ANZIAM J. 52(2011), 309–317
doi:10.1017/S1446181111000721

GENERAL THREE-POINT QUADRATURE FORMULAS OF EULER TYPE

IVA FRANJIĆ ✉ [1], JOSIP PEČARIĆ [2] and IVAN PERIĆ [1]

(Received 17 April, 2009; revised 4 September, 2011)

Abstract

General three-point quadrature formulas for the approximate evaluation of an integral of a function f over $[0, 1]$, through the values $f(x)$, $f(1/2)$, $f(1-x)$, $f'(0)$ and $f'(1)$, are derived via the extended Euler formula. Such quadratures are sometimes called "corrected" or "quadratures with end corrections" and have a higher accuracy than the adjoint classical formulas, which only include the values $f(x)$, $f(1/2)$ and $f(1-x)$. The Gauss three-point, corrected Simpson, corrected dual Simpson, corrected Maclaurin and corrected Gauss two-point formulas are recaptured as special cases. Finally, sharp estimates of error are given for this type of quadrature formula.

2010 *Mathematics subject classification*: primary 26D15; secondary 65D30, 65D32.

Keywords and phrases: general three-point quadrature formulas, corrected quadrature formulas, sharp estimates of error, Bernoulli polynomials, extended Euler formula.

1. Introduction

The objects of interest in this paper are quadratures of the form

$$\int_0^1 f(t)\,dt \approx w(x)f(x) + (1-2w(x))f(1/2) + w(x)f(1-x).$$

Some of the most famous quadrature rules belong to this group: the Simpson rule, the dual Simpson rule, the Maclaurin rule and even the Gauss two-point formula (the case where $w(x) = 1/2$). These formulas are accurate for all polynomials of order at most three and have been studied previously by the authors [5].

This paper is a continuation and simultaneously a generalization of our previous results. The aim is to derive three-point quadrature formulas which are accurate for all polynomials of order at most five. What might be considered as a downside is

[1]Faculty of Food Technology and Biotechnology, University of Zagreb, Pierottijeva 6, 10000 Zagreb, Croatia; e-mail: ifranjic@pbf.hr, iperic@pbf.hr.
[2]Faculty of Textile Technology, University of Zagreb, Prilaz baruna Filipovića 28a, 10000 Zagreb, Croatia; e-mail: pecaric@element.hr.

that these formulas will contain values of the first derivative at the end points of the interval. Such quadratures are sometimes called "corrected" or "quadratures with end corrections" [8, 11].

The main tool used here is the extended Euler formula, obtained by Dedić et al. [2]: if $f : [a, b] \to \mathbb{R}$ is such that $f^{(n-1)}$ is continuous and of bounded variation on $[a, b]$ for some $n \geq 1$, then, for every $y \in [a, b]$,

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(t)\,dt = f(y) &- \sum_{k=1}^{n-1}\frac{(b-a)^{k-1}}{k!}B_k\Big(\frac{y-a}{b-a}\Big)[f^{(k-1)}(b) - f^{(k-1)}(a)] \\ &+ \frac{(b-a)^{n-1}}{n!}\int_a^b\Big[B_n^*\Big(\frac{y-t}{b-a}\Big) - B_n\Big(\frac{y-a}{b-a}\Big)\Big]\,df^{(n-1)}(t),\end{aligned} \tag{1.1}$$

where $B_k(t)$ is the kth Bernoulli polynomial and $B_k^*(t) = B_k(t - \lfloor t\rfloor)$, $t \in \mathbb{R}$.

We recall some basic properties of Bernoulli polynomials. Bernoulli polynomials $B_k(t)$ are uniquely determined by

$$B_k'(t) = kB_{k-1}(t); \quad B_k(t+1) - B_k(t) = kt^{k-1}, k \geq 0; \quad B_0(t) = 1.$$

For the kth Bernoulli polynomial, we have

$$B_k(1-t) = (-1)^k B_k(t), \quad t \in \mathbb{R}, k \geq 1. \tag{1.2}$$

The kth Bernoulli number B_k is defined by $B_k = B_k(0)$. From (1.2), it follows that for $k \geq 2$, $B_k(1) = B_k(0) = B_k$. Note that $B_{2k-1} = 0$, $k \geq 2$, and $B_1(1) = -B_1(0) = 1/2$.

The $B_k^*(t)$ are periodic functions of period 1 and are related to the Bernoulli polynomials by $B_k^*(t) = B_k(t)$, $0 \leq t < 1$. The function $B_0^*(t)$ is a constant equal to 1, while $B_1^*(t)$ is a discontinuous function with a jump of -1 at each integer. For $k \geq 2$, $B_k^*(t)$ is a continuous function. Further details of the Bernoulli polynomials are given by Abramowitz and Stegun [1] and Krylov [7].

2. Main results

Let $x \in [0, 1/2)$ and let $f : [0, 1] \to \mathbb{R}$ be such that $f^{(n-1)}$ is continuous and of bounded variation on $[0, 1]$ for some $n \geq 1$. Take $y = x$, $y = 1/2$, $y = 1 - x$ in (1.1), multiply by $w(x)$, $1 - 2w(x)$, $w(x)$, respectively, and add. We obtain

$$\begin{aligned}\int_0^1 f(t)\,dt = w(x)f(x) + (1 - 2w(x))f(1/2) + w(x)f(1-x) - T_{n-1}(x) \\ + \frac{1}{n!}\int_0^1 F_n(x, t)\,df^{(n-1)}(t),\end{aligned} \tag{2.1}$$

where, for $t \in \mathbb{R}$,

$$\begin{aligned}T_{n-1}(x) &= \sum_{k=1}^{n-1}\frac{G_k(x, 0)}{k!}[f^{(k-1)}(1) - f^{(k-1)}(0)], \\ G_n(x, t) &= w(x)[B_n^*(x-t) + B_n^*(1-x-t)] + (1 - 2w(x))B_n^*(1/2 - t), \\ F_n(x, t) &= G_n(x, t) - G_n(x, 0).\end{aligned}$$

https://doi.org/10.1017/S1446181111000721 Published online by Cambridge University Press

The functions G_n have the following properties:

$$G_n(x, 1-t) = (-1)^n G_n(x, t), \quad t \in [0, 1], \tag{2.2}$$

$$\frac{\partial G_n(x, t)}{\partial t} = -nG_{n-1}(x, t). \tag{2.3}$$

Further, we have $G_{2k-1}(x, 0) = 0$ for $k \geq 1$, and so $F_{2k-1}(x, t) = G_{2k-1}(x, t)$. Note that this is independent of the weight $w(x)$. On the other hand, in general $G_{2k}(x, 0) \neq 0$.

In order to obtain from (2.1) the highest accuracy quadrature formula which at the same time does not include derivatives in the quadrature, the condition $G_2(x, 0) = 0$ has to be imposed. This condition, and an appropriate choice of the node x, produce the Simpson, dual Simpson, Maclaurin and Gauss two-