V \leqslant n$. Their regularized Siegel–Weil formula was refined by Ikeda [Ike96] and Ichino [Ich01], and was extended to the case of unitary groups by Tan [Tan98] and Ichino [Ich04], and to the case of orthogonal groups by Moeglin [Moe97] and Jiang and Soudry [JS07]. Ichino [Ich07] obtained an analogous result of Kudla and Rallis [KR88b] for unitary groups, following their techniques. When V is anisotropic, the theta integral is of course well defined and is related by [KR88a] to a holomorphic value of the Eisenstein series. The metaplectic anisotropic analogue was proven by Sweet [Swe90] in many cases.

For symplectic, unitary and orthogonal groups, the present paper proves that if the point at which the Eisenstein series is evaluated lies within the left half-plane, then the Siegel–Weil formula is valid with no restrictions. Here, the Eisenstein series attached to the standard sections coming from the Weil representation are holomorphic there, and our formula involves the regularized theta integral in the isotropic case.

Received 30 April 2010, accepted in final form 13 December 2010, published online 4 May 2011.
2010 Mathematics Subject Classification 11F27 (primary), 11F70 (secondary).
Keywords: Eisenstein series, Siegel–Weil formula, singular forms.
The author is supported by a Grant-in-Aid for JSPS Fellows.

For simplicity, we now confine ourselves to the case of symplectic groups. Let F be a number field and V a quadratic space over F of dimension m. Let $H = O(V)$ be the orthogonal group of V, $G = \mathrm{Sp}(n)$ the symplectic group of rank n and P the maximal parabolic subgroup of G with abelian unipotent radical. For a fixed non-trivial additive character ψ of $\mathbb{A}/F$, let $\omega = \omega_\psi$ denote the Weil representation of $\widetilde{G(\mathbb{A})}$ on the space $S(V^n(\mathbb{A}))$ of the Schwartz functions on $V^n(\mathbb{A})$ which correspond to polynomials in the Fock model at every archimedean place of F. We consider the linear action of $H(\mathbb{A})$ on the space of Schwartz functions on $V^n(\mathbb{A})$.

For $g \in \widetilde{G(\mathbb{A})}$ and $\Phi \in S(V^n(\mathbb{A}))$, let

$$f_\Phi^{(s)}(g) = |a(g)|^{s-s_0}\omega(g)\Phi(0),$$

where $s_0 = (m-n-1)/2$ and the quantity $|a(g)|$ is defined in the text. Then the Eisenstein series is defined, for $\Re s > (n+1)/2$, by

$$E(g; f_\Phi^{(s)}) = \sum_{\gamma \in P(F)\backslash G(F)} f_\Phi^{(s)}(\gamma g)$$

and needs to be defined by the meromorphic continuation otherwise.

Next consider the integral

$$I(g; \Phi) = \int_{H(F)\backslash H(\mathbb{A})} \Theta(g, h; \Phi)\, dh$$

of the theta function

$$\Theta(g, h; \Phi) = \sum_{x \in V^n(F)} \omega(g)\Phi(h^{-1}x),$$

where dh is an invariant measure on $H(F)\backslash H(\mathbb{A})$ normalized to have total volume 1 except in the split binary case. If $m \leqslant n+1$ and V is isotropic, then $I(g; \Phi)$ diverges for general Φ, but there exists a natural extension of this integral for any element of $S(V^n(\mathbb{A}))$ (see § 1.3). By abuse of notation, we write $I(g; \Phi)$ for this extension.

Theorem. *Assume that $m \leqslant n+1$. Let $\Phi \in S(V^n(\mathbb{A}))$.*

(i) *$E(g; f_\Phi^{(s)})$ is holomorphic at $s = s_0$.*

(ii) *If V is not a split binary quadratic space, then*

$$E(g; f_\Phi^{(s)})|_{s=s_0} = \varkappa I(g; \Phi)$$

with

$$\varkappa = \begin{cases} 1 & \text{if } m = 1, \\ 2 & \text{if } m > 1. \end{cases}$$

(iii) *If V is a split binary quadratic space, then*

$$E(g; f_\Phi^{(s)})|_{s=s_0} = 0, \quad \frac{\partial}{\partial s}E(g; f_\Phi^{(s)})\bigg|_{s=s_0} = 2I(g; \Phi).$$

As a consequence, $E(g; f_\Phi^{(s)})|_{s=s_0}$ for $m < n$ is a singular form. As was pointed out by Kudla and Rallis, the factor $\varkappa$ occurs in Siegel's earlier work (see the introduction of [KR88a] and [Sie35, Satz 2, p. 555]). Analogous results for unitary and orthogonal groups are included in Theorems 2.1 and 2.2. When V is even dimensional and anisotropic, this formula has already been proven by Kudla and Rallis [KR88a], but their proof is rather long and is not transparent. This paper gives a natural proof in a general situation. We should remark that when $m = n + 1$, the formula above has been known except in the split binary case (cf. [Kud97]).

We now give an outline of the proof. Suppose that $m \leqslant n$. Applying the functional equation, we have

$$E(g; f_\Phi^{(s)}) = \frac{a(s)}{b(s)} E(g; h^{(-s)}), \quad h^{(-s)} = \frac{b(s)}{a(s)} M(s) f_\Phi^{(s)},$$

where $a(s)$ and $b(s)$ are certain products of abelian L-functions and $M(s)$ is the global intertwining operator. In § 3, we will show that $h^{(s)}$ is holomorphic at $s = -s_0$. We should mention that when m is even, Kudla and Rallis have already proven this crucial fact in [KR88a] and that our proof depends heavily on their results. They assumed that V is anisotropic, but their calculations are completely local and therefore applicable to the general case, and even to incoherent Eisenstein series. Then it follows at once that $E(g; h^{(s)})$ has at most a simple pole at $s = -s_0$. Since $a(s)/b(s)$ has a simple zero at $s = s_0$ unless V is a split binary quadratic space, it turns out that $E(g; f_\Phi^{(s)})$ is holomorphic at $s = s_0$. If V is a split binary quadratic space, then $a(s)/b(s)$ has a double zero at $s = s_0$ and, as such, $E(g; f_\Phi^{(s)})|_{s=s_0}$ is identically zero. As will be seen in § 5, the residue of $E(g; h^{(s)})$ is related to $I(g; \Phi)$ by the regularized Siegel–Weil formula, so that the desired formula holds up to a constant. In § 6, we shall determine the constant of proportionality by comparing the Fourier coefficients of rank $m - 1$.

1. Preliminaries

1.1 We treat symplectic, unitary and orthogonal groups, which we refer to as Cases 1–3, respectively. Let $E = F$ be a number field and $x \mapsto \bar{x}$ the trivial automorphism of E in Cases 1 and 3, and let E be a quadratic extension of a number field F and $x \mapsto \bar{x}$ the non-trivial automorphism of E over F in Case 2. Let G be an algebraic group defined by

$$G = \left\{ g \in \mathrm{R}_{E/F}\mathrm{GL}_{2n} \;\middle|\; g \begin{pmatrix} 0 & \mathbf{1}_n \\ -\epsilon \mathbf{1}_n & 0 \end{pmatrix} {}^t\bar{g} = \begin{pmatrix} 0 & \mathbf{1}_n \\ -\epsilon \mathbf{1}_n & 0 \end{pmatrix} \right\},$$

where $\mathrm{R}_{E/F}\mathrm{GL}_{2n}$ denotes the general linear group over E, regarded as an algebraic group over F by restricting scalars, and $\epsilon = 1$ in Cases 1, 2 and $\epsilon = -1$ in Case 3. Put

$$S_n = \{ b \in \mathrm{R}_{E/F}\mathrm{M}_n \mid {}^t\bar{b} = \epsilon b \}.$$

Let $P = MN$ be the parabolic subgroup of G which has a Levi factor

$$M = \left\{ m(a) = \begin{pmatrix} a & 0 \\ 0 & {}^t\bar{a}^{-1} \end{pmatrix} \;\middle|\; a \in \mathrm{R}_{E/F}\mathrm{GL}_n \right\}$$

and the unipotent radical

$$N = \left\{ n(b) = \begin{pmatrix} \mathbf{1}_n & b \\ 0 & \mathbf{1}_n \end{pmatrix} \;\middle|\; b \in S_n \right\}.$$

Let $V = E^m$ be a space of column vectors equipped with an E-valued non-degenerate form $\varphi : V \times V \to E$ such that $\overline{\varphi(x, y)} = \epsilon \varphi(y, x)$ and $\varphi(xa, yb) = \bar{a} \varphi(x, y) b$ for $a, b \in E$ and $x, y \in V$. Let

$$H = \{ h \in \mathrm{R}_{E/F}\mathrm{GL}_m \mid \varphi(hx, hy) = \varphi(x, y) \text{ for all } x, y \in V \}$$

be the isometry group of φ. Throughout this paper, we put

$$\varrho = n + 1, \quad s_0 = (m - n - 1)/2, \qquad \text{(Case 1)}$$

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$$\varrho = n, \qquad s_0 = (m-n)/2, \tag{Case 2}$$
$$\varrho = n-1, \quad s_0 = (m-n+1)/2. \tag{Case 3}$$

Let $\mathbb{A} = \mathbb{A}_F$ be the ring of adeles of F and $\mathbb{A}_E$ that of E. Fix a non-trivial additive character $\psi = \bigotimes_v \psi_v$ of $\mathbb{A}/F$. For each place v of F, let F_v be the v-completion of F and $E_v = E \otimes_F F_v$. We write $V_v = V \otimes_F F_v$, $G_v = G(F_v)$ and $H_v = H(F_v)$, for simplicity. Let $\mathfrak{o}_E$ be the ring of integers of E and, for each finite place v of F, let $\mathfrak{o}_{E_v}$ be the closure of $\mathfrak{o}_E$ in E_v and put $K_v = G_v \cap \mathrm{GL}_{2n}(\mathfrak{o}_{E_v})$. For each infinite place v of F, let K_v be the maximal compact subgroup of G_v, which is defined as in § 3.3 or [Shi99, § 5]. Put $K = \prod_v K_v$.

