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HAMILTONIAN CYCLES IN STRONG PRODUCTS OF GRAPHS

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ABSTRACT. Let $\bar{\times} G^k$ denote the graph $G \bar{\times} G \bar{\times} \cdots \bar{\times} G$ (k times) where $G \bar{\times} H$ is the strong product of the two graphs G and H. In this paper we prove the conjecture of J. Zaks [3]: For every connected graph G with at least two vertices there exists an integer $k = k(G)$ for which the graph $\bar{\times} G^k$ is hamiltonian.

Let G be a graph (undirected) and let $V(G)$ and $E(G)$ denote the vertex set and the edge set of G. The strong product $G \bar{\times} H$ of two graphs G and H is defined by

$$V(G \bar{\times} H) = V(G) \times V(H)$$

$E(G \bar{\times} H) = \{\{(u_1, v_1)(u_2, v_2)\} \mid u_1, u_2 \in V(G), v_1, v_2 \in V(H)$ and
either $u_1 = u_2$ and $\{v_1, v_2\} \in E(H)$
either $v_1 = v_2$ and $\{u_1, u_2\} \in E(G)$
either $\{u_1, u_2\} \in E(G)$ and $\{v_1, v_2\} \in E(H)\}$.

This product is commutative and associative and following [3] we shall denote by $\bar{\times} G^k$ the graph $G \bar{\times} G \bar{\times} \cdots \bar{\times} G$ (k times).

In [3] J. Zaks proved that: "For every h and k, $h \geq 1$, $k \geq 1$, there exists an h-connected graph $G = G(h, k)$, such that the graph $\bar{\times} G^k$ is non-hamiltonian" and asked:

"Is it true that for every connected graph G with at least two vertices there exists an integer $k = k(G)$ for which the graph $\bar{\times} G^k$ is hamiltonian".

We give an affirmative answer to this question in Theorem 11, the proof of which needs Lemmata and Propositions 1 to 10.

Our notations are as follows:

— P_n the path with n vertices.
— $d_G(x)$ the degree of the vertex x in G.
— $\Delta(G)$ the maximum degree of the vertices of G.
— For a in $V(H)$ (resp. b in $V(G)$) G_a (resp. H_b) denote the subgraph $G \bar{\times} \{a\}$ of $G \bar{\times} H$ (resp. $\{b\} \bar{\times} H$ of $G \bar{\times} H$).

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The reader is referred to C. Berge [1] for any graph theory terms not defined here.

LEMMA 1. *The strong product of two connected graphs is connected.*

LEMMA 2. *For every graph G and every integer $n \geq 2$, there exists a covering of the vertices of $G \bar{\times} P_n$ by vertex-disjoint subgraphs isomorphic to P_n.*

Proof. The subgraphs of the covering are $(P_n)_a$ with $a \in V(G)$. ■

LEMMA 3. *For every n and m, $2 \leq n < m$, there exists a covering of the vertices of $K_{1,n} \bar{\times} K_{1,m}$ by vertex-disjoint subgraphs isomorphic to $K_{1,n}$.*

Proof. The subgraphs of the covering are $(K_{1,n})_a$ with $a \in V(K_{1,m})$. ■

LEMMA 4. *For every n, $n \geq 3$, there exists a covering of the vertices of $\bar{\times} K_{1,n}^2$ by vertex-disjoint subgraphs isomorphic to K_{1,n_i} with $n_i < n$.*

Proof. The general construction is an easy generalization of decomposition shown in Fig. 1 for $n = 5$. ■

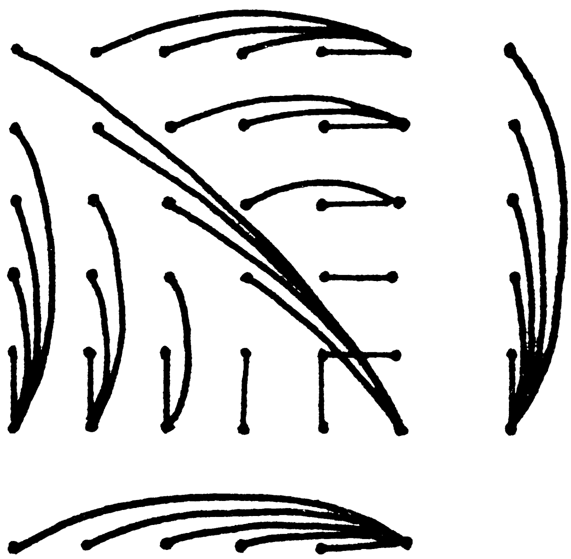

Figure 1

PROPOSITION 5. *For every connected graph G, there exists an integer $k_1 = k_1(G)$ such that $V(\bar{\times} G^{k_1})$ can be covered by vertex-disjoint paths of positive length.*

