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The Orthonormal Dilation Property for Abstract Parseval Wavelet Frames

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Abstract. In this work we introduce a class of discrete groups containing subgroups of abstract translations and dilations, respectively. A variety of wavelet systems can appear as $\pi(\Gamma)\psi$, where π is a unitary representation of a wavelet group and Γ is the abstract pseudo-lattice Γ. We prove a sufficent condition in order that a Parseval frame $\pi(\Gamma)\psi$ can be dilated to an orthonormal basis of the form $\tau(\Gamma)\Psi$, where τ is a super-representation of π. For a subclass of groups that includes the case where the translation subgroup is Heisenberg, we show that this condition always holds, and we cite familiar examples as applications.

1 Introduction and Preliminaries

Given a Parseval frame $\{\psi_\alpha\}$ in a Hilbert space $\mathcal{H}$, it is known that there is a Hilbert space $\mathcal{K}$ and an orthornomal basis $\{\Psi_\alpha\}$ for $\mathcal{K}$ such that $\mathcal{H} \subset \mathcal{K}$ and $\psi_\alpha = P_{\mathcal{H}}(\Psi_\alpha)$, where $P_{\mathcal{H}}$ is the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$ [11]. In this case it is said that $\{\Psi_\alpha\}$ is an orthonormal dilation of $\{\psi_\alpha\}$. If $\{\psi_\alpha\}$ is of the form $\pi(G)\psi$ where G is group and π is a unitary representation of G, then it is also known [11] that there is an orthonormal dilation of the form $\tau(G)\Psi$, where τ is a unitary representation of G acting in $\mathcal{K}$ such that $\tau(g)|_{\mathcal{H}} = \pi(g)$ for all $g \in G$ and such that $P_{\mathcal{H}}(\Psi) = \psi$. An affine wavelet system is not of the form $\pi(G)\psi$, but there is nevertheless an underlying group structure that can be regarded as having the form $\pi(\Gamma)\psi$, where Γ is a discrete *pseudo-lattice* in a group G. For the wavelet system $\{2^{j/2}\psi(2^j \cdot -k) : j \in \mathbb{Z}, k \in \mathbb{Z}\}$ in $L^2(\mathbb{R})$, one can take G to be the connected Lie group of affine transformations of the line with π the quasiregular representation induced from the dilation subgroup, or (as in [8]) one can take G to be the Baumslag–Solitar group $BS(1,2) = \langle u, t : utu^{-1} = t^2\rangle$ with $\pi(u)$ and $\pi(t)$ the 2-dilation and unit translation, respectively. When a Parseval wavelet frame has such a structure, it is natural to ask if there is an orthonormal dilation with the same structure; more precisely, if $\{\psi_\alpha\} = \pi(\Gamma)\psi$, is there a unitary representation τ of G acting in a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$, and a vector $\Psi \in \mathcal{K}$, such that $\tau(g)|_{\mathcal{H}} = \pi(g)$ for all $g \in G$ and such that $P_{\mathcal{H}}(\Psi) = \psi$? In this case we say that $\pi(\Gamma)\psi$ has the G-dilation property, and it was then natural to ask for an explicit description of a G-dilation of $\pi(\Gamma)\psi$. For the 2-wavelet system on the line, it was shown in [8] that for the $G = BS(1,2)$, every system $\pi(\Gamma)\psi$ has the G-dilation property, and an explicit description of G-dilations

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is carried out for Shannon-type wavelets. More recently, various generalizations of results in [8, 11] have been obtained in [2].

In this paper we introduce a natural and general class of groups G for which a number of well-known function systems, including both affine wavelet systems and shearlet systems, can be viewed as systems of the form $\pi(\Gamma)\psi$, where Γ is a pseudo-lattice in G. We generalize the methods of [8] in this direction to prove a sufficient condition on the group G in order that every such system has the G-dilation property. We then describe two natural families of wavelet groups and prove that they satisfy this sufficient condition. As one example, we exhibit a natural group G and representation π such that a shearlet system is of the form $\pi(\Gamma)\psi$ and has the G-dilation property.

