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ONE-DIMENSIONAL MONOID RINGS WITH n-GENERATED IDEALS

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ABSTRACT. A commutative ring R is said to have the n-generator property if each ideal of R can be generated by n elements. Rings with the n-generator property have Krull dimension at most one. In this paper we consider the problem of determining when a one-dimensional monoid ring $R[S]$ has the n-generator property where R is an artinian ring and S is a commutative cancellative monoid. As an application, we explicitly determine when such monoid rings have the three-generator property.

Let R be a commutative ring with identity. If an ideal I of R can be generated by n elements, then we say that I is *n-generated*; and, if every ideal of R is n-generated, we say that R has the *n-generator property*. Determining when a group or monoid ring has the n-generator property has been studied in [1], [3], [4], [6], [7], [8], [9], [11] and [12]. The case $n = 1$ can be found in [4] or Chapter 19 of [2]. Monoid (and group) rings with the two-generator property were determined in [8] and [9]. In this note we consider the problem of determining when a one-dimensional monoid ring $R[S]$ has the n-generator property where R is an artinian ring and S is a commutative cancellative monoid.

All rings and groups will be commutative and the groups will be written additively. All monoids S will be commutative and cancellative and the quotient group of a monoid S will be written $G(S)$ while the invertible elements of S will be denoted by $\tau(S)$. We refer to [2] or [5] for elementary properties of group or monoid rings. A typical element of $R[S]$ will be written $a_1X^{s_1} + \cdots + a_kX^{s_k}$ where $a_j \in R$ and $s_j \in S$. When we refer to (R, M) being a local ring, we mean R is a noetherian ring with unique maximal ideal M; $l(M)$ will denote the length of M. The integers will be denoted by Z while the nonnegative integers will be denoted by Z_+.

It is well-known that a ring with the n-generator property has Krull dimension zero or one. From this restriction on dimension we have $1 \geq \dim(R[S]) = \dim\big(R[G(S)]\big) = \dim(R) + \alpha$ where α denotes the torsionfree rank of $G(S)$ and dim is Krull dimension. Thus, three cases naturally arise: $\dim(R) = 0$, $\alpha = 0$; $\dim(R) = 1$, $\alpha = 0$; and $\dim(R) = 0$ and $\alpha = 1$. In this note we are interested in the last case. For the first two cases $S = G(S)$ is a finite group and in the last case $G(S) = Z \oplus H$ where H is a finite group. Group rings with Krull dimension zero having the n-generator property were studied in [9] and determined when the base ring R is a field. Also, it was determined in [9] when $R[Z \oplus H]$ has the n-generator property where R is a field. Note that the arguments needed for the second case seem to be of a different nature (see [8, Section 2]) than the other two.

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Assume S is a submonoid of $Z_+ \oplus H$ with quotient group $G(S) = Z \oplus H$ where H is a finite group. If $R = R_1 \oplus \cdots \oplus R_s$ is a direct sum of rings, then the monoid ring $R[S]$ has the n-generator property if and only if each $R_i[S]$ has the n-generator property. Thus, to determine when $R[S]$ has the n-generator property, it suffices to consider the case of when R is an artinian local ring.

Let S be as defined in the previous paragraph. Section 1 contains results for an arbitrary n. In particular, Theorem 1.2 yields necessary and sufficient conditions for the monoid ring $R[S]$ to have the n-generator property where R is a local ring and $\tau(S) = 0$. In Theorem 1.4 we are able, in certain situations, to remove the assumption of $\tau(S) = 0$. In Section 2 we apply these theorems to determine when the monoid ring $R[S]$ has the three-generator property.

1. **The general case.** Let S be a cancellative monoid with quotient group $G(S) = Z \oplus H$ where H is a finite group. If $\phi\colon G(S) \longrightarrow Z$ is the projection mapping, then we may assume that $\phi(S) = Z$ or $\phi(S) \subseteq Z_+$ since $G\big(\phi(S)\big) = Z$. Also, $R[S]$ is graded by the first coordinates of S and the homogeneous elements of degree j are sums of elements of the form rX^s where $r \in R$ and $s = (j, h) \in Z \oplus H$. We begin with a lemma concerning submonoids of $Z_+ \oplus H$.