Proof. Every non empty graph G with no isolated vertices is vertex covered by disjoint paths of positive length and by stars, as can be easily shown by induction on the number of vertices of G (or by considering a maximal matching of G). Let the disjoint paths P_{n_i}, $i \in I$, of positive length and stars

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K_{1,n_j}, $j \in J$, cover all vertices of G. If $J = \emptyset$ the proposition is true for $k_1 = 1$. If $J \neq \emptyset$, $V(\overline{\times} G^2)$ can be covered by vertex-disjoint subgraphs of the following types: $P_{n_i} \overline{\times} P_{n_j}$, $P_{n_i} \overline{\times} K_{1,n_j}$, $K_{1,n_i} \overline{\times} K_{1,n_j}$.

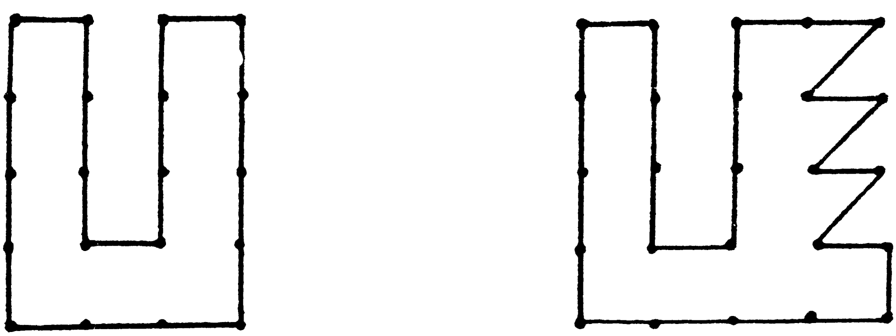
Then, as a consequence of Lemmata 2, 3, 4, $V(\overline{\times} G^2)$ admits a covering by vertex-disjoint subgraphs isomorphic to paths of positive length and stars K_{1,n_l}, $l \in L$ with

$$\max\{n_l, l \in L\} \leq \max\{n_j, j \in J\} - 1.$$

Since $\overline{\times} G^2$ is connected (Lemma 1) an easy induction on $\max\{n_j, j \in J\}$ shows that an integer k_1 exists, as required; in fact, k_1 can be chosen to satisfy $k_1 \leq 2^{\Delta(G)-2}$. ■

LEMMA 6. *For every n and m, n, $m \geq 2$, $P_n \overline{\times} P_m$ admits a hamiltonian cycle (of length nm).*

Proof. The construction of a hamiltonian cycle in $P_n \overline{\times} P_m$ is an immediate generalization of one of the two following constructions of Fig. 2. ■



n or m even   n and m odd

Figure 2

PROPOSITION 7. *Let G be a graph of maximum degree $\Delta(G)$; if there exists a covering of $V(G)$ by vertex-disjoint paths of positive length, then there exists an integer $k_2 = k_2(G)$ such that $V(\overline{\times} G^{k_2})$ can be covered by vertex-disjoint cycles of length at least $\Delta(G)$.*

Proof. Let us consider a covering of $V(G)$ by vertex-disjoint paths P_{n_i}, $i \in I$, $n_i \geq 2$. Then, by Lemma 6, $V(\overline{\times} G^2)$ admits a covering by vertex-disjoint cycles of length at least $(\text{Inf}_{i \in I} n_i)^2$ and thus by paths with at least $(\text{Inf}_{i \in I} n_i)^2$ vertices. Then, by induction, $\overline{\times} G^p$ can be covered by vertex-disjoint cycles of length at least $(\text{Inf } n_i)^p$. As $\text{Inf}_{i \in I} n_i \geq 2$, there exists an integer k_2 such that $(\text{Inf}_{i \in I} n_i)^{k_2} \geq \Delta(G)$ ■

As pointed out by the referee the following lemma is similar to lemma 5 of [2] (see also the proof of theorem 4 in [2]).

LEMMA 8. *Let T be a tree of maximum degree $\Delta(T)$ and C be a cycle of length $k \geq \Delta(T)$, then $T \bar{\times} C$ is hamiltonian and there exists a hamiltonian cycle which uses, for any vertex $u \in V(T)$ exactly $k - d_T(u)$ edges of C_u.*

Proof. By induction on $|V(T)|$.

When $T = \{u\}$, $\{u\} \bar{\times} C$ is isomorphic to C, so the result is true.

If $|V(T)| > 1$, let u be an end vertex of T and v its neighbour in T, let T' be the subtree induced by $V(T) - \{u\}$. By induction hypothesis, $T' \bar{\times} C$ admits a hamiltonian cycle which uses $k - d_{T'}(v) = k - (d_T(v) - 1)$ edges of C_v. Since $k \geq \Delta(T)$, $k - d_{T'}(v) \geq 1$, so there exists an edge $\{(v, c_1)(v, c_2)\}$ of this hamiltonian cycle, with $\{c_1, c_2\}$ in $E(C)$.