For the remainder of this paper, all groups are automatically countable and discrete. By *representation* of a group G, we shall mean a homomorphism of G into the group of unitary operators on some Hilbert space $\mathcal{H}$ that is continuous in the strong operator topology. Representations will be assumed to *faithful*, that is, one-to-one mappings.

Let Γ_0 be a countable discrete group and $\alpha\colon \Gamma_0 \to \Gamma_0$ a monomorphism. Define

$$G(\alpha, \Gamma_0) := \left\langle u, \Gamma_0 : u\gamma u^{-1} = \alpha(\gamma),\ \ \forall\, \gamma \in \Gamma_0 \right\rangle.$$

The subset $\Gamma = \Gamma_1\Gamma_0$, where $\Gamma_1 = \{u^j : j \in \mathbb{Z}\}$ will be called the *standard pseudo-lattice* in G. As an example, observe that if $\Gamma_0 = \mathbb{Z}$ and α_2 is the monomorphism of $\mathbb{Z}$ defined by $\alpha(1) = 2$, then $G(\alpha, \mathbb{Z}) = BS(1, 2)$.

In the following section we use positive-definite maps to obtain a sufficient condition on the group G in order that every Parseval wavelet frame $\pi(\Gamma)\psi$ has the G-dilation property. Then in Section 3 we prove that our condition holds for two families of groups $G(\alpha, \Gamma_0)$ and describe three examples.

2 The Group Dilation Property

A map $K\colon X \times X \to \mathbb{C}$ is called a *positive definite map* if for all finite sequences $\{\gamma_1, \gamma_2, \dots, \gamma_k\}$ in X and $\{\xi_1, \xi_2, \dots, \xi_k\} \subset \mathbb{C}$,

$$\sum_{1\le i,j\le k} K(\gamma_i, \gamma_j)\xi_i\overline{\xi}_j \ge 0.$$

If $X = G$ is a group, then, following [7], we say that $K\colon G \times G \to \mathbb{C}$ is a *group positive definite map* if K is a positive definite map and $K(sx, sy) = K(x, y)$ holds for all s, x, and y in G. By [7, Theorem 2.8], every group positive definite map has the form

$$K_{\rho,\eta}(x, y) = \left\langle \rho(x)\eta, \rho(y)\eta \right\rangle,$$

where ρ is a representation of G and η is a cyclic vector for ρ.

For the remainder of this section, we fix a group $G = G(\alpha, \Gamma_0)$, with $\Gamma = \Gamma_1\Gamma_0$, and write the element $u^j\gamma \in \Gamma$ as (j, γ). Let ρ be a representation of Γ_0; we say that a representation T is an *α-root of ρ* if $T \circ \alpha = \rho$. In the following abstract version of [8, Theorem 2.1], we use this notion to formulate a sufficient condition in order that a positive definite map on Γ extends to a group positive definite map.

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Proposition 2.1 *Suppose that every representation of Γ_0 has an α-root. Let $K\colon \Gamma \times \Gamma \to \mathbb{C}$ be a positive definite mapping such that for any (j,γ) and (j',γ') in Γ, and $\gamma_0 \in \Gamma_0$, the relations*

$$\tag{2.1} \begin{aligned} K\big((j+1,\gamma),(j'+1,\gamma')\big) &= K\big((j,\gamma),(j',\gamma')\big)\\ K\big((j,\alpha^{-j}(\gamma_0)\gamma),(j',\alpha^{-j'}(\gamma_0)\gamma')\big) &= K\big((j,\gamma),(j',\gamma')\big), \quad j,j' \leqslant 0, \end{aligned}$$

both hold. Then K is the restriction of a group positive definite map $K_{\tau,\psi}$. More explicitly, there is a representation τ of G acting in a Hilbert space $\mathcal{H}$ and a vector $\psi \in \mathcal{H}$, such that $\mathcal{H} = \overline{\operatorname{span}}\{\tau(\Gamma)\psi\}$ and

$$K\big((j,\gamma),(j',\gamma')\big) = \big\langle \tau(j,\gamma)\psi, \tau(j',\gamma')\psi \big\rangle .$$