In Case 1 we denote the two-fold metaplectic cover of $G(\mathbb{A})$ by $\widetilde{G(\mathbb{A})}$ and identify $\widetilde{G(\mathbb{A})}$ as a set with $G(\mathbb{A}) \times \{\pm 1\}$ in the standard way and, for each place v of F, let $\widetilde{G}_v$ be the two-fold metaplectic cover of G_v. There is a splitting $G(F) \to \widetilde{G(\mathbb{A})}$, a canonical splitting $N(\mathbb{A}) \to \widetilde{G(\mathbb{A})}$ and, if v is finite and does not divide 2, a canonical splitting $K_v \to \widetilde{G}_v$. We still use $G(F)$, $N(\mathbb{A})$ and K_v to denote the images of these splittings. To make our exposition uniform, we write $\widetilde{G(\mathbb{A})} = G(\mathbb{A})$ and $\widetilde{G}_v = G_v$ in the other cases. We write $\widetilde{K}$ (respectively $\widetilde{P(\mathbb{A})}$) for the pull-back of K (respectively $P(\mathbb{A})$) in $\widetilde{G(\mathbb{A})}$. The standard norm of an idele $x \in \mathbb{A}_E^\times$ is denoted by $|x|_{\mathbb{A}_E}$. We define $|a(g)|$ by writing $g = pk \in \widetilde{G(\mathbb{A})}$ with $p = (m(a)n(b), \zeta) \in \widetilde{P(\mathbb{A})}$ and $k \in \widetilde{K}$, and taking $|a(g)| = |\det a|_{\mathbb{A}_E}$.

1.2 Siegel Eisenstein series

In Case 1 we write $\det V$ for the image in $F^\times/F^{\times 2}$ of the determinant of the matrix $\frac{1}{2}(\varphi(v_i, v_j)) \in S_m(F)$, where $v_1, \dots, v_m$ is any basis for $V(F)$. Then we define a quadratic character χ_V of $\mathbb{A}^\times/F^\times$ by

$$\chi_V(x) = \prod_v (x_v, (-1)^{m(m-1)/2} \det V)_{F_v}$$

for $x = (x_v) \in \mathbb{A}^\times$, where $(\ ,\ )_{F_v}$ denotes the Hilbert symbol for F_v. Let $\gamma_{F_v}(\cdot, \psi_v)$ denote the Weil index associated to ψ_v. In Case 2 we write $\epsilon_{E/F}$ for the quadratic character of $\mathbb{A}^\times/F^\times$ associated to E/F by class field theory and fix a character χ of $\mathbb{A}_E^\times/E^\times$ such that $\chi|_{\mathbb{A}^\times} = \epsilon_{E/F}^m$.

Define a character χ_φ of $\widetilde{P(\mathbb{A})}$ by

$$\chi_\varphi\left(\left(\begin{pmatrix} a & * \\ 0 & {}^t a^{-1} \end{pmatrix}, \zeta\right)\right) = \chi_V(\det a), \tag{Case 1, $2|m$}$$

$$\chi_\varphi\left(\left(\begin{pmatrix} a & * \\ 0 & {}^t a^{-1} \end{pmatrix}, \zeta\right)\right) = \zeta \chi_V(\det a) \prod_v \gamma_{F_v}(\det a_v, \psi_v)^{-1}, \tag{Case 1, $2\nmid m$}$$

$$\chi_\varphi\left(\begin{pmatrix} a & * \\ 0 & {}^t \bar{a}^{-1} \end{pmatrix}\right) = \chi(\det a), \tag{Case 2}$$

$$\chi_\varphi\left(\begin{pmatrix} a & * \\ 0 & {}^t a^{-1} \end{pmatrix}\right) = 1 \tag{Case 3}$$

for $a = (a_v) \in \mathrm{GL}_n(\mathbb{A}_E)$. For $s \in \mathbb{C}$, let $I(s, \chi_\varphi)$ be the space of right $\widetilde{K}$-finite functions $f^{(s)} : \widetilde{G(\mathbb{A})} \to \mathbb{C}$ satisfying

$$f^{(s)}(pg) = \chi_\varphi(p)|a(p)|^{s+\varrho/2} f^{(s)}(g)$$

for all $g \in \widetilde{G(\mathbb{A})}$ and $p \in \widetilde{P(\mathbb{A})}$.

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We call a right $\widetilde{K}$-finite function $f^{(s)}$ on $\mathbb{C} \times \widetilde{G(\mathbb{A})}$ a holomorphic section of $I(s, \chi_\varphi)$ if $f^{(s)}(g)$ is holomorphic in s for each $g \in \widetilde{G(\mathbb{A})}$ and $f^{(s)} \in I(s, \chi_\varphi)$ for each $s \in \mathbb{C}$. A holomorphic section of $I(s, \chi_\varphi)$ is called standard if its restriction to $\widetilde{K}$ is independent of s.

For a holomorphic section $f^{(s)}$ of $I(s, \chi_\varphi)$, we form the Eisenstein series $E(g; f^{(s)})$ by

$$E(g; f^{(s)}) = \sum_{\gamma \in P(F)\backslash G(F)} f^{(s)}(\gamma g).$$

Such a series converges absolutely for $\Re s > \varrho/2$ and admits a meromorphic continuation to the whole plane and a functional equation

$$E(g; f^{(s)}) = E(g; M(s) f^{(s)}),$$

where $M(s) : I(s, \chi_\varphi) \to I(-s, \chi_\varphi)$ is the global intertwining operator defined, for $\Re s > \varrho/2$, by

$$M(s) f^{(s)}(g) = \int_{S_n(\mathbb{A})} f^{(s)}\left(\begin{pmatrix} 0 & -\epsilon \mathbf{1}_n \\ \mathbf{1}_n & 0 \end{pmatrix} n(b) g \right) db$$

and by the meromorphic continuation otherwise. Here, we take the Haar measure db on $S_n(\mathbb{A})$ so that $S_n(F)\backslash S_n(\mathbb{A})$ has volume 1. The poles of $E(g; f^{(s)})$ in $\Re s \geqslant 0$ are at most simple, and their location is completely determined (cf. [Ike96, KR90b, KR94, Tan99]).

Remark 1.1. In [Tan99], Tan assumes that F is totally real, but he does not use this assumption.

1.3 Regularization of theta integrals

Let $\omega = \omega_\psi$ or $\omega = \omega_{\psi,\chi}$ be the Weil representation of $\widetilde{G(\mathbb{A})} \times H(\mathbb{A})$ associated to ψ. Recall that ω can be realized on the space $\mathcal{S}(V^n(\mathbb{A}))$ of Schwartz functions on $V^n(\mathbb{A})$ with

$$\begin{gathered} \omega(h)\Phi(x) = \Phi(h^{-1}x) \quad h \in H(\mathbb{A}), \\ \omega(m(a))\Phi(x) = \chi_\varphi(m(a)) |\det a|_{\mathbb{A}_E}^{m/2} \Phi(xa) \quad a \in \mathrm{GL}_n(\mathbb{A}_E), \\ \omega(n(b))\Phi(x) = \psi(\mathrm{tr}(bQ(x)))\Phi(x) \quad b \in S_n(\mathbb{A}). \end{gathered}$$

Here, for $x \in V^n(\mathbb{A})$, let $Q(x) = \frac{1}{2}(\varphi(x_i, x_j)) \in S_n(\mathbb{A})$ be the matrix of inner products of the components of x. Let $S(V^n(\mathbb{A}))$ be the subspace of $\mathcal{S}(V^n(\mathbb{A}))$ consisting of functions which correspond to polynomials in the Fock model at every archimedean place of F.