LEMMA 1.1. *Let $S \subseteq Z_+ \oplus H$ be monoid with quotient group $G(S) = Z \oplus H$ where H is a nonzero finite abelian group. Then there exists $N \geq 1$ so that $(m, h) \in S$ for all $m \geq N$ and $h \in H$.*

PROOF. Write $H = Z/n_1Z \oplus \cdots \oplus Z/n_tZ$ and let i be the exponent of H. We first note that $(m, 0) \in S$ for all large m. Indeed, write $(1, 0) = (n+1, \alpha) - (n, \alpha)$ where $(n+1, \alpha), (n, \alpha) \in S$ and observe that $(in, 0) = i(n, \alpha) \in S$ and $(in+1, 0) = (i-1)(n, \alpha) + (n+1, \alpha) \in S$.

Next write $(0, 1, 0, \ldots, 0) = s_1 - s_2$ where $s_1 = (n, \alpha, \gamma)$ and $s_2 = (n, \beta, \gamma)$ belong to S and $\alpha - \beta = 1$ in Z/n_1Z. Thus, $(in, 1, 0, \ldots, 0) = s_1 + (i-1)s_2 \in S$. Thus, for all large m, $(m, \alpha, 0, \ldots, 0) \in S$ for all $\alpha \in Z/n_1Z$. Since H is a finite group, the lemma follows.

Before stating our first theorem we need additional notation. Let (R, M) be an artinian local ring with $l(M) = m$ and let u be the smallest integer so that $M^{u+1} = 0$. If $m \geq 1$, then there exist $r_1, \ldots, r_m \in R$ and $0 < i_1 < \cdots < i_u = m$ so that $r_{i_{v-1}+1} + M^{v+1}, \ldots, r_{i_v} + M^{v+1}$ is a basis for the R/M-vector space M^v/M^{v+1} for $1 \leq v \leq u$. Also, for the proof of the following theorem we need to define a congruence $\sim$ on the monoid S. Let $\sim$ be the equivalence relation defined on S by $s_1 \sim s_2$ if $s_1 = s_2 + s$ for some $s \in \tau(S)$. Note that $\tau(S/\sim) = 0$. See [2, Theorem 4.4].

THEOREM 1.2. *Assume (R, M) is an artinian local ring and let S be a submonoid of $Z_+ \oplus H$ with quotient group $G(S) = Z \oplus H$ where H is a finite abelian group. Let i be the smallest positive integer so that $(i, h) \in S$ for some $h \in H$. Set $l(M) = m$.*

(a) Let k be the order of H/U where $\tau(S) = 0 \oplus U$. If $R[S]$ has the n-generator property, then $(1+m)ik \leq n$.

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(b) Let k be the order of H. If $(1+m)ik \leq n$, then $R[S]$ has the n-generator property.

PROOF. Fix $g \in H$ so that $\alpha = (i, g) \in S$. Also, let $r_1, \ldots, r_m \in R$ be the elements defined in the paragraph preceding the statement of the theorem.

(a) Let $\sim$ be the equivalence relation defined above. Then $\tau(S/\sim) = 0$, and $R[S/\sim]$ has the n-generator property. If we write $G(S/\sim) = Z \oplus K$ for K a finite abelian group, then we have $H/U \simeq K$. Thus, we may assume $\tau(S) = 0$.

Suppose $(1+m)ik > n$. Now write $H = \{h_1, \ldots, h_k\}$ and let $a_j = r_{m-j+1}$ for $1 \leq j \leq m$. By the above lemma there exists $N \geq 1$ so that $(j, h) \in S$ for all $j \geq N$ and $h \in H$. Now set $s_1 = (N, h_1), s_2 = (N, h_2), \ldots, s_k = (N, h_k), s_{k+1} = (N+1, h_1), \ldots, s_{ik} = (N+i-1, h_k)$. Let I be the ideal generated by

$$\begin{gathered} a_1X^{s_1}, \ldots, a_1X^{s_{ik}}, \\ a_2X^{s_1+\alpha}, \ldots, a_2X^{s_{ik}+\alpha}, \\ \vdots \\ a_mX^{s_1+(m-1)\alpha}, \ldots, a_mX^{s_{ik}+(m-1)\alpha}, \\ X^{s_1+m\alpha}, \ldots, X^{s_{ik}+m\alpha}. \end{gathered}$$