Proof By a theorem attributed to Kolmogorov (see, for example, [7]), we have a Hilbert space $\mathcal{H}$ and a mapping $v\colon \Gamma \to \mathcal{H}$, such that $\operatorname{span}\{v(j,\gamma) : (j,\gamma) \in \Gamma\}$ is dense in $\mathcal{H}$, and

$$K\big((j,\gamma),(j',\gamma')\big) = \big\langle v(j,\gamma), v(j',\gamma')\big\rangle$$

holds for all (j,γ) and (j',γ') belonging to Γ. Define the operator $D\colon \mathcal{H} \to \mathcal{H}$ by $Dv(j,\gamma) = v(j+1,\gamma)$ and by extending to all of $\mathcal{H}$ by linearity and density as usual. The first of the relations (2.1) shows that D is unitary. For each $n = -1,0,1,2,\dots,$ set

$$\mathcal{H}_n = \overline{\operatorname{span}}\{v(j,\gamma) : (j,\gamma) \in \Gamma,\ j \le n\}.$$

Note that $D\mathcal{H}_n = \mathcal{H}_{n+1}$ and $\mathcal{H}_n \subset \mathcal{H}_{n+1}$. Set $\mathcal{K}_n = \mathcal{H}_n \ominus \mathcal{H}_{n-1}$, $n \ge 0$. For $\gamma_0 \in \Gamma_0$, define the operator $T_0(\gamma_0)$ on $\mathcal{H}_0$ by

$$T_0(\gamma_0)\big(v(j,\gamma)\big) = v\big(j,\alpha^{-j}(\gamma_0)\gamma\big)$$

and again extending to all of $\mathcal{H}_0$; the second relation in (2.1) shows that $\gamma \mapsto T_0(\gamma)$ is a (unitary) representation of Γ_0. Since the subspace $\mathcal{K}_0$ is invariant under T_0, we can define the representation ρ_1 of Γ_0 acting in $\mathcal{K}_1$ by $\rho_1(\gamma) = DT_0(\gamma)D^{-1}$. Now by our hypothesis, ρ_1 has an α-root T_1, since T_1 acts in $\mathcal{K}_1$ and satisfies $T_1 \circ \alpha = \rho_1$. Now the representation $\gamma \mapsto \rho_2(\gamma) = DT_1(\gamma)D^{-1}$ of Γ_0 acting in $\mathcal{K}_2$ has an α-root T_2 acting in $\mathcal{K}_2$. Continuing in this way, we obtain, for each positive integer n, a representation T_n of Γ_0 acting in $\mathcal{K}_n$, so that

$$T_n \circ \alpha = DT_{n-1}D^{-1}.$$

(Again in the preceding, T_0 is restricted to $\mathcal{K}_0$.) Now write

$$\mathcal{H} = \mathcal{H}_0 \oplus \big(\bigoplus_{n\ge 1} \mathcal{K}_n\big)$$

and define the representation T of Γ_0 by $T = T_0 \oplus \big(\bigoplus_{n\ge1} T_n\big)$.

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Next we must verify the relation $DT(\gamma)D^{-1} = T(\alpha(\gamma))$. Fix $\gamma_0 \in \Gamma_0$; for $v(j,\gamma)$ with $j \leq 0$,

$$\begin{aligned}\big(DT_0(\gamma_0)D^{-1}\big)\big(v(j,\gamma)\big) &= \big(DT_0(\gamma_0)\big)\big(v(j-1,\gamma)\big)\\ &= D\big(v(j-1,\alpha^{-j+1}(\gamma_0)\gamma)\big)\\ &= T_0\big(\alpha(\gamma_0)\big)\big(v(j,\gamma)\big),\end{aligned}$$

and hence the relation $DT_0(\gamma)D^{-1} = T_0\big(\alpha(\gamma)\big)$ holds on $\mathcal{H}_0$. Now for $v \in \mathcal{H}$, write $v = \sum_{n\geq 0} v_n$. We have $DT(\gamma)D^{-1}v_0 = T\big(\alpha(\gamma)\big)\,v_0$ and for $n \geq 1$,

$$DT(\gamma)D^{-1}v_n = DT_{n-1}(\gamma)D^{-1}v_n = T_n\big(\alpha(\gamma)\big)\,v_n,$$

so

$$DT(\gamma)D^{-1}v = \sum_{n\geq 0} DT(\gamma)D^{-1}v_n = \sum_{n\geq 0} T_n\big(\alpha(\gamma)\big)\,v_n = T\big(\alpha(\gamma)\big).$$

It follows that the mapping τ defined by $\tau(u) = D$ and $\tau(\gamma) = T(\gamma)$ is a representation of G.