For $g \in \widetilde{G(\mathbb{A})}$, $h \in H(\mathbb{A})$ and $\Phi \in S(V^n(\mathbb{A}))$, let

$$\Theta(g, h; \Phi) = \sum_{x \in V^n(F)} \omega(g)\Phi(h^{-1}x).$$

This function is left invariant under $G(F)$ and $H(F)$ and is slowly increasing on $G(F)\backslash \widetilde{G(\mathbb{A})}$ and $H(F)\backslash H(\mathbb{A})$. We consider the integral

$$I(g; \Phi) = \int_{H(F)\backslash H(\mathbb{A})} \Theta(g, h; \Phi)\, dh,$$

where dh is a Haar measure on $H(\mathbb{A})$ such that $H(F)\backslash H(\mathbb{A})$ has volume 1 unless φ is a split binary quadratic form. If φ is a split binary quadratic form, then $H \simeq \mathrm{GL}_1 \rtimes \mu_2$ with $\mu_2 = \{\pm 1\}$. In this case the measure above is $dh = dh_1 dc$, where dh_1 is the Tamagawa measure on $\mathrm{GL}_1(\mathbb{A})$ and dc is the Haar measure on $\mu_2(\mathbb{A})$ such that $\mathrm{vol}(\mu_2(\mathbb{A})) = 1$. Let r be the dimension of a maximal totally isotropic subspace of $V(F)$. By Weil's convergence criterion [Wei65], $I(g; \Phi)$ converges absolutely for all Φ, provided that either:

- $m \leqslant \varrho$ and φ is anisotropic; or
- $m - r > \varrho$.

Kudla and Rallis [KR94] discovered that the first condition is not essential for defining theta integrals. We here use the regularization in terms of Hecke operators instead of differential operators from the universal enveloping algebra at a real place as in [Ich01, Ich04, JS07, Tan98].

From now on we assume that $m \leqslant \varrho$. Suppose that φ is isotropic. We first choose a suitable finite place v, dependent on ψ and the fixed function $\Phi \in \mathcal{S}(V^n(\mathbb{A}))$. Then there exists an element α of the spherical Hecke algebra of H_v satisfying $\int_{H_v} \alpha(h)\, dh = 1$ and such that $\Theta(g, h; \omega(\alpha)\Phi)$ is rapidly decreasing on $H(F)\backslash H(\mathbb{A})$. The regularized theta integral is defined by

$$I(g; \Phi) = \int_{H(F)\backslash H(\mathbb{A})} \Theta(g, h; \omega(\alpha)\Phi)\, dh.$$

This is independent of the choice of v and α, and is a unique $H(\mathbb{A})$-invariant extension of the theta integral (see [Ich01, Ich04, JS07, Yam10] for details).

2. Statement of the main results

For $\Phi \in S(V^n(\mathbb{A}))$, we define a standard section $f_\Phi^{(s)}$ of $I(s, \chi_\varphi)$ by

$$f_\Phi^{(s)}(g) = |a(g)|^{s-s_0}\omega(g)\Phi(0).$$

We study the resulting Eisenstein series

$$E(g; f_\Phi^{(s)}) = \sum_{\gamma \in P(F)\backslash G(F)} f_\Phi^{(s)}(\gamma g).$$

Theorem 2.1. *Suppose that $m \leqslant \varrho$. Let $\Phi \in S(V^n(\mathbb{A}))$.*

(i) *$E(g; f_\Phi^{(s)})$ is holomorphic at $s = s_0$.*

(ii) *If φ is a split binary quadratic form, then $E(g; f_\Phi^{(s)})|_{s=s_0} = 0$.*

Theorem 2.2. *Suppose that $m \leqslant \varrho$. Let $\Phi \in S(V^n(\mathbb{A}))$.*

(i) *Except when φ is a split binary quadratic form,*

$$E(g; f_\Phi^{(s)})|_{s=s_0} = \varkappa I(g; \Phi)$$

with

$$\varkappa = \begin{cases} 1 & \text{if } m = 1 \text{ in Case 1,} \\ 2 & \text{otherwise.} \end{cases}$$

(ii) *If φ is a split binary quadratic form, then*

$$\frac{\partial}{\partial s} E(g; f_\Phi^{(s)})\bigg|_{s=s_0} = 2I(g; \Phi).$$

3. The intertwining operator

3.1 Fix a place v of F and suppress it from the notation. Thus, $F = F_v$ is a local field of characteristic zero and, in Case 2, $E = E_v$ is a quadratic extension of F or $E = F \oplus F$ according

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as v is inert or split. We can define χ_V, $\epsilon_{E/F}$, $I(s,\chi_\varphi)$ and $M(s)$ locally. These local objects occur in the factorizations of the global ones defined in § 1.2. We define holomorphic sections and standard sections similarly.

The product group $\widetilde{G}\times H$ acts on the Schwartz space $\mathcal{S}(V^n)$ via the local Weil representation ω. When $F=\mathbb{R}$ or $\mathbb{C}$, let $\mathfrak{g}$ and $\mathfrak{h}$ be the complexified Lie algebras of G and H, respectively. Put $S(V^n)=\mathcal{S}(V^n)$ if F is a p-adic field, and let $S(V^n)$ be the subspace of $\mathcal{S}(V^n)$ which corresponds to the space of polynomials in a Fock model compatible with $\widetilde{K}$ and some maximal compact subgroup K_H of H if $F=\mathbb{R}$ or $\mathbb{C}$. Let $R(V)$ be the image of the intertwining map

$$S(V^n)\to I(s_0,\chi_\varphi),\quad \Phi\mapsto f_\Phi^{(s_0)}(g)=\omega(g)\Phi(0).$$

We extend $f_\Phi^{(s_0)}$ to a standard section $f_\Phi^{(s)}$ of $I(s,\chi_\varphi)$. Rallis's theorem on coinvariants, which is extended to almost all cases (cf. [KR90a, LZ98, LZ08, MVW87, Yam]), states that if F is a p-adic field (respectively $F=\mathbb{R}$ or $\mathbb{C}$), then $R(V)$ coincides with the maximal quotient of $S(V^n)$ on which H (respectively $(\mathfrak{h},K_H)$) acts trivially. In particular, if $m\leqslant\varrho$, then that maximal quotient is irreducible (cf. [Li89]) and so is $R(V)$.

For a quadratic character η of $F^\times$, let $L(s,\eta)$ denote the local abelian L-factor. Set $\zeta(s)=L(s,1)$. Put

$$a(s)=L\left(s-\frac{n-1}{2},\chi_V\right)\prod_{j=1}^{[n/2]}\zeta(2s-n+2j),\qquad \text{(Case 1, } 2\mid m)$$
$$b(s)=L\left(s+\frac{n+1}{2},\chi_V\right)\prod_{j=1}^{[n/2]}\zeta(2s+n+1-2j),$$

$$a(s)=\prod_{j=1}^{[(n+1)/2]}\zeta(2s-n-1+2j),\qquad \text{(Case 1, } 2\nmid m)$$
$$b(s)=\prod_{j=1}^{[(n+1)/2]}\zeta(2s+n+2-2j),$$

$$a(s)=\prod_{j=1}^{n}L(2s-n+j,\epsilon_{E/F}^{m+j-1}),\qquad \text{(Case 2)}$$
$$b(s)=\prod_{j=1}^{n}L(2s+n+1-j,\epsilon_{E/F}^{m+j-1}),$$

$$a(s)=\prod_{j=1}^{[n/2]}\zeta(2s-n+2j),\qquad \text{(Case 3)}$$
$$b(s)=\prod_{j=1}^{[n/2]}\zeta(2s+n+1-2j).$$

A normalized intertwining operator $M^*(s)$ is defined by

$$M^*(s)=a(s)^{-1}M(s).$$

When $m \leqslant \varrho - 1$, let $U = V \oplus \mathcal{H}^{\varrho - m}$, where $\mathcal{H}$ is a hyperbolic plane. It is noteworthy that the special value of s for the space U is

$$-s_0 = (\dim U - \varrho)/2.$$

The proof of the following two propositions will be given on a case-by-case basis in the rest of this section.

Proposition 3.1. *If $m \leqslant \varrho - 1$, then $b(s)M^*(s)f_\Phi^{(s)}$ is holomorphic at $s = s_0$ for every $\Phi \in S(V^n)$.*

Proposition 3.2. *$M^*(s)$ is holomorphic in the right half-plane $\Re s \geqslant 0$. If $m \leqslant \varrho - 1$, then $M^*(-s_0)$ maps $R(U)$ onto $R(V)$.*

3.2 The non-archimedean case

Let F be a p-adic field. Then Proposition 3.2 is well known. Moreover, $M^*(s)$ is entire and $M^*(s_0)$ annihilates $R(V)$, which proves Proposition 3.1 since $b(s)$ has a simple pole at $s = s_0$. For proofs of these facts, we refer to [KR92] for symplectic groups, to [Swe95] for metaplectic groups, to [KS97] for unitary groups and to [Yam] for orthogonal groups.

We can prove the following proposition by the standard Gindikin–Karpelevich argument.