Then $I = (f_1, \ldots, f_n)$ for some $f_1, \ldots, f_n \in R[S]$. We may write for $1 \leq j \leq n$

$$f_j = b_j(1,1)X^{s_1} + \cdots + b_j(ik,1)X^{s_k} + b_j(1,2)X^{s_1+\alpha} + \cdots + b_j(ik,2)X^{s_{ik}+\alpha} + \cdots$$

where

$$\begin{gathered} b_j(1,1), \ldots, b_j(ik,1) \in (a_1)R; \\ b_j(1,2), \ldots, b_j(ik,2) \in (a_1, a_2)R; \\ \vdots \\ b_j(1,m), \ldots, b_j(ik,m) \in (a_1, a_2, \ldots, a_m)R. \end{gathered}$$

We now consider the matrix B formed by using the coefficients of $f_1, \ldots, f_n$. If one performs the usual row operations (interchanging two rows, replacing a row by a unit (of R) multiple of itself; replacing a row with itself added to a unit multiple of another row) on B to form B', then the resulting elements of $R[S]$ obtained in the obvious way from the rows of B' will still generate I. Now perform the procedure described on page 25 of [6] in order to row reduce B. The resulting matrix will have the upper triangular pattern

$$\begin{bmatrix} D_1 & S_{12} & S_{13} & S_{14} & \cdots \\ 0 & D_2 & S_{23} & S_{24} & \cdots \\ 0 & 0 & D_3 & S_{34} & \cdots \\ \vdots & \vdots & & & \end{bmatrix}$$

where $D_j = a_jI_{ik}$, I_{ik} is the ik by ik identity matrix, and S_{uv} is an ik by ik diagonal matrix whose entries belong to $(a_u)R$. Since $(1+m)ik > n$, let ξ be the $(n+1)$-st generator in the original list of generators of I. Then $\xi \in (f_1, \ldots, f_n)$ leads to a contradiction.

(b) As above, write $H = \{h_1, \ldots, h_k\}$. Now suppose there is an ideal $I = (f_1, \ldots, f_{n+1})$ which requires $n + 1$ generators, and assume that the f_j have the form $f_j = b_{j1}X^{s_{j1}} + \cdots + b_{jk}X^{s_{jk}} +$ lower degree terms where $b_{uv} \in R$, $s_{jv} = (e_j, h_v) \in S$ and at least one member of the k-tuple $(b_{j1}, \ldots, b_{jk})$ is nonzero. Furthermore, we assume these generators are chosen so that $e_1 + \cdots + e_{n+1}$ is minimal among all such $n + 1$ generators of I and $e_1 \leq \cdots \leq e_{n+1}$. We now show how to select $w = (1 + m)k + 1$ of these generators so that we can use elimination techniques to reduce the degree of a generator. Viewing $e_1, \ldots, e_{n+1}$ modulo i, determine the number of the e_j which belong to each of the residue classes $0, 1, \ldots, i-1$. Since at least one of the residue classes contains at least w elements, there is a subsequence $e_{j_1}, \ldots, e_{j_w}$ with the property that the difference of any two terms of this subsequence is divisible by i. We now restrict our attention to the corresponding $f_{j_1}, \ldots, f_{j_w}$. For notational convenience we relabel these as $f_1, \ldots, f_w$. Now let $t_j = (e_{j+1} - e_j)/i$ and set $\alpha_j = t_j\alpha$ where $1 \leq j \leq w-1$. Then for $2 \leq j \leq w$ we can rearrange and relabel s_{uv} so that $s_{uv} = (e_1, h_v) + \alpha_1 + \cdots + \alpha_{u-1}$. Of course, we also rearrange and relabel the corresponding coefficients of the f_j for $2 \leq j \leq w$. Note that the coefficients of $f_1, \ldots, f_w$ can be written as $b_{uv} = c_{uv}(0) + c_{uv}(1)r_1 + c_{uv}(2)r_2 + \cdots + c_{uv}(m)r_m$ where $c_{uv}(j)$ is zero or a unit of R. Now consider the w by $w - 1$ matrix C which consists of the initial coefficients of the $f_1, \ldots, f_w$:

$$\begin{bmatrix} c_{11}(0) & c_{11}(1)r_1 & c_{11}(2)r_2 & \cdots & c_{11}(m)r_m & c_{12}(0) & \cdots & c_{2k}(m)r_m \\ c_{21}(0) & c_{21}(1)r_1 & c_{21}(2)r_2 & \cdots & c_{21}(m)r_m & c_{22}(0) & \cdots & c_{2k}(m)r_m \\ \vdots & & & & & & & \\ c_{w1}(0) & c_{w1}(1)r_1 & c_{w1}(2)r_2 & \cdots & c_{w1}(m)r_m & c_{w2}(0) & \cdots & c_{wk}(m)r_m \end{bmatrix}.$$