Finally, take $\psi = v(0,0)$. Then

$$v(j,\gamma) = D^j v(0,\gamma) = D^j T(\gamma)v(0,0) = D^j T(\gamma)\psi,$$

so ψ is cyclic for τ. Hence the group positive definite map defined for all $x, y \in G$ by $K_{\tau,\psi}(x,y) = \langle \tau(x)\psi, \tau(y)\psi\rangle$ is an extension of K. ■

We combine the preceding with general results also from [8] to obtain our condition for the G-dilation property.

Theorem 2.2 *Suppose that every representation of Γ_0 has an α-root, and let π be any representation of $G(\alpha,\Gamma_0)$. Then every Parseval wavelet frame $\pi(\Gamma)\psi$ has the G-dilation property.*

Proof Let $\Gamma = \Gamma_1\Gamma_0 \subset G$ as above and recall that we write $u^j\gamma = (j,\gamma)$. Define

$$K\big((j,\gamma),(j',\gamma')\big) = \delta_{j,j'}\delta_{\gamma,\gamma'} - \big\langle \pi(j,\gamma)\psi, \pi(j',\gamma')\psi\big\rangle.$$

Observe that $\delta_{j+1,j'+1} = \delta_{j,j'}$ and

$$\delta_{j,j'}\delta_{(\alpha^{-j}\gamma_0)\gamma,(\alpha^{-j'}\gamma_0)\gamma'} = \delta_{j,j'}\delta_{\gamma,\gamma'},$$

and that in the group G, $u^{-j}\gamma_0 u^j = \alpha^{-j}(\gamma_0)$ holds for $j, \gamma_0 \in \Gamma_0$. Hence

$$\begin{aligned}K\big((j+1,\gamma),(j'+1,\gamma')\big) &= \delta_{j+1,j'+1}\delta_{\gamma,\gamma'} - \big\langle \pi(j+1,\gamma)\psi, \pi(j'+1,\gamma')\psi\big\rangle\\ &= \delta_{j,j'}\delta_{\gamma,\gamma'} - \big\langle D\pi(j,\gamma)\psi, D\pi(j',\gamma')\psi\big\rangle\\ &= K\big((j,\gamma),(j',\gamma')\big),\end{aligned}$$

and for $j, j' \le 0$,

$$
\begin{aligned}
&K\big(\big(j,(\alpha^{-j}\gamma_0)\gamma\big),\big(j',(\alpha^{-j'}\gamma_0)\gamma'\big)\big)\\
&\quad=\delta_{j,j'}\delta_{\alpha^{-j}(\gamma_0)\gamma,\alpha^{-j'}(\gamma_0)\gamma'}-\big\langle \pi\big(j,\alpha^{-j}(\gamma_0)\gamma\big)\,\psi,\pi\big(j',\alpha^{-j'}(\gamma_0)\gamma'\big)\,\psi\big\rangle\\
&\quad=\delta_{j,j'}\delta_{\gamma,\gamma'}-\big\langle \pi(\gamma_0 u^j\gamma)\psi,\pi(\gamma_0 u^{j'}\gamma')\psi\big\rangle\\
&\quad=\delta_{j,j'}\delta_{\gamma,\gamma'}-\big\langle \pi(\gamma_0)\pi(j,\gamma)\psi,\pi(\gamma_0)\pi(j',\gamma')\psi\big\rangle\\
&\quad=\delta_{j,j'}\delta_{\gamma,\gamma'}-\big\langle \pi(j,\gamma)\psi,\pi(j',\gamma')\psi\big\rangle\\
&\quad=K\big((j,\gamma),(j',\gamma')\big)\,.
\end{aligned}
$$

