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BOOK REVIEW

MINIMAL FLOWS AND THEIR EXTENSIONS

By J. AUSLANDER: North-Holland 1988, 265 pages, ISBN 0444 704531.

Topological dynamics has its origins in the work of Poincaré and G. D. Birkhoff on the qualitative theory of differential equations [**1**]. The underlying structure is that of a transformation group, i.e. a triple (X, T, π) where X is a topological space, T a topological group, and $(x, t) \to^{\pi} xt : X \times T \to X$ a continuous map such that $xe = x$ and $(xt)s = x(ts)$ $(x \in X,\ t, s \in T)$; where e is the identity of the group T. The set X is called the phase space and T the phase group of the triple (X, T, π).

For the rest of this review all phase spaces will be assumed compact Hausdorff. Also the letter 'π' will be omitted and the pair (X, T) or simply X referred to as a flow.

Originally (in the work of Poincaré and Birkhoff) the group T was either the reals, $\mathbf{R}$ or the integers, $\mathbf{Z}$, and topological methods were used to study such questions as stability and recurrence. These results were then generalized to arbitrary groups and codified in the colloquium volume by Gottschalk and Hedlund [**7**]. There the various notions involved were abstracted and analyzed in an attempt to determine their essential properties and to develop a theory which would shed some light on the original situation.

Of the many concepts discussed in [**7**] the one most germane to the book under review is that of a minimal flow i.e. one in which the orbit of every point is dense. Such flows are important because every flow (X, T) contains at least one closed, not empty, T-invariant subset Y such that the induced flow (Y, T) is minimal.

Let G be a compact topological group, T a dense subgroup and π the restriction of the group multiplication to $G \times T$. Then it is immediate that (G, T) is a minimal flow. If H is a closed subgroup of G, then π induces an action of T on the homogeneous space $G/H = \{Hg \mid g \in G\}$ which is again a minimal flow.

Minimal flows of the form above may be characterized intrinsically. Thus the minimal flow (X, T) is isomorphic to one of the form $(G/H, T)$ if and only if the family of maps $(\pi^t \mid t \in T)$ is equicontinuous. (Such flows are naturally called equicontinous ones.)

Let (X, T) be a flow with metrizable phase space X, and let d be a metric on X. Set $\rho(x, y) = \sup\{d(xt, yt) \mid t \in T\}$ and $\delta(x, y) = \inf\{d(xt, yt) \mid t \in T\}$. Then (X, T) is equicontinuous if and only if ρ is continuous. (Note that ρ always defines a metric not necessarily continuous on X such that $\rho(xt, yt) = \rho(x, y)(x, y \in X, t \in T)$.)

It is also clear that (X, T) equicontinuous implies that $\delta(x, y) > 0$ $(x, y \in X$ with $x \neq y)$. For a while it was conjectured that this latter condition (which became known as distality) together with minimality implied equicontinuity. However, counter examples were soon found, but distality proved to be an extremely fruitful concept.

Intuitively speaking the flow (X, T) is distal if two distinct points cannot be brought arbitrarily close together by the elements of T. It may be characterized without reference to a metric by means of the diagonal action $((x, y), t) \to (xt, yt): X \times X \times T \to X \times X$ of T on $X \times X$, viz (X, T) is distal if and only if $\overline{(x, y)T} \cap \Delta = \emptyset$ ($x, y \in X$ with $x \neq y$), where $\Delta = \{(x, x) \mid x \in X\}$.

A fundamental result in abstract topological dynamics is the Furstenberg structure theorem which classified minimal distal flows in terms of isometric extensions.

The flow (X, T) is an *extension* of the flow (Y, T) ((Y, T) is a factor of (X, T)) if there exists an epimorphism ϕ of (X, T) onto (Y, T); i.e. a continuous map ϕ of X onto Y such that $\phi(xt) = \phi(x)t$ ($x \in X$, $t \in T$). The extension $\phi : (X, T) \to (Y, T)$ is *isometric* if given $\varepsilon > 0$ there exists $\delta > 0$ such that $x_1, x_2 \in X$ with $\phi x_1 = \phi x_2$ implies that $d(x_1, t, x_2 t) < \varepsilon$ ($t \in T$). Notice that the flow (X, T) is equicontinuous if and only if (X, T) is an isometric extension of the 'trivial' or one-point flow. (For ease of exposition I have assumed X metrizable, but this is unnecessary in the definition and the following result.)