Proposition 3.3. *If the residual characteristic of F is not 2 and if $I(s, \chi_\varphi)$ contains a section $f_0^{(s)}$ which is identically 1 on K, then*

$$b(s)M^*(s)f_0^{(s)} = f_0^{(-s)}.$$

3.3 The cases of $\mathbf{Sp}(n, \mathbb{R})$ and $U(n, n; \mathbb{R})$

This subsection concerns the cases in which $F = \mathbb{R}$ in Case 1 or $E/F = \mathbb{C}/\mathbb{R}$ in Case 2. Let (p, q) be the signature of φ. Put

$$\iota = 1, \quad l_0 = (p - q)/2, \qquad \text{(Case 1)}$$

$$\iota = 2, \quad l_0 = p - q. \qquad \text{(Case 2)}$$

Note that the maximal compact subgroups are defined by

$$K = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in \mathrm{GL}_{2n}(\mathbb{R}) \;\middle|\; a\,{}^t b = b\,{}^t a,\ a\,{}^t a + b\,{}^t b = \mathbf{1}_n \right\}, \qquad \text{(Case 1)}$$

$$K = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in \mathrm{GL}_{2n}(\mathbb{C}) \;\middle|\; a\,{}^t \bar{b} = b\,{}^t \bar{a},\ a\,{}^t \bar{a} + b\,{}^t \bar{b} = \mathbf{1}_n \right\} \qquad \text{(Case 2)}$$

in the present cases. We remind the reader that the group $\widetilde{K}$ in Case 1 has a character $(\det)^{1/2}$ whose square descends to the determinant character of $K \simeq U(n)$. In Case 2, let k be the integer with $k \equiv m \pmod 2$ such that $\chi(z) = (z/\bar{z})^{k/2}$. Provided that $l \in \frac{1}{2}\mathbb{Z}$ satisfies $l \equiv l_0 \pmod 2$, we can define a standard section $f_l^{(s)}$ of $I(s, \chi_\varphi)$ by

$$f_l^{(s)}((u, 1)) = \det(a + \sqrt{-1}b)^l, \qquad \text{(Case 1)}$$

$$f_l^{(s)}(u) = \det(a + \sqrt{-1}b)^{(k+l)/2} \det(a - \sqrt{-1}b)^{(k-l)/2} \qquad \text{(Case 2)}$$

for $u = \left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right) \in K$.

Proof of Proposition 3.1. Note that $a(s)^{-1}$ is entire and that $a(s_0)^{-1} \neq 0$ except in the quadratic case in which m is even and $n \equiv p \pmod 2$. The archimedean cases are more delicate because of

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$$\operatorname{ord}_{s=s_0} b(s) = -[(n-m)/2] - 1, \qquad \text{(Case 1)}$$
$$\operatorname{ord}_{s=s_0} b(s) = -(n-m). \qquad \text{(Case 2)}$$

We proceed as in [KR88a, § 4], which covers the quadratic case in which m is even. Since the $(\mathfrak{g}, \widetilde{K})$-module $R(V)$ is generated by $f_{l_0}^{(s_0)}$ (see [How89, KR90a, Li89, LZ98]), by virtue of [KR88a, Proposition 4.9] it suffices to show that $b(s)M^*(s)f_{l_0}^{(s)}(\mathbf{1}_{2n})$ is holomorphic at $s = s_0$. Put

$$\Gamma_n^{\iota}(s) = \pi^{\iota n(n-1)/4} \prod_{j=0}^{n-1} \Gamma\left(s - \frac{\iota j}{2}\right).$$

From [Shi82, (1.31)], we find that

$$\begin{aligned} M(s)f_l^{(s)}(\mathbf{1}_{2n}) &= \int_{S_n(\mathbb{R})} \det(x + \sqrt{-1}\mathbf{1}_n)^{-\{\iota(s+\varrho/2)+l\}/2} \det(x - \sqrt{-1}\mathbf{1}_n)^{-\{\iota(s+\varrho/2)-l\}/2}\, dx \\ &= \frac{(\sqrt{-1})^{-nl} 2^{n(1-\iota s)} \pi^{\iota n\varrho/2} \Gamma_n^{\iota}(\iota s)}{\Gamma_n^{\iota}(\frac{1}{2}\{\iota(s+\varrho/2)+l\})\Gamma_n^{\iota}(\frac{1}{2}\{\iota(s+\varrho/2)-l\})}. \end{aligned}$$

Now one can directly check that $\operatorname{ord}_{s=s_0} b(s)M^*(s)f_{l_0}^{(s)}(\mathbf{1}_{2n}) = 0$ in both cases. □

Lemma 3.4. *If $m \leqslant \varrho - 1$, then the $(\mathfrak{g}, \widetilde{K})$-module $I(-s_0, \chi_\varphi)$ is generated by the following vectors:*

$$\{f_{l_0+2j}^{(-s_0)} \mid j \in \mathbb{Z}, -[p/2] \leqslant j \leqslant [q/2]\}, \qquad \text{(Case 1)}$$
$$\{f_{l_0+2j}^{(-s_0)} \mid j \in \mathbb{Z}, -p \leqslant j \leqslant q\}. \qquad \text{(Case 2)}$$

Proof. Since our assertion in Case 2 readily follows from the module structure of $I(-s_0, \chi_\varphi)$ described in [LZ98], we consider Case 1. Recall that all $\widetilde{K}$-types of $I(s, \chi_\varphi)$ have multiplicity one. Moreover, the $\widetilde{K}$-types occurring in $I(s, \chi_\varphi)$ are precisely the irreducible representations σ_λ of $\widetilde{K}$ whose highest weights $\lambda = (\lambda_1, \ldots, \lambda_n)$ satisfy

$$\lambda_i \equiv l_0 \pmod 2$$

for all i. For integers j as listed above, we put

$$p_j = p + 2j + \varrho - m, \quad q_j = q - 2j + \varrho - m, \quad m' = 2\varrho - m.$$

Let U_j be a quadratic space of dimension m' over $\mathbb{R}$ and signature (p_j, q_j). Recall that $R(U_j)$ is a cyclic submodule of $I(-s_0, \chi_\varphi)$ generated by $f_{l_0+2j}^{(-s_0)}$. By [KR90a, Corollary 2.5], the $\widetilde{K}$-type σ_λ does not occur in $R(U_j)$ if and only if either:

- $p_j < n$ and $\lambda_{p_j+1} > l_0 + 2j$; or
- $q_j < n$ and $\lambda_{n-q_j} < l_0 + 2j$.

Assume that σ_λ does not occur in any $R(U_j)$. Since $p_j \geqslant n$ (respectively $q_j \geqslant n$) if $j = [q/2]$ (respectively $j = -[p/2]$), if l is the largest integer that satisfies the first condition, then

$$-[p/2] \leqslant l < [q/2], \quad l_0 + 2(l+1) > \lambda_{n-q_{l+1}} \geqslant \lambda_{p_l+1} > l_0 + 2l,$$

which contradicts the condition on λ_i. Thus, $\widetilde{K}$-types of submodules $R(U_j)$ exhaust those of $I(-s_0, \chi_\varphi)$, thereby completing the proof. □

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Proof of Proposition 3.2. We limit ourselves to examining $M^*(s)$ for its property at $s=-s_0$. The calculation in the proof of Proposition 3.1 confirms that $\mathrm{ord}_{s=-s_0} M^*(s)f_{l_0}^{(s)}(\mathbf{1}_{2n})=0$ in all cases and so, by [KR88a, Proposition 4.9] and Lemma 3.4, the first claim follows. Consequently, the operator $M^*(-s_0):I(-s_0,\chi_\varphi)\to I(s_0,\chi_\varphi)$ is $(\mathfrak{g},\widetilde{K})$-intertwining. Since $M^*(-s_0)f_{l_0}^{(-s_0)}$ equals $f_{l_0}^{(s_0)}$ up to a non-zero constant, the remaining part is evident. □

3.4 The remaining cases

Let F be $\mathbb{R}$ or $\mathbb{C}$. We will complete the proof of Propositions 3.1 and 3.2 for the cases not covered by §§ 3.2 and 3.3, i.e. the cases of complex symplectic groups, general linear groups and orthogonal groups. In the first case, Proposition 3.1 is included in [KR88a, § 4]. Recall that when $E=F\oplus F$, we arrive at $G\simeq \mathrm{GL}_{2n}(F)$ and $M\simeq \mathrm{GL}_n(F)\times \mathrm{GL}_n(F)$. Clearly, we may assume that $\chi=1$ in this case.

The structure of the degenerate principal series $I(s,\chi_\varphi)$ is completely known in almost all cases, but we first exclude the complex orthogonal case of odd n, for which we have not been able to find a reference. We refer to [HL99] for general linear groups, to [Lok06] for real orthogonal groups and to [LZ08] for other complex groups. Note that $I(s,\chi_\varphi)$ contains the unique vector $f_0^{(s)}$ that is identically 1 on K. The fact we use is that $I(-s_0,\chi_\varphi)$ is generated as a $(\mathfrak{g},K)$-module by $f_0^{(-s_0)}$ in Cases 1, 2 and by $f_0^{(-s_0)}$ and its twist by the determinant character of G in Case 3.