Then we construct a hamiltonian cycle in $T \bar{\times} C$ by replacing this edge by the following path:

— the edge $\{(v, c_1)(u, c_1)\}$
— the path obtained by removing the edge $\{(u, c_1)(u, c_2)\}$ of the cycle C_u
— the edge $\{(u, c_2)(v, c_2)\}$.

This cycle uses for any vertex u in T exactly $k - d_T(u)$ edges of C_u. ■

REMARK. Lemma 8 gives an iterative method to construct a hamiltonian cycle of $T \bar{\times} C$. Furthermore, by starting the construction with u and v we have that, for any edge $\{u, v\}$ of T and for any edge $\{c_1, c_2\}$ of C, there exists a hamiltonian cycle in $T \bar{\times} C$ which uses the two following edges $\{(u, c_1)(v, c_1)\}$ and $\{(u, c_2)(v, c_2)\}$. ■

COROLLARY 9. If $\bar{\times} G^k$ *is hamiltonian, then for every* $p \geq 0$, $\bar{\times} G^{k+p}$ *is hamiltonian too.*

Proof. By induction on p. Lemma 8 applied with T a spanning tree of G and C a hamiltonian cycle in $\bar{\times} G^k$ shows that $\bar{\times} G^{k+1}$ is hamiltonian. ■

Corollary 9 is true even for just the cartesian product and it has been proved in [3] as theorem 1.

PROPOSITION 10. *If G is connected and if there exists a covering of $V(G)$ by α vertex-disjoint cycles of length at least l, with $\alpha \geq 2$, then for every tree T with at least two vertices and with $\Delta(T) \leq l$, there exists a covering of $V(G \bar{\times} T)$ by β vertex-disjoint cycles of length at least l with $\beta \leq \alpha - 1$.*

Proof. Since G is connected, there exists two cycles of the covering C_1 and C_2, and vertices c_1 in $V(C_1)$ and c_2 in $V(C_2)$ with $\{c_1, c_2\}$ in $E(G)$.

By Lemma 8, there exists a covering of $V(G \bar{\times} T)$ by α hamiltonian cycles of $C_i \bar{\times} T$. To prove the proposition it suffices to construct a hamiltonian cycle in $V(C_1 \bar{\times} T) \cup V(C_2 \bar{\times} T)$.

Let $\{u, v\}$ be an edge of T. By the remark of Lemma 8, we can construct a

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hamiltonian cycle H_1 in $C_1 \bar{\times} T$ (resp. H_2 in $C_2 \bar{\times} T$) which uses the edge $\{(u, c_1)(v, c_1)\}$ (resp. $\{(u, c_2)(v, c_2)\}$). We obtain a cycle in $V(C_1 \bar{\times} T) \cup V(C_2 \bar{\times} T)$ by replacing in $E(H_1) \cup E(H_2)$ the preceding edges by $\{(u, c_1)(u, c_2)\}$ and $\{(v, c_1)(v, c_2)\}$. ■

THEOREM 11. *For any connected graph G with at least two vertices there exists an integer $k = k(G)$ for which the graph $\bar{\times} G^k$ is hamiltonian.*

Proof. By Proposition 5, there exists an integer k_1 such that there exists a covering of $V(\bar{\times} G^{k_1})$ by vertex-disjoint paths, with at least two vertices; then by Proposition 7 applied to $\bar{\times} G^{k_1}$, there exists an integer k_2 such that $V(\bar{\times} G^{k_1 k_2})$ can be covered by α vertex-disjoint cycles of length at least $\Delta(G)$ (since $\Delta(G^{k_1}) \geq \Delta(G)$).

Repeated applications of Proposition 10 show that there exists an integer $k_3 \leq \alpha - 1$ such that $\bar{\times} G^{k_1 k_2 k_3}$ is hamiltonian. Thus Theorem 11 is proved with $k = k_1 k_2 k_3$. ■

REMARK. The integer k found in the proof of theorem 11 is not the best possible. We conjecture that:

CONJECTURE. For any connected graph G with at least two vertices $\bar{\times} G^{\Delta(G)}$ is hamiltonian.

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REFERENCES

1. C. Berge. *Graphs and Hypergraphs*, North-Holland. Amsterdam 1973.

2. M. Rosenfeld and D. Barnette, *Hamiltonian Circuits in Certain Prisms*, Discrete Math. **5,** 1973, 389-394.

3. J. Zaks. *Hamiltonian cycles in products of graphs*, Canadian Math. Bull. vol. **17** (5), 1975, 763–765.

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