The calculations show that the map K satisfies both of the conditions in (2.1). By Proposition 2.1 we conclude that K is a positive definite map and hence there exists a representation τ of G with Hilbert space $\mathcal{K}$ and $\eta \in \mathcal{K}$ such that $K = K_{\tau,\eta}$ on $\Gamma \times \Gamma$. Then by [8, Lemma 2.5, proof of Theorem 2.6] $\pi \oplus \tau$ is a super-representation of π (acting in $\mathcal{H} \oplus \mathcal{K}$) for which $\tilde{\psi} = \psi \oplus \eta$ is a G-dilation vector for ψ and $\tilde{\pi}(x)\psi = \pi(x)\psi$. ■

Observe that in the case of $BS(1,2) = G(\alpha_2,\mathbb{Z})$, the fact that every representation of Γ_0 has an α-root is a simple consequence of the Borel functional calculus. For every unitary operator T on a Hilbert space $\mathcal{H}$, there is a unitary operator S such that $S^2 = T$. However, in general it seems difficult to prove that a pair (α,Γ_0) has the property that every representation of Γ_0 has an α-root. In the following section we describe two families of groups $G(\alpha,\Gamma_0)$ for which this property does in fact hold.

3 Examples

We begin with the case where Γ_0 is a finitely-generated abelian group. A variety of fundamental results for countable abelian groups have been obtained by Baggett, Bownik, Merrill, Furst, Packer, and many others. See, for example, [1].

Example 3.1 (**A**-wavelet system) Let Γ_0 be the free abelian group generated by $t_1, t_2, \dots, t_n$, and let $\alpha(t_j) = t_1^{a_{1j}} t_2^{a_{2j}} \cdots t_n^{a_{nj}}$, where $\mathbf{A} = [a_{i,j}] \in GL(n,\mathbb{Z})$.

We claim that every representation of Γ_0 has an α-root. Let ρ be any representation of Γ_0, and write $\mathbf{A}^{-1} = [b_{i,j}]$. Since the $b_{i,j}$ are rational, the Borel functional calculus obtains operators $V_{i,j}$, $1 \le i, j \le n$ such that $V_{i,j} = \rho(t_1)^{b_{i,j}}$. Define $T(t_j)$, $1 \le j \le n$ by

$$T(t_j) = V_{1,j} V_{2,j} \cdots V_{n,j}.$$

An easy computation shows that $T \circ \alpha = \rho$.

Next we consider wavelet groups where the subgroup Γ_0 is nilpotent, but not abelian. Nearest to the abelian case is the case where Γ_0 is Heisenberg: let $\Gamma_0 = \langle t_1, t_2, t_3\rangle$ with relations $t_3t_2 = t_1t_2t_3$, $t_1t_2 = t_2t_1$, $t_1t_3 = t_3t_1$. Then Γ_0 is isomorphic

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with the discrete Heisenberg group

$$\mathbb{H} = \left\{ \begin{bmatrix} 1 & k & m \\ 0 & 1 & l \\ 0 & 0 & 1 \end{bmatrix} : k, l, \text{and } m \text{ are integers} \right\}$$

via the map $t_1 \mapsto t_1^m, t_2 \mapsto t_2^l, t_3 \mapsto t_3^k$, and we identify $\Gamma_0 = \mathbb{H}$. For any positive numbers a and b, the mapping α defined by $\alpha(t_3) = t_3^a, \alpha(t_2) = t_2^b, \alpha(t_1) = t_1^{ab}$ is a monomorphism of $\mathbb{H}$.

When α is of the form above, we use the notation $G(\alpha, \mathbb{H}) = G(a, b, \mathbb{H})$. The following lemma shows that, at least where a and b are integers, $G(a, b, \mathbb{H})$ has the α-root property.