On the other hand, submodules $R(V)$ and $R(U)$ are generated by $f_0^{(s_0)}$ and $f_0^{(-s_0)}$, respectively. Indeed, if $\mathscr{H}(K)$ (respectively $\mathscr{H}(K_H)$) denotes the space of K-harmonics (respectively K_H-harmonics), then the basic result of Howe [How89] states that $\mathscr{H}(K)\cap\mathscr{H}(K_H)$ generates $S(V^n)$ as a $(\mathfrak{g},K)\times(\mathfrak{h},K_H)$-module. Thus, $R(V)$ is generated as a $(\mathfrak{g},K)$-module by the images of K_H-invariants in the space $\mathscr{H}(K)\cap\mathscr{H}(K_H)$, i.e. the constants in the Fock model.

Since the Gindikin–Karpelevich argument shows that

$$b(s)M^*(s)f_0^{(s)}=f_0^{(-s)},$$

we can prove Propositions 3.1 and 3.2 by arguing exactly as in § 3.3.

Finally, we complete the complex orthogonal case. For the same reason as in [Yam, Proof of Proposition 8.10], the holomorphy of $M^*(s)$ in the case of odd n can be reduced to that in the case of even n. Now we can repeat what we have just done.

4. Holomorphy of $E(g;f_\Phi^{(s)})$ at $s=s_0$

We return to the global situation and prove a slightly stronger result than Theorem 2.1. Provided that $m=\varrho$, the general theory of Langlands contains (i) and Proposition 5.8(ii) includes (ii). We suppose that $m\leqslant\varrho-1$. We define $a(s)$ and $b(s)$ by taking the complete Hecke L-functions in place of the local L-factors in the definitions of $a_v(s)$ and $b_v(s)$ in § 3.1. Taking Proposition 3.3 into account, we define a normalized global intertwining operator by

$$M^\circ(s)=\frac{b(s)}{a(s)}M(s).$$

Let $\mathcal{C}=\{W_v\}$ be a collection of local ϵ-hermitian spaces of dimension m over E_v which satisfies $\chi_{W_v}=\chi_{V_v}$ for all v in Case 1 and such that W_v is isometric to V_v for almost all v. We form a restricted direct product $\Pi(\mathcal{C})=\bigotimes'_v R(W_v)$, which we can regard as a subrepresentation of $I(s_0,\chi_\varphi)$. When $\mathcal{C}=\{V_v\}$, we write $\Pi(V)$ in place of $\Pi(\mathcal{C})$.

Let $f^{(s)}$ be a standard section of $I(s, \chi_\varphi)$ such that $f^{(s_0)} \in \Pi(\mathcal{C})$. Put $h^{(-s)} = M^\circ(s) f^{(s)}$. By the functional equation referred to in § 1.2, we have

$$E(g; f^{(s)}) = E(g; M(s) f^{(s)}) = \frac{a(s)}{b(s)} E(g; h^{(-s)}).$$

Recall that for any holomorphic section $f^{(s)}$ of $I(s, \chi_\varphi)$, the Eisenstein series $E(g; f^{(s)})$ has at most a simple pole at $s = -s_0$ (cf. [Ike96, KR90b, KR94, Tan99]). Proposition 3.1 enables us to choose an entire function $\gamma(s)$ with $\gamma(-s_0) \neq 0$ so that $\gamma(s) h^{(s)}$ is a holomorphic section of $I(s, \chi_\varphi)$. Then $E(g; \gamma(s) h^{(s)})$ has at most a simple pole at $s = -s_0$ and hence so does $E(g; h^{(s)})$. Note that $a(s)/b(s)$ has a simple zero at $s = s_0$ except when φ is a split binary quadratic form, in which case it has a double zero. Thus, $E(g; f^{(s)})$ is holomorphic at $s = s_0$.

When V is not a split binary quadratic space, the identity obtained by evaluating at $s = s_0$ is

$$E(g; f^{(s)})|_{s=s_0} = -\gamma \operatorname{Res}_{s=-s_0} E(g; h^{(s)}), \quad \gamma = \frac{a(s_0)}{\operatorname{Res}_{s=s_0} b(s)}. \tag{4.1}$$

5. The regularized Siegel–Weil formula

Assume that $m \leqslant \varrho - 1$. Let $\mathcal{H}^j$ be the direct sum of j hyperbolic planes and set $U = V \oplus \mathcal{H}^{\varrho - m}$. We can identify the restricted tensor product $\Pi(V) = \bigotimes_v R(V_v)$ (respectively $\Pi(U) = \bigotimes_v R(U_v)$) with a subrepresentation of $I(s_0, \chi_\varphi)$ (respectively $I(-s_0, \chi_\varphi)$). To state the regularized Siegel–Weil formula in a form which is more suitable for our purpose, we start with the following proposition.

Proposition 5.1. *Assume that $m \leqslant \varrho - 1$. Then $\Pi(U)$ has a unique irreducible quotient $\Pi(V)$. Moreover, $M^\circ(s)$ is holomorphic in the right half-plane $\Re s \geqslant 0$ and $M^\circ(-s_0)$ induces a non-zero intertwining map $\Pi(U) \to \Pi(V)$.*

Proof. Recall that $R(U_v)$ has a unique irreducible quotient $R(V_v)$ for each place v of F. By Proposition 3.2, the quotient map $R(U_v) \to R(V_v)$ can be realized as the restriction of $M_v^*(-s_0)$ to $R(U_v)$. Since $b_v(s)$ has no poles or zeroes in the right half-plane, the proof is complete in view of Proposition 3.3. □

Theorem 5.2 (See [Ich01, Ich04, JS07, KR94]). *Assume that $m \leqslant \varrho - 1$. Then there is a non-zero constant c_0 such that for all holomorphic sections $f^{(s)}$ of $I(s, \chi_\varphi)$ satisfying $f^{(-s_0)} \in \Pi(U)$,*

$$\operatorname{Res}_{s=-s_0} E(g; f^{(s)}) = c_0 I(g; \Phi),$$

where Φ is any element of $S(V^n(\mathbb{A}))$ satisfying the relation

$$M^\circ(-s_0) f^{(-s_0)} = f_\Phi^{(s_0)}.$$

Remark 5.3. (i) Though [Ich04] concerns a totally imaginary quadratic extension E of a totally real field F, no new proof is necessary for the case of general quadratic extensions (cf. Remark 1.1).

(ii) Our formulation of the regularized Siegel–Weil formula differs slightly from that by Kudla and Rallis [KR94]. They start with U and call U and V complementary. Our formulation is natural in that there are no complementary spaces in the quaternion case (cf. [Yam, Yam10]).

(iii) Theorem 5.2 is stated in [Ich01, Ich04, JS07] in terms of $S(U^n(\mathbb{A}))$ instead of $\Pi(U)$. One can see that the map $\Phi \mapsto \operatorname{Res}_{s=-s_0} E(g; f_\Phi^{(s)})$ factors through the quotient

$$S(U^n(\mathbb{A})) \to \Pi(U) \to \Pi(V),$$

so that the principle of [KR94, Theorem 3.1] coupled with Proposition 5.1 leads to the desired formula.

Corollary 5.4. *Assume that $m \leqslant \varrho - 1$.*

(i) *Except when φ is a split binary quadratic form, there exists a non-zero constant c such that, for all $\Phi \in S(V^n(\mathbb{A}))$,*

$$E(g; f_\Phi^{(s)})|_{s=s_0} = cI(g; \Phi).$$

(ii) *If φ is a split binary quadratic form, then there exists a non-zero constant c' such that, for all $\Phi \in S(V^n(\mathbb{A}))$,*

$$\frac{\partial}{\partial s} E(g; f_\Phi^{(s)})\bigg|_{s=s_0} = c' I(g; \Phi).$$

Proof. Proposition 3.1 enables us to define a $\widetilde{K}$-intertwining map $A : \Pi(V) \to I(-s_0, \chi_\varphi)$ by

$$A(f^{(s_0)}) = \lim_{s \to s_0} M^\circ(s) f^{(s)},$$

where $f^{(s)}$ is a standard section defined by $f^{(s_0)} \in \Pi(V)$. We have

$$M^\circ(-s_0) \circ A(f_\Phi^{(s_0)}) = \delta f_\Phi^{(s_0)}, \quad \delta = \frac{b(-s_0)}{a(s_0)} \lim_{s \to s_0} \frac{b(s)}{a(-s)}$$

for all $\Phi \in S(V^n(\mathbb{A}))$ by the functional equation $M(-s) \circ M(s) = 1$. Since $a(-s)$ is equal to $b(s)$ up to an exponential factor, we see that $\delta \neq 0$.