THEOREM (Furstenberg) [**5**]. *Let (X, T) be minimal distal. Then there exists an ordinal η, a family of flows $(X_\alpha \mid \alpha \leq \eta)$ and isometric extensions $\phi_\alpha : X_{\alpha+1} \to X_\alpha$ ($0 \leq \alpha$, $\alpha + 1 \leq \eta$) such that X_0 is the trivial flow, $X_\eta = X$, and $X_\beta = \varprojlim X_\alpha$ ($\alpha < \beta$) for limit ordinals $\beta \leq \eta$.*

This result gave rise to several other structure theorems in the same vein (see [**2**], [**4**], [**8**]) and it soon became clear that it would prove fruitful to 'relativize' the various notions involved. Thus, for example, instead of distal flows the 'proper' objects of study were distal extensions of flows. (The extension $\phi : X \to Y$ is *distal* if $\inf\{d(x_1 t, x_2 t) \mid t \in T\} > 0$ whenever $x_1, x_2 \in X$ with $x_1 \neq x_2$ and $\phi(x_1) = \phi(x_2)$.) This may be the reason that the author appropriately entitled the book under review 'minimal flows and their extensions' rather than simply 'minimal flows'.

Abstract topological dynamics is only a small part of the vast field of dynamical systems which has a flow as its basic model. Flows seem to occur everywhere: in the study of differential equations, harmonic analysis on groups, combinatorial number theory, random walks on groups, to mention but a few. However, till now the methods and results of the abstract theory have had little impact on these areas of research. This is unfortunate for many of the problems considered in the latter have to do with the 'asymptotic' properties of the acting group involved, and methods have been developed in abstract dynamics to study these by means of the algebraic properties of the 'points at infinity.'

Thus to each flow (X, T) is canonically 'attached' its enveloping semigroup $E(X) = \text{cls}\{\pi^t \mid t \in T\} \subset X^X$. This semigroup has a rich algebraic structure which reflects many of the dyamical properties of the flow (X, T). (For example a minimal-flow (X, T) is equicontinuous if and only if $E(X)$ is a compact topological group and an arbitrary one is distal if and only if $E(X)$ is a group.)

Let $\mathscr{M}$ denote the category the objects of which are minimal flows with phase group, T and maps are homomorphisms of flows. Then $\mathscr{M}$ has a universal object i.e. a minimal flow (M, T) which is an extension of (X, T) $((X, T) \in \mathcal{M})$. This allows one to study not only an individual element of $\mathcal{M}$ but also how it 'sits' in $\mathcal{M}$.

Let G denote the group of automorphisms of $\mathcal{M}$. Then to each $X \in \mathcal{M}$ may be assigned canonically a subgroup, $\mathrm{gp}(X)$ of G consisting of those elements of G which 'act as the identity' on X. Clearly these groups would not appear if one studied (X, T) in isolation. However, they turn out to be very important in the study of extensions of minimal flows.

The book under review is a pleasant, leisurely account of the topics discussed. In places it reads like a novel. The author's approach to the subject might be termed 'topological' as opposed to 'algebraic'; he stresses the former and down-plays the latter wherever possible.

There are two disadvantages to this approach. First, the author is unable to prove some results in their full generality. (This is not really a serious objection since the book is clearly not intended as a research monograph.)

A more serious objection is that it will tend to confirm many people's belief that topological dynamics is more a point of view, a method of attack rather than a subject in its own right. Thus one couches one's problem in the language of flows and then attacks it from scratch as though there were no general theory which might be applicable to all dynamical systems. Judging from the introduction this was surely not the author's intention.

Again in this vein the author might have served his apparent purpose better by choosing examples to illustrate the power of the abstract approach to elucidate what is going on even when the ab-initio approach can be pushed through.

However, in the final analysis these objections boil down to a matter of taste: how best to prove the results and how to attract more people to this beautiful area of research. There are two other books which cover much of the same material, the reviewer's 'Lectures on Topological Dynamics' [3] and Glasner's 'Proximal Flows' [6]. The former may be too austere and the latter too specialized as a general introduction. Hopefully the Auslander text will be more successful in introducing the subject to students and researchers from other areas.

REFERENCES

[1] G. D. Birkhoff. Dynamical systems. *A.M.S. Coll. Publ.* **9** (1927).
[2] R. Ellis. The Veech structure theorem. *Trans. A.M.S.* **186** (1973), 203-218.
[3] R. Ellis. Lecture on topological dynamics. (Benjamin, New York, 1969).
[4] R. Ellis, S. Glasner, L. Shapiro & P. J. Flows. *Adv. Math.* **17** (1975), 213-260.
[5] H. Firstenberg. The structure of distal flows. *Amer. J. Math.* **85** (1963), 477-515.
[6] S. Glasner. Proximal flows. *Springer Verlag Lect. Notes in Math.* **517** (1976).
[7] W. H. Gottschalk & G. H. Hedlund. Topological dynamics. *A.M.S. Coll. Publ.* **36** (1955).
[8] W. A. Veech. Point distal flows. *Amer. J. Math.* **92** (1970), 205-242.

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