Lemma 3.2 *Let A, B, and C be unitary operators on a Hilbert space $\mathcal{H}$ satisfying $AB = CBA$, $AC = CA$, $BC = CB$, and let a, b, and c be positive integers such that $c = ab$. Suppose that U and V are unitary operators belonging to the von-Neumann algebra generated by A and B, and satisfying $U^a = A$ and $V^b = B$. Then the element $W = UVU^{-1}V^{-1}$ satisfies $UW = WU$, $VW = WV$, and $W^c = C$.*

Proof Let $\mathcal{A}$ be the von Neumann algebra generated by A and B. The group N generated by A and B is isomorphic with the Heisenberg group $\mathbb{H}$, and so for any P and Q in N, $[P, Q] = PQP^{-1}Q^{-1}$ belongs to the center of N. It follows that $[\mathcal{A}, \mathcal{A}] \subset \operatorname{cent}(\mathcal{A})$ and in particular $W \in \operatorname{cent}(\mathcal{A})$. It remains to show that $W^c = C$. To prove this, we proceed by induction on $c = ab$. If $c = 1$, then $a = b = 1$, and there is nothing to prove. Suppose that $c > 1$ and that for any a', b', c' with $a'b' = c'$ and $c' < c$, we have

$$W^{c'} = U^{a'} V^{b'} U^{-a'} V^{-b'}.$$

If $a > 1$, then we have

$$W^{(a-1)b} = U^{a-1} V^b U^{-a+1} V^{-b}.$$

Observe that U commutes with $V^b U^{-a+1} V^{-b}$. Indeed, by definition of W, $UV^b = W^b V^b U$, so $UV^{-b} = W^{-b} V^{-b} U$, from which the observation follows. Hence

$$\begin{aligned} W^{ab} = W^{(a-1)b} W^b &= \left(U^{a-1} V^b U^{-a+1} V^{-b}\right)\left(UV^b U^{-1} V^{-b}\right) \\ &= U^{a-1}\left(V^b U^{-a+1} V^{-b}\right) U \left(V^b U^{-1} V^{-b}\right) \\ &= U^{a-1} U \left(V^b U^{-a+1} V^{-b}\right)\left(V^b U^{-1} V^{-b}\right) \\ &= U^a V^b U^{-a} V^{-b}. \end{aligned}$$

If $a = 1$, then $b > 1$, and the proof is similar. ■

It is almost immediate that for α as in the preceding, every representation of $\mathbb{H}$ has an α-root. More generally, we consider the following class of groups that includes $G(\alpha, \mathbb{H})$. Let n be a positive integer, and let $t_1, t_2, \dots, t_n$, and z_{ij}, $1 \le i, j \le n$ satisfy the relations for all i, j and k:

$$t_i t_j = z_{i,j} t_j t_i, \quad \text{and} \quad z_{ij} t_k = t_k z_{ij}.$$

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Observe that the relation $z_{ji} = z_{ij}^{-1}$ follows from the above. The group

$$F_n = \langle t_1, t_2, \dots t_n, z_{ij}, 1 \leq i, j \leq n \rangle$$

is the free, two-step (discrete) nilpotent group generated by the n elements $t_k, 1 \leq k \leq n$.

Theorem 3.3 *Define $\alpha \colon F_n \to F_n$ by $\alpha(t_k) = t_k^{a_k}$ and $\alpha(z_{ij}) = z_{ij}^{a_i a_j}$, where the a_k are integers. Then every representation of F_n has an α-root.*

Proof Let ρ be any representation of F_n acting in $\mathcal{H}$, put $A_k = \rho(t_k)$, $C_{ij} = \rho(z_{ij}), 1 \leq i, j, k \leq n$ and let $\mathcal{A}$ be the von-Neumann algebra generated by $\{A_1, \dots, A_n\}$. An argument similar to that of Lemma 3.2 applied to the group N generated by $\{A_1, \dots, A_n\}$ shows that $[\mathcal{A}, \mathcal{A}] \subset \operatorname{cent}(\mathcal{A})$. By the Borel functional calculus, for each k we have $U_k \in \mathcal{A}$ such that $U_k^{a_k} = A_k$. Now for each i and j put $W_{ij} = U_i U_j U_i^{-1} U_j^{-1}$. By the preceding we have that W_{ij} is central, and by Lemma 3.2, $W_{ij}^{a_i a_j} = C_{i,j}$. Put $T(t_k) = U_k$ and $T(z_{ij}) = W_{ij}, 1 \leq i, j, k \leq n$. Since