We claim that $A(f_\Phi^{(s_0)}) \in \Pi(U)$. We may suppose that $\Phi = \bigotimes_v \Phi_v$ is factorizable and have only to show that $A_v(f_{\Phi_v}^{(s_0)}) \in R(U_v)$ in obvious notation. First assume that v is infinite. Since $I_v(s, \chi_{\varphi_v})$ is multiplicity free as a representation of $\widetilde{K}_v$ and since each $\widetilde{K}_v$ type of $R(V_v)$ occurs in $R(U_v)$ by Proposition 3.2, our claim follows. Next suppose that v is finite. Then our claim is obvious since $R(U_v)$ coincides with the inverse image of $R(V_v)$ under $M_v^*(-s_0)$ (cf. [KR92, KS97, Swe95, Yam]). Theorem 5.2 applied to $h^{(-s)} = M^\circ(s) f_\Phi^{(s)}$ gives

$$E(g; f_\Phi^{(s)})|_{s=s_0} = \lim_{s \to s_0} \frac{a(s)}{b(s)} E(g; h^{(-s)}) = -\gamma \operatorname{Res}_{s=-s_0} E(g; h^{(s)}) = -\gamma \delta c_0 I(g; \Phi),$$

provided that φ is not a split binary quadratic form, where we used (4.1). Thus, $c = -\gamma\delta c_0$ works. One can prove the case of a split binary quadratic form similarly. □

Corollary 5.5. *Notation being as in § 4, if $\mathcal{C}$ cannot be the set of localizations of any global space, then $E(g; f^{(s)})|_{s=s_0}$ is identically zero.*

Proof. Theorem 4.9 of [KR94] coupled with (4.1) proves this corollary. □

Remark 5.6. Notation and assumption being as above, we put $h^{(-s)} = M^\circ(s) f^{(s)}$. Then

$$\frac{\partial}{\partial s} E(g; f^{(s)})\bigg|_{s=s_0} = \gamma E(g; h^{(s)})|_{s=-s_0}.$$

The nature of this derivative remains to be determined. In this direction, we refer to [Kud97, KRY06] for the quadratic case in which $m = \varrho$.

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Let D_F denote the absolute value of the discriminant of F. The residue of $\zeta(s)$ at $s=1$ is denoted by ρ_F. Once we establish $c=\varkappa$ and $c'=2$ in the next section, the following result immediately follows from the proof above.

COROLLARY 5.7. *Let c_0 be the constant defined in Theorem 5.2. Then*

$$c_0=\varkappa b(-s_0)^{-1}\mathrm{Res}_{s=-s_0}a(s)$$

unless φ is a split binary quadratic form, in which case

$$c_0=\frac{D_F^{1/2}\rho_F^2\prod_{j=2}^{[n/2]}\zeta(2j-1)}{\zeta(n)\prod_{j=1}^{[n/2]}\zeta(2(n-j))}.$$

We end this section by proving the special case of Theorem 2.2 in which $m=\varrho$.

PROPOSITION 5.8 (See [GT, Ich04, Kud97]). (i) *Suppose that $m=\varrho$, and exclude the case of a split binary quadratic form. Then for all $\Phi\in S(V^n(\mathbb{A}))$*

$$E(g;f_\Phi^{(s)})|_{s=0}=2I(g;\Phi).$$

(ii) *If $n=1$ and φ is a split binary quadratic form, then*

$$E(g;f_\Phi^{(s)})|_{s=0}=0,\quad \frac{\partial}{\partial s}E(g;f_\Phi^{(s)})\bigg|_{s=0}=2I(g;\Phi).$$

Proof. First of all, we define the general notation that we will use in the following. For $0\leqslant j\leqslant n$, we define a Weyl element $w_j\in G(F)$ by

$$w_j=\left(\begin{array}{cc|cc}\mathbf{1}_{n-j}&&&\\&&&\mathbf{1}_j\\\hline&&\mathbf{1}_{n-j}&\\&-\epsilon\mathbf{1}_j&&\end{array}\right)$$

and a maximal parabolic subgroup Q_j^n of $\mathrm{R}_{E/F}\mathrm{GL}_n$ by

$$Q_j^n=\left\{\begin{pmatrix}a&*\\0&d\end{pmatrix}\in\mathrm{R}_{E/F}\mathrm{GL}_n\;\middle|\;a\in\mathrm{R}_{E/F}\mathrm{GL}_{n-j},d\in\mathrm{R}_{E/F}\mathrm{GL}_j\right\}.$$

For $\beta\in S_n(F)$ and an automorphic form A on $\widetilde{G(\mathbb{A})}$, let

$$A_\beta(g)=\int_{S_n(F)\backslash S_n(\mathbb{A})}A(n(b)g)\psi(-\mathrm{tr}(\beta b))\,db,\quad g\in\widetilde{G(\mathbb{A})}$$

denote the βth Fourier coefficient of A.

Let us turn to the proof of (i). We refer to [Kud97, Theorem 3.1] for symplectic groups and to [Ich04, Theorem 4.2] for unitary groups. Strictly speaking, these assume F to be totally real, but the same proof works over any number field. As for orthogonal groups, we give a sketch of the proof for the convenience of the reader.

By the general theory of Langlands, both $E(g;f_\Phi^{(s)})$ and $M(s)$ are holomorphic at $s=0$. Let G_∞ (respectively $G(\mathbb{A}_\mathrm{f})$) be the infinite (respectively finite) part of $G(\mathbb{A})$ and $\mathfrak{g}_\infty$ be the complexified Lie algebra of G_∞. Since the two maps $\Phi\mapsto E(g;f_\Phi^{(s)})|_{s=0}$ and $\Phi\mapsto I(g;\Phi)$ are $H(\mathbb{A})$-invariant and respect the action of $(\mathfrak{g}_\infty,K_{G_\infty})\times G(\mathbb{A}_f)$, they define intertwining maps from the representation $\Pi(V)$ to the space of automorphic forms on $G(\mathbb{A})$. Theorem 3.1

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of [KR94], which one can readily extend to the case of orthogonal groups, tells us that such maps must be proportional.

To determine the constant of proportionality, we consider the constant term with respect to the parabolic subgroup P. Recall that the constant term of $I(g;\Phi)$ is given by

$$\int_{H(F)\backslash H(\mathbb{A})}\sum_{i=0}^{(n-1)/2}\sum_{x\in V^n(F),Q(x)=0,\mathrm{rank}(x)=i}\omega(g)\omega(\alpha)\Phi(h^{-1}x)\,dh,$$

where $\mathrm{rank}(x)$ denotes the dimension of the subspace of $V(F)$ spanned by the components of $x\in V^n(F)$. As in [KR88b, § 6], we can show that the ith term, viewed as an automorphic form on $\mathrm{GL}_n(\mathbb{A})$, has central character given by

$$z\mapsto |z|_{\mathbb{A}}^{n(n-1)/2-i(n-i)}.$$

Note that the zeroth term, which is given by

$$\int_{H(F)\backslash H(\mathbb{A})}\omega(g)\omega(\alpha)\Phi(0)\,dh=\omega(g)\Phi(0),$$

has central character distinct from those of the remaining terms.

On the other hand, the constant term of $E(g;f_\Phi^{(s)})$ is given by

$$\sum_{j=0}^{n}\sum_{\gamma\in Q_j^n(F)\backslash\mathrm{GL}_n(F)}\int_{S_j(\mathbb{A})}f_\Phi^{(s)}\left(w_jn\left(\begin{pmatrix}0&0\\0&b\end{pmatrix}\right)m(\gamma)g\right)db$$

(cf. [KR88a, Lemma 2.4]). The zeroth term is $f_\Phi^{(s)}(g)$ and the nth term is $M(s)f_\Phi^{(s)}(g)$. As in [KR88a, Proof of Proposition 7.1], we can see that there is a constant $\mu\in\mathbb{C}$ satisfying $M(0)f^{(0)}=\mu f^{(0)}$ for all $f^{(0)}\in\Pi(V)$. We next note that as $M(0)^2=1$, $\mu\in\{\pm1\}$. Moreover, $\mu=1$ is easy to see by replacing $f^{(0)}$ by the vector of $\Pi(V)$ that is identically 1 on K. Thus, $M(0)$ acts as an identity on $\Pi(V)$.

Since $f_\Phi^{(0)}$ and $M(0)f_\Phi^{(0)}$ have central character $z\mapsto|z|_{\mathbb{A}}^{n(n-1)/2}$, which is distinct from those of intermediate terms (see the calculation in § 6), the zeroth term in the constant term of $I(g;\Phi)$ matches

$$f_\Phi^{(0)}(g)+M(0)f_\Phi^{(0)}(g)=2\omega(g)\Phi(0).$$

Thus, the constant of proportionality must be 2. This result is compatible with the calculations of Gan and Takeda [GT], where they proved (i) among other things when Φ is a particular spherical vector.

