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ON r^*-INVARIANT MEASURE ON A LOCALLY COMPACT SEMIGROUP WITH RECURRENCE

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0. **Introduction.** A regular Borel measure μ is said to be r^*-invariant on a locally compact semigroup if $\mu(Ba^{-1}) = \mu(B)$ for all Borel sets B and points a of S, where $Ba^{-1} = \{x \in S, xa \in B\}$. In [1] Argabright conjectured that *the support of an r^*-invariant measure on a locally compact semigroup is a left group*, Mukherjea and Tserpes [4] proved this conjecture in the case that the measure is finite; however their method of proof fails when the measure is infinite. In this paper the following theorem is proved:

Let F be the support of an r^*-invariant measure on a locally compact semigroup S and let F contain a compact set C of positive finite measure such that $\mu(C^* - C^*c^{-1}) = 0$, where $C^* = \bigcup_{i=0}^{\infty} Cc^{-1}$, $c \in C$ and $(cc^{-1}) \neq \phi$. Then F is a left group.

1. **Recurrence.** In a letter J. M. Rosenblatt supplied us with the following example. Let S be the additive group of real numbers and $x \to x+1$ be the translation to the right by one unit distance. Let μ be the Lebesgue measure on S. Now for any compact set C, $\bigcap_{n=1}^{\infty} C-(n+t)$ is empty, for if the contrary is true there is a real number x and a sequence $\{d_t\}$ in C, with $x = d_t \times (n+t)$ for all $t = n, n+1, \ldots$. The compactness of C yields a subsequence of $]d_t\}$ convergent to a real number. This implies that a subsequence of $\{-i\}$ converges, which is impossible. Therefore, $\mu[C - \bigcup_{n=1}^{\infty} \bigcap_{t=n}^{\infty} C-(n+t)] = \mu(C-\phi) = \mu(C) > 0$. Therefore, $\mu[C - \bigcup_{n=1}^{\infty} \bigcap_{t=n}^{\infty} C-(n+t)] = \mu(C-\phi) = \mu(C) > 0$. Also, there is no compact set of positive measure such that $\mu(\bigcap_{t=1}^{\infty} C \setminus C-i) = 0$.

Let $C^* = \bigcup_{t=0}^{\infty} Cc^{-t}$. Then $C^* - C^*c^{-1} = C - \bigcup_{t=1}^{\infty} Cc^{-t}$ is the set of points which under $x \to xc$ never return to C. Since $x \to xc$ is continuous, then

$$\begin{aligned}\liminf Cc^{-n} &= \bigcap_{n=1}^{\infty} \bigcup_{j=n}^{\infty} Cc^{-1} \\ &= \limsup Cc^{-n} \\ &= \bigcap_{n=i}^{\infty} \bigcup_{j=n}^{\infty} Cc^{-i}\end{aligned}$$

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is the set of points which under $x \to xc$ return infinitely often to C. Hence, $C - \bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} Cc^{-1}$ is the set of points of C which under $x \to xc$ do not return infinitely often to C [5, p. 334].

LEMMA. *If the measure μ is r^*-invariant on a locally compact semigroup S and there exist a compact set C of positive finite measure and $c \in C$ such that $\mu(C^* - C^*c^{-1}) = 0$, then $\mu(C - \bigcup_{n=1}^{\infty} \bigcap_{j=n}^{\infty} Cc^{-j}) = 0$.*

Proof. Following Wright [7] it is clear that $C - \bigcap_{n=1}^{\infty} \bigcup_{j=n}^{\infty} Cc^{-j} = (C - C^*c^{-1}) \cup [C \cap \bigcup_{n=1}^{\infty} (C^*c^{-n+1} - C^*c^{-n+2})]$. Since $\mu(C^* - C^*c^{-1}) = 0$ and the measure μ is r^*-invariant, then $\mu(C^*c^{-(n+1)} - C^*c^{-(n+2)}) = 0$, for each $n \geq 0$. Hence, $\mu[C - \bigcup_{n=1}^{\infty} \bigcap_{t=n}^{\infty} Cc^{-(n+t)}] = 0$.

2. **Argabright Conjecture.** Since F is a right ideal in S, then aF is a right ideal and a semigroup in S. The measure μ induces a r^*-invariant measure on aF which is the restriction of the measure μ to aF. Since Tserpes and Mukherjea [4] proved the conjecture when μ is finite, we may assume that the induced measure is infinite on aF, which is left cancellative [4]. In view of [4] and [6], F **is now assumed to be left cancellative**.

THEOREM. *Let F be the support of an r^*-invariant on a locally compact semigroup S and let F contain a compact set C of positive finite measure such that $\mu(C^* - C^*c^{-1}) = 0$, where $C^* = \bigcup_{i=0}^{\infty} Cc^{-i}$, $c \in C$ and $(cc^{-1}) \neq \phi$. Then F is a left group.*

Proof. The lemma implies that $\mu(C - \bigcup_{n=1}^{\infty} \bigcap_{t=n}^{\infty} Cc^{-(n+t)}) = 0$ and $\mu(\bigcap_{t=n_0}^{\infty} Cc^{-t}) > 0$, for some n_0. Since $\mu(\bigcap_{t=n_0}^{\infty} Cc^{-t}) = \mu(\bigcap_{t=0}^{\infty} Cc^{-t})c^{-n} = 0$, then the r^*-invariance of the measure μ implies that $\mu(\bigcap_{t=0}^{\infty} Cc^{-t}) > 0$. Let $K = \bigcap_{t=0}^{\infty} Cc^{-t}$, then $Kc \subset Cc \cap K$, $Kc \subset K$, and K is a compact set in C. If we set $L = \bigcap_{s=1}^{\infty} Kc^s$, then Lemma 2 of Łos and Schwarz [3] implies that $Lc = L$. Since $L(cc^{-1})c = Lc = L$, then $L(cc^{-1}) \subset Lc^{-1}$. The mapping $\rho_c : Lc^{-1} \to L$ defined by $\rho_c(\ell c^{-1}) = (\ell c^{-1})c$ is bijective. Hence, $(L(cc^{-1})c)c^{-1} = L(cc^{-1})$, $(Lc)c^{-1} = L(cc^{-1})$ and $Lc^{-1} = L(cc^{-1})$. Since $\ell(cc^{-1}) \subset (\ell c)c^{-1}$ for all $\ell \in L$ and $L(cc^{-1}) = (Lc)c^{-1}$, there must exist an ℓ_1 such that $\ell_1 \subset \ell_1(cc^{-1})$. Therefore, $\ell_1 = \ell_1 e$ where $e \in (cc^{-1})$ and $\ell_1 e = \ell_1 e^2$. Left cancellability implies that $e = e^2$ is an idempotent.

The fact that the support F of the measure μ contains an idempotent enables us to use the result of Berglund and Hofmann [2, p. 94] to prove that F is a left group.

3. **Remarks.** We have not succeeded in constructing an example of a locally compact semigroup with an r^*-invariant measure whose support does not contain an idempotent. Since F is a locally compact left group, then $F = E \times G$, where E is a locally compact left zero semigroup and G is a locally compact

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group. Argabright [1] showed that $\mu = v + \lambda$, where v is a positive Borel measure on E and λ is a right Haar measure on G.

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