$$T(z_{ij}) = T(t_i) T(t_j) T(t_i)^{-1} T(t_j)^{-1}$$

holds for all i and j, then T is a representation of F_n. Since

$$T\big(\alpha(t_k)\big) = T(t_k^{a_k}) = T(t_k)^{a_k} = A_k = \rho(t_k),$$

and

$$T\big(\alpha(z_{ij})\big) = T(z_{ij}^{a_i a_j}) = T(z_{ij})^{a_i a_j} = C_{ij} = \rho(z_{ij}),$$

then $T \circ \alpha = \rho$. ∎

The following are two examples of representations of $G(a, a, \mathbb{H})$, where $\mathbb{H}$ is the simply connected Heisenberg group.

Example 3.4 Let π be the representation of $G(2, 2, \mathbb{H})$ acting in $L^2(\mathbb{R}^2)$ by $t_1 \mapsto e^{2\pi i \lambda} I$, $t_2 \mapsto M$, and $t_3 \to T$, where I is the identity operator, and M and T are the operators on $\mathcal{H} = L^2(\mathbb{R}^2)$ given by

$$M f(\lambda, t) = e^{-2\pi i \lambda t} f(\lambda, t), \quad T f(\lambda, t) = f(\lambda, t - 1).$$

Now define $\pi(u) f(\lambda, t) = f(4\lambda, 2^{-1} t) 2^{3/2}$. The systems $\pi(\Gamma)\psi$ are Fourier transforms of wavelet systems of multiplicity one subspaces of $L^2(\mathbb{H})$, and large classes of Parseval wavelet frames have been found in our earlier work [4].

Example 3.5 (Shearlet system) Let π be the representation of $G(a, a, \mathbb{H})$ given by $u \mapsto D, t_1 \mapsto T_1, t_2 \mapsto T_2$, and $t_3 \mapsto M$, where D, T_1, T_2, M are the unitary operators on $L^2(\mathbb{R}^2)$ defined by

$$D f(x) = a^{-3/2} f(a^{-2} x_1, a^{-1} x_2) \qquad M f(x) = f(x_1 - x_2, x_2)$$
$$T_1 f(x) = f(x_1 - 1, x_2) \qquad T_2 f(x) = f(x_1, x_2 - 1).$$

Systems of this form have been well studied; see, for example, [6, 9, 10].

Remark Lemma 3.2 can be used to prove that for other nilpotent groups Γ_0, every representation has an α-root. For example, let

$$\Gamma_0 = \langle t_1, t_2, t_3, t_4, t_5 : t_5 t_4 = t_4 t_5 t_2, t_5 t_3 = t_3 t_5 t_1, t_i t_j = t_j t_i, 1 \le i, j \le 4\rangle;$$

Γ_0 is the integer lattice in a two-step simply-connected Lie group whose Lie algebra has basis $\{X_1, X_2, \dots, X_5\}$ with $[X_5, X_4] = X_2$ and $[X_5, X_3] = X_1$, $[X_i, X_j] = 0, 1 \le i, j \le 4$. Let a and b be integers and define $\alpha\colon \Gamma_0 \to \Gamma_0$ by $\alpha(t_5) = t_5^a$, $\alpha(t_k) = t_k^b$, $k = 3, 4$ and for $k = 1, 2$, $\alpha(t_k) = t_k^{ab}$. By application of Lemma 3.2 to $\{\pi(t_5), \pi(t_3), \pi(t_1)\}$ and $\{\pi(t_5), \pi(t_4), \pi(t_2)\}$, we find that π has an α-root. One example of π is the following. Let $\pi\colon G \to \mathcal{U}\big(L^2(\mathbb{R}^4)\big)$ be given by $u \mapsto D$, $t_k \mapsto T_k$, $k = 1, 2, 3, 4$, and $t_5 \mapsto M$, where T_k is the translation operator $T_k f(x) = f(x_1, \dots, x_k - 1, \dots x_4)$ and D, M are defined by

$$Df(x) = a^{-3/2} f\big((ab)^{-1}x_1, (ab)^{-1}x_2, a^{-1}x_3, a^{-1}x_4\big)$$
$$Mf(x) = f(x_1 - x_3, x_2 - x_4, x_3, x_4).$$

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