To prove (ii), we have only to show that

$$\frac{\partial}{\partial s}E_t(g;f_\Phi^{(s)})\bigg|_{s=0}=2I_t(g,\Phi)$$

for all $t\in F^\times$ by the irreducibility and non-singularity of $\Pi(V)$. We proceed as in [KRY06, § 5.3]. We identify V with F^2 in such a way that

$$\varphi\left(\begin{pmatrix}\xi\\\eta\end{pmatrix},\begin{pmatrix}\xi'\\\eta'\end{pmatrix}\right)=\xi\eta'+\eta\xi'.$$

For $\Phi=\bigotimes_v\Phi_v\in S(V(\mathbb{A}))$, which we assume is invariant under $\mu_2(\mathbb{A})$, an easy calculation shows that

$$I_t(g;\Phi)=(2D_F^{1/2}\rho_F)^{-1}\prod_v O_{t,v}(\omega(g_v)\Phi_v),$$

https://doi.org/10.1112/S0010437X11005379 Published online by Cambridge University Press

where

$$O_{t,v}(\omega(g_v)\Phi_v) = \zeta_v(1)\int_{F_v^\times} \omega(g_v)\Phi_v\left(\begin{pmatrix} bt \\ b^{-1}\end{pmatrix}\right)\frac{|db|_v}{|b|_v}.$$

Recall that $\{\zeta_v(1)^{-1}\}$ is a set of convergence factors for GL_1. Note that

$$E_t(g; f_\Phi^{(s)}) = \prod_v W_{t,v}(g_v; f_{\Phi_v}^{(s)}),$$

$$W_{t,v}(g_v; f_{\Phi_v}^{(s)}) = \int_{F_v} f_{\Phi_v}^{(s)}\left(\begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix} n(b) g_v\right)\psi_v(-tb)\, d_v b,$$

where $d_v b$ is the self-dual Haar measure on F_v with respect to ψ_v. The gauge form $d\xi \wedge d\eta$ on V determines a measure $|d\xi\wedge d\eta|_v = c_v(\psi)\, d_v z$ for a positive constant $c_v(\psi)$, where $d_v z$ is the self-dual Haar measure on $V(F_v)$ with respect to the pairing $(x,y)\mapsto \psi_v(\varphi(x,y))$. Similarly, $|db|_v = c'_v(\psi)\, d_v b$ for a positive constant $c'_v(\psi)$. Now arguing exactly as in [KRY06, Proof of Proposition 5.3.3], we can show that

$$\zeta_v(1) W_{t,v}(g_v; f_{\Phi_v}^{(0)}) = \frac{c'_v(\psi)}{c_v(\psi)}\, O_{t,v}(\omega(g_v)\Phi_v)$$

for all places v of F. Therefore,

$$\begin{aligned}\lim_{s\to 0} s^{-1} E_t(g; f_\Phi^{(s)}) &= \lim_{s\to 0}\frac{1}{s\zeta(1+s)}\prod_v \zeta_v(1+s) W_{t,v}(g_v; f_{\Phi_v}^{(s)}) \\ &= \rho_F^{-1}\prod_v \frac{c'_v(\psi)}{c_v(\psi)}\, O_{t,v}(\omega(g_v)\Phi_v) = 2I_t(g;\Phi)\end{aligned}$$

since $\prod_v c_v(\psi) = D_F$ and $\prod_v c'_v(\psi) = D_F^{1/2}$. □

6. The constant of proportionality

To complete our picture, it remains to prove that $c = \varkappa$ and $c' = 2$. In addition to the notation from the proof of Proposition 5.8, we need some more notation. Put

$$\begin{aligned} &\ell = m-1, && \ell_0 = m-1, && \text{(Case 1)}\\ &\ell = m, && \ell_0 = m, && \text{(Case 2)}\\ &\ell = m, && \ell_0 = m+1. && \text{(Case 3)}\end{aligned}$$

Note that $n > \ell_0$ by assumption. For $\gamma \in \mathrm{GL}_{n-\ell}(\mathbb{A}_E)$, we put

$$m_0(\gamma) = m\left(\begin{pmatrix} \gamma & 0 \\ 0 & \mathbf{1}_\ell\end{pmatrix}\right)$$

to make our exposition smooth. Put

$$G_0 = \left\{\left(\begin{array}{cc|cc} \mathbf{1}_{n-\ell_0} & & & \\ & a & & b \\ \hline & & \mathbf{1}_{n-\ell_0} & \\ & c & & d \end{array}\right) \in G\right\}.$$

Let $\widetilde{G_0(\mathbb{A})}$ be the pull-back of $G_0(\mathbb{A})$ in $\widetilde{G(\mathbb{A})}$. Then $\widetilde{G_0(\mathbb{A})}\times H(\mathbb{A})$ acts on $S(V^{\ell_0}(\mathbb{A}))$ via the Weil representation associated to ψ. Define $\Phi_0 \in S(V^{\ell_0}(\mathbb{A}))$ by $\Phi_0(y) = \Phi(0,y)$ for $y \in V^{\ell_0}(\mathbb{A})$.

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Let $E(g_0; f^{(s)}_{\Phi_0})$ denote the Eisenstein series on $\widetilde{G_0(\mathbb{A})}$ attached to the standard section defined by Φ_0.

Fix $\beta_1 \in S_\ell(F) \cap \mathrm{GL}_\ell(E)$ and put

$$\beta = \begin{pmatrix} \mathbf{0}_{n-\ell} & 0 \\ 0 & \beta_1 \end{pmatrix} \in S_n(F), \quad \beta_0 = \begin{pmatrix} \mathbf{0}_{\ell_0-\ell} & 0 \\ 0 & \beta_1 \end{pmatrix} \in S_{\ell_0}(F).$$

If V is not a split binary quadratic space, then Corollary 5.4(i) yields

$$E_\beta(g; f^{(s)}_\Phi)|_{s=s_0} = cI_\beta(g; \Phi).$$

The argument of [KR94, Proof of Lemma 6.10] shows that in all cases $\Theta_\beta(g, h; \Phi)$ is termwise absolutely integrable on $H(F)\backslash H(\mathbb{A})$. Therefore, for $g_0 \in \widetilde{G_0(\mathbb{A})}$,

$$\begin{aligned} I_\beta(g_0; \Phi) &= \int_{H(F)\backslash H(\mathbb{A})} \Theta_\beta(g_0, h; \omega(\alpha)\Phi)\, dh \\ &= \int_{H(F)\backslash H(\mathbb{A})} \int_{H_v} \alpha(h')\Theta_\beta(g_0, hh'; \Phi)\, dh'\, dh \\ &= \int_{H(F)\backslash H(\mathbb{A})} \Theta_\beta(g_0, h; \Phi)\, dh \\ &= \int_{H(F)\backslash H(\mathbb{A})} \sum_{x \in V^n(F), Q(x)=\beta} \omega(g_0)\Phi(h^{-1}x)\, dh \\ &= \int_{H(F)\backslash H(\mathbb{A})} \sum_{y \in V^{\ell_0}(F), Q(y)=\beta_0} \omega(g_0)\Phi(0, h^{-1}y)\, dh = I_{\beta_0}(g_0; \Phi_0). \end{aligned}$$

The argument of [KR88a, Proof of Lemma 2.4] tells us that in all cases

$$E_\beta(g; f^{(s)}_\Phi) = \sum_{j=0}^{n-\ell} E^j_\beta(g; f^{(s)}_\Phi),$$

where

$$E^j_\beta(g; f^{(s)}_\Phi) = \sum_{\gamma \in Q^{n-\ell}_j(F)\backslash \mathrm{GL}_{n-\ell}(E)} f^{(s)}_{\Phi,\beta,j}(m_0(\gamma)g)$$

with

$$f^{(s)}_{\Phi,\beta,j}(g) = \int_{S_{\ell+j}(\mathbb{A})} f^{(s)}_\Phi\left(w_{\ell+j} n\left(\begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}\right) g\right) \psi\left(-\mathrm{tr}\left(\begin{pmatrix} 0 & 0 \\ 0 & \beta_1 \end{pmatrix} b\right)\right) db.$$

Observe that for $z \in \mathbb{A}^\times_E$,

$$E^j_\beta(m_0(z\mathbf{1}_{n-\ell})g; f^{(s)}_\Phi) = \chi^{(s)}_j(z) E^j_\beta(g; f^{(s)}_\Phi),$$

where

$$\chi^{(s)}_j(z) = \chi_\varphi\left(m_0\left(\begin{pmatrix} z\mathbf{1}_{n-\ell-j} & 0 \\ 0 & \bar{z}^{-1}\mathbf{1}_j \end{pmatrix}\right)\right) |z|^{j(j+\ell-n-2s)+(n-\ell)(s+\varrho/2)}_{\mathbb{A}_E}.$$

One can easily check that $\chi^{(s_0)}_0$ in Cases 1, 2 is distinct from $\chi^{(s_0)}_j$ with $j > 0$. On the other hand, for $z \in \mathbb{A}^\times_E$,

$$I_\beta(m_0(z\mathbf{1}_{n-\ell})g; \Phi) = \chi^{(s_0)}_0(z) I_\beta(g; \Phi).$$

If $m=1$ in Case 1, then we obtain $c\Phi(0)=\Phi(0)$ and hence $c=1$ as claimed. We assume in Case 1 that $m>1$ hereafter. It follows that

$$cI_{\beta_0}(g_0;\Phi_0)=E^0_\beta(g_0;f^{(s)}_\Phi)|_{s=s_0}=E_{\beta_0}(g_0;f^{(s)}_{\Phi_0})|_{s=0}. \tag{Cases 1, 2}$$

Provided that $V(F)$ represents β_0, an easy application of the moment map shows that both $E_{\beta_0}(g_0;f^{(s)}_{\Phi_0})|_{s=0}$ and $I_{\beta_0}(g_0;\Phi_0)$ are non-zero for a suitable choice of Φ (cf. [KR94, Proposition 2.7]) and therefore $c=2$ thanks to Proposition 5.8(i). Using Proposition 5.8(ii), we can prove $c'=2$ in the same fashion.