We reach a contradiction if the above matrix is equivalent to a matrix with a row of zeros. (By equivalent we mean performing the latter two of the three row operations described above. Interchanging two rows is not allowed because of the restriction $e_1 \leq \cdots \leq e_w$.) Locate the first nonzero entry, if any, in the first column of C and use it to eliminate the nonzero entries below it. At this point one must rearrange the entries of the rows which have been altered. Indeed, if one considers the sum $\epsilon_1 r_j + \epsilon_2 r_j$ where ϵ_1, ϵ_2 are units of R, then $\epsilon_1 + \epsilon_2$ may be a unit of R, zero or a nonzero nonunit of R. For the first two possibilities, no rearranging is needed. However, for the third one, $\epsilon_1 r_j + \epsilon_2 r_j = \epsilon_{j+1} r_{j+1} + \cdots + \epsilon_m r_m$ where the ϵ's are either units of R or 0. Note that for all three cases, the terms $r_1, \ldots, r_{j-1}$ were not involved, *i.e.*, the rearrangement of the rows does not affect prior columns. So, proceed to the second column and repeat the process. Because of the size of C, we will eventually obtain a row of zeros, and hence arrive at our contradiction.

The following example illustrates that the lack of the assumption "$\tau(S) = 0$" in part (b) of the above theorem can lead one to overestimate the number of generators. Let F be a field of odd characteristic and let $S = \langle (1,0), (2,1), (0,2) \rangle$ be a submonoid of $Z_+ \oplus Z/4Z$. Since $m = 0$, $i = 1$ and $k = 4$, $F[S]$ has the four-generator property; however, $F[S]$ has the two-generator property by [8, Theorem 3.1].

The following corollary shows that when $\tau(S) = 0$ we do obtain necessary and sufficient conditions for when $R[S]$ has the n-generator property. Also note that Theorem 5.17 of [6] is just the case of $k = 1$ in the following corollary.

COROLLARY 1.3. *Assume (R, M) is an artinian local ring and let S be a submonoid of $Z_+ \oplus H$ with quotient group $G(S) = Z \oplus H$ where H is a finite abelian group of order k. Assume $\tau(S) = 0$. Let i be the smallest positive integer so that $(i, h) \in S$ for some $h \in H$. Set $l(M) = m$. Then $R[S]$ has the n-generator property if and only if $(1 + m)ik \leq n$.*

We now generalize an argument found in the proof of [8, Theorem 3.1] to show the assumption $\tau(S) = 0$ in Corollary 1.3 may be removed under certain circumstances.

THEOREM 1.4. *Let F be a field and S a submonoid of $Z_+ \oplus H$ with quotient group $G(S) = Z \oplus H$ where H is a finite abelian group whose order is a unit of F. Let k be the order of H/U where $\tau(S) = 0 \oplus U$, and i the smallest positive integer so that $(i, h) \in S$ for some $h \in H$. Then $F[S]$ has the n-generator property if and only if $ik \leq n$.*

PROOF. If $F[S]$ has the n-generator property, then apply Theorem 1.2. For the converse, $F[Z_+][H]$ is an etale $F[Z_+]$-algebra by [8, Lemma 1.5] and is integrally closed by [10, p. 75, Proposition 2]. Thus, $F[Z_+][H]$ is the integral closure of $F[S]$. Choose representatives $h_1, \ldots, h_k \in H$ of H/U. Let $B = \{(u, h_v) \in S : 0 \leq u \leq i - 1, 1 \leq v \leq k\}$. Then $F[Z_+][H] = F[S][X^b : b \in B]$. Thus, the integral closure of $F[S]$ is generated as a module by ik elements over $F[S]$. Since $F[S]$ is reduced, $F[S]$ has the ik-generator property by [3, Theorem 2.3].