Finally, we consider Case 3. Then $\chi^{(s_0)}_0$ coincides with $\chi^{(s_0)}_1$ and is distinct from $\chi^{(s_0)}_j$ with $j>1$. In particular, for $g\in G(\mathbb{A})$,

$$cI_\beta(g;\Phi)=E^0_\beta(g;f^{(s)}_\Phi)|_{s=s_0}+E^1_\beta(g;f^{(s)}_\Phi)|_{s=s_0}. \tag{Case 3}$$

We temporarily put $Q_1=Q^{n-\ell}_1$ and write N_1 for the unipotent radical of Q_1. To eliminate extraneous terms in $E^1_\beta(g;f^{(s)}_\Phi)|_{s=s_0}$, we consider its constant term along the parabolic subgroup Q_1. Notice that the constant terms of $I_\beta(g;\Phi)$ and $E^0_\beta(g;f^{(s)}_\Phi)$ along Q_1 are just given by restriction. Put

$$w=\begin{pmatrix}\mathbf{1}_{n-\ell-2}&&\\&&1\\&1&\end{pmatrix},\quad T=\left\{\begin{pmatrix}\mathbf{1}_{n-\ell-2}&0&0\\&1&t\\&&1\end{pmatrix}\;\middle|\;t\in\mathrm{M}_1\right\}.$$

Then $\{1,w\}$ is a set of double coset representatives for $Q_1\backslash\mathrm{GL}_{n-\ell}/Q_1$ and $w^{-1}Q_1w\cap N_1\backslash N_1$ is represented by T. By the standard calculation, the constant term of $E^1_\beta(g;f^{(s)}_\Phi)$ along Q_1 is given by

$$f^{(s)}_{\Phi,\beta,1}(g)+\sum_{\gamma\in Q^{n-\ell_0}_1(F)\backslash\mathrm{GL}_{n-\ell_0}(F)}\int_{T(\mathbb{A})}f^{(s)}_{\Phi,\beta,1}\left(m_0\left(wt\begin{pmatrix}\gamma&0\\0&1\end{pmatrix}\right)g\right)dt.$$

The central character of the second of the two terms above, viewed as a function on $\mathrm{GL}_{n-\ell_0}(\mathbb{A})$, is given by

$$z\mapsto|z|^{1+\ell+(n-\ell-3)(s+\varrho/2)}_{\mathbb{A}},$$

which is distinct at $s=s_0$ from that of $I_\beta(g;\Phi)$. Therefore

$$cI_{\beta_0}(g_0;\Phi_0)=f^{(s_0)}_{\Phi,\beta,0}(g_0)+f^{(s_0)}_{\Phi,\beta,1}(g_0)=E_{\beta_0}(g_0;f^{(s)}_{\Phi_0})|_{s=0} \tag{Case 3}$$

and Proposition 5.8(i) again completes the proof in Case 3.

References

GT W. T. Gan and S. Takeda, *The regularized Siegel–Weil formula: the second term identity and non-vanishing of theta lifts from orthogonal groups*, J. Reine Angew. Math., to appear.

How89 R. Howe, *Transcending classical invariant theory*, J. Amer. Math. Soc. **2** (1989), 535–552.

HL99 R. Howe and S. T. Lee, *Degenerate principal series representations of* $\mathrm{GL}_n(\mathbb{C})$ *and* $\mathrm{GL}_n(\mathbb{R})$, J. Funct. Anal. **166** (1999), 244–309.

Ich01 A. Ichino, *On the regularized Siegel–Weil formula*, J. Reine Angew. Math. **539** (2001), 201–234.

Ich04 A. Ichino, *A regularized Siegel–Weil formula for unitary groups*, Math. Z. **247** (2004), 241–277.

Ich07 A. Ichino, *On the Siegel–Weil formula for unitary groups*, Math. Z. **255** (2007), 721–729.

Ike96 T. Ikeda, *On the residue of the Eisenstein series and the Siegel–Weil formula*, Compositio Math. **103** (1996), 183–218.

https://doi.org/10.1112/S0010437X11005379 Published online by Cambridge University Press

JS07 D. Jiang and D. Soudry, *On the genericity of cuspidal automorphic forms of* $\mathrm{SO}(2n+1)$*, II*, Compositio Math. **143** (2007), 721–748.

Kud97 S. Kudla, *Central derivatives of Eisenstein series and height pairings*, Ann. of Math. (2) **146** (1997), 545–646.

KR88a S. Kudla and S. Rallis, *On the Weil–Siegel formula*, J. Reine Angew. Math. **387** (1988), 1–68.

KR88b S. Kudla and S. Rallis, *On the Weil–Siegel formula II. The isotropic convergent case*, J. Reine Angew. Math. **391** (1988), 65–84.

KR90a S. Kudla and S. Rallis, *Degenerate principal series and invariant distributions*, Israel J. Math. **69** (1990), 25–45.

KR90b S. Kudla and S. Rallis, *Poles of Eisenstein series and L-functions*, in *Festschrift in honor of I. I. Piatetski-Shapiro on the occasion of his sixtieth birthday*, Part II, Israel Mathematical Conference Proceedings, vol. 3 (Weizmann, Jerusalem, 1990), 81–110.

KR92 S. Kudla and S. Rallis, *Ramified degenerate principal series representations for* $\mathrm{Sp}(n)$, Israel J. Math. **78** (1992), 209–256.

KR94 S. Kudla and S. Rallis, *A regularized Siegel–Weil formula: the first term identity*, Ann. of Math. (2) **140** (1994), 1–80.

KRY06 S. Kudla, M. Rapoport and T. Yang, *Modular forms and special cycles on Shimura curves*, Annals of Mathematics Studies, vol. 161 (Princeton University Press, Princeton, NJ, 2006).

KS97 S. Kudla and W. J. Sweet Jr, *Degenerate principal series representations for* $U(n,n)$, Israel J. Math. **98** (1997), 253–306.

LZ98 S. T. Lee and C.-B. Zhu, *Degenerate principal series and local theta correspondence*, Trans. Amer. Math. Soc. **350** (1998), 5017–5046.

LZ08 S. T. Lee and C.-B. Zhu, *Degenerate principal series and local theta correspondence III*, J. Algebra **319** (2008), 336–359.

Li89 J.-S. Li, *Singular unitary representations of classical groups*, Invent. Math. **97** (1989), 237–255.

Lok06 H. Y. Loke, *Howe quotients of unitary characters and unitary lowest weight modules*, Represent. Theory **10** (2006), 21–47, with an appendix by S. T. Lee.

Moe97 C. Moeglin, *Non nullité de certains relêvements par séries théta*, J. Lie Theory **7** (1997), 201–229.

MVW87 C. Moeglin, M.-F. Vignera and J.-L. Waldspurger, *Correspondence de Howe sur un corps p-adique*, Lecture Notes in Mathematics, vol. 1291 (Springer, New York, 1987).

Shi82 G. Shimura, *Confluent hypergeometric functions on tube domains*, Math. Ann. **260** (1982), 269–302.

Shi99 G. Shimura, *Generalized Bessel functions on symmetric spaces*, J. Reine Angew. Math. **509** (1999), 35–66.

Sie35 C. L. Siegel, *Über die analytische Theorie der quadratishen Formen*, Ann. of Math. (2) **36** (1935), 527–606.

Sie51 C. L. Siegel, *Indefinite quadratische Formen und Funktionentheorie I*, Math. Ann. **124** (1951), 17–54; *II*, (1952), 364–387.

Swe90 W. J. Sweet Jr, *The metaplectic case of the Siegel–Weil formula*, Thesis, University of Maryland (1990).

Swe95 W. J. Sweet Jr, *Functional equations of p-adic zeta integrals and representations of the metaplectic group*, Preprint (1995).

Tan98 V. Tan, *A regularized Siegel–Weil formula on* $U(2,2)$ *and* $U(3)$, Duke Math. J. **94** (1998), 341–378.

Tan99 V. Tan, *Poles of Siegel Eisenstein series on* $U(n,n)$, Canad. J. Math. **51** (1999), 164–175.

https://doi.org/10.1112/S0010437X11005379 Published online by Cambridge University Press

Wei65 A. Weil, *Sur la formule de Siegel dans la théorie des groupes classiques*, Acta Math. **113** (1965), 1–87.

Yam S. Yamana, *Degenerate principal series representations for quaternionic unitary groups*, Israel J. Math., to appear.

Yam10 S. Yamana, *On the Siegel–Weil formula for quaternionic unitary groups*, Preprint (2010).

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