2. **The case $n = 3$.** In the case of $n = 2$, Theorem 3.1 of [8] now follows easily from the results of Section 1 and [8, Theorem 2.7]. We illustrate this by obtaining the corresponding theorems for $n = 3$. In particular, the following Theorems 2.1 and 2.4 are the $n = 3$ versions of [8, Theorems 2.7 and 3.1], respectively.

THEOREM 2.1. *Let R be a commutative ring and let S be a cancellative monoid whose quotient group $G(S)$ has torsionfree rank one, say $G(S) = Z \oplus H$ where H is a finite group of order $m_0 = 2^u 3^v m_1$ where m_1 is not divisible by 2 or 3. If $H \subset S$, then $R[S]$ has the three-generator property if and only if the following hold.*

(i) *$R = R_1 \oplus \cdots \oplus R_s$ where each (R_j, M_j) is a local artinian ring with the two-generator property.*

(ii) *m_0 is a unit in each R_j which is not a field of characteristic 2 or 3.*

(iii) *If any R_j is a field of characteristic 2 or 3, then u or v is less than or equal to 1, repectively.*

(iv) *If M_j is not principal, then $M_j^2 = 0$; but if M_j is principal, then $M_j^3 = 0$.*

(v) *One of the following holds:*

  (a) *$S \simeq Z \oplus H$.*

  (b) *$S \simeq Z_+ \oplus H$.*

  (c) *$S \simeq T \oplus H$ where T is a submonoid of $Z_+ \setminus \{1\}$ containing 2, each R_j is a field, and if any R_j has characteristic 2 or 3, then u or v equals 0, respectively.*

  (d) *$S \simeq T \oplus H$ where T is a submonoid of $Z_+ \setminus \{1, 2\}$ containing 3, each R_j is a field, and if any R_j has characteristic 2 or 3, then u or v equals 0, respectively.*

PROOF. Note that if $S = G(S)$, then we are in the case treated by [9, Theorem 5.1]. Thus, since $R[S]$ having the three-generator property implies $R[G(S)]$ has the three-generator property, conditions (i)–(iv) follow from [9, Theorem 5.1]. It suffices to show parts (c) and (d) of (v). Consider $R_j[T]$. By Corollary 1.3, $(1+m)i \leq 3$ where $m = l(M)$ and $i = \min(T \setminus \{0\})$. (Here $k = 1$.) If $i = 2$ or 3, then $m = 0$, *i.e.*, R_j is a field. Since $R_j[H][T]$ has the three-generator property, apply the above argument to the decomposition of $R_j[H]$ as a direct sum of local rings to obtain that $R_j[H]$ is a direct sum of fields. Hence, m_0 is a unit of R_j. Thus, (c) and (d) hold.

For the converse, since $R[Z]$ has the three-generator property if and only if the polynomial ring $R[x]$ has the three-generator property, if $S \simeq Z \oplus H$ or $Z_+ \oplus H$, then we are done by [9, Theorem 5.1]. Assume part (c) or (d) of (v) hold. Since the order of H is a unit of R, $R[H]$ is a direct sum of fields. Thus, it suffices to show $R[T]$ has the three-generator property where R is a field. Using the notation of Corollary 1.3, $i = 2$ or 3 and $m = 0$. Thus, $(1+m)i \leq 3$ and $R[T]$ has the three-generator property.

With the next theorem we complete the characterization by assuming H is not contained in S. The proof depends upon the next two lemmas, the results of Section 1 and Theorem 2.1. When $p = 2$ the first lemma is just [8, Lemma 3.4]. Its proof is omitted since it requires only minor modifications to the proof of [8, Lemma 3.4]. Note that when we say S is irreducible we mean S has no nontrivial finite direct summands.

LEMMA 2.2. *Let S be a submonoid of $Z_+ \oplus H$ with quotient group $G(S) = Z \oplus H$ where H is a finite abelian p-group, p a prime. If S is irreducible and* $0 \oplus pH \subseteq \tau(S)$, *then* $0 \oplus pH = \tau(S)$.

The next lemma follows from the one above and Theorem 1.2.

LEMMA 2.3. *Let R be a commutative ring and S a submonoid of $Z_+ \oplus H$ with quotient group $G(S) = Z \oplus H$ where H is a finite abelian group. If S is irreducible and $R[S]$ has the three-generator property, then $H = Z/2^jZ$ or $Z/3^jZ$ for some $j \geq 0$. Furthermore, if $\tau(S) = 0$, then $j \leq 1$.*

THEOREM 2.4. *Let R be a ring and let S be a submonoid of $Z_+ \oplus H$ with quotient group $G(S) = Z \oplus H$ where H is a finite abelian group not contained in S. Then $R[S]$ has the three-generator property if and only if the following hold:*

(i) $R = R_1 \oplus \cdots \oplus R_s$ where R_j is a field.

(ii) $S = S_1 \oplus K$, K a finite abelian group of order m and either $m = 0$ or is a unit in each of the R_j.

(iii) One of the following holds:

(a) S_1 is a submonoid of $Z_+ \oplus Z/2^uZ$ with $u \geq 1$ containing $(0,2)$ *and* $(1,g)$ *for some $g \in Z/2^uZ$. If some R_j has characteristic* 2, *then* $u = 1$.

(b) S_1 is a submonoid of $Z_+ \oplus Z/3^uZ$ with $u \geq 1$ containing $(0,3)$ *and* $(1,g)$ *for some $g \in Z/3^uZ$. If some R_j has characteristic* 3, *then* $u = 1$.

PROOF. Just as in the proof of [8, Theorem 3.1] we may assume S is irreducible, *i.e.*, $m = 0$ in (ii). For the sufficiency of the conditions we may assume R is a field. If

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(a) of (iii) holds, then $R[S]$ has the two-generator property by [8, Theorem 3.1]. Now assume (b) holds. Depending on whether the characteristic of R is or is not 3, $R[S]$ has the three-generator property by either Corollary 1.3 or Theorem 1.4, respectively.

For the necessity of (i)—(iii) we have, by Lemma 2.3, $H = Z/2^uZ$ or $Z/3^uZ$ with $u \geq 1$ and $G(S) = Z \oplus H$ (since we have assumed S is irreducible). Now write $R = R_1 \oplus \cdots \oplus R_s$ where each R_j is a local artinian ring and then consider $R_j[S/\sim]$ where $\sim$ is the equivalence relation defined in Section 1. Using Corollary 1.3 (and its notation) we have R_j is a field, $i = 1$ and $G(S/\sim) = Z \oplus Z/2Z$ or $Z \oplus Z/3Z$. Thus, we have the description of S found in (iii). It remains only to show if $G(S) = Z \oplus Z/2^uZ$ or $Z \oplus Z/3^uZ$ with the characteristic of some R_j being 2 or 3, respectively, then $u = 1$. But this follows from Theorem 2.1.

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References

1. J. T. Arnold and R. Matsuda, *The n-generator property for semigroup rings*, Houston J. Math. **12**(1986), 345–356.
2. R. Gilmer, *Commutative Semigroup Rings*, University of Chicago Press, Chicago, 1984.
3. C. Greither, *On the two generator problem for ideals of a one-dimensional ring*, J. Pure Appl. Algebra **24**(1982), 265–276.
4. B. Hardy and T. Shores, *Arithmetical semigroup rings*, Canad. J. Math. **32**(1980), 1361–1371.
5. G. Karpilovsky, *Commutative Group Algebras*, Marcel Dekker, New York, 1983.
6. R. Matsuda, *Torsion-free abelian semigroup rings, V*, Bull. Fac. Sci., Ibaraki Univ., Math. (A) **11**(1979), 1–37.
7. ———, *n-Generator property of a polynomial rings*, Bull. Fac. Sci., Ibaraki Univ., Math. (A) **16**(1984), 17–23.
8. J. Okon, D. Rush and J. P. Vicknair, *Commutative semigroup rings with two-generated ideals*, J. London Math. Soc., (2) **45**(1992), 417–432.
9. J. Okon and J. P. Vicknair, *Group rings with n-generated ideals*, Comm. Algebra **20**(1992), 189–217.
10. M. Raynaud, *Anneaux locaux Henseliens*, Lecture Notes in Math. (169), Springer-Verlag, Berlin, 1970.
11. D. E. Rush, *Rings with two-generated ideals*, J. Pure Appl. Algebra **73**(1991), 257–275.
12. A. Shalev, *Dimension subgroups, nilpotency indices, and the number of generators of ideals in p-group algebras*, J. Algebra **129**(1990), 412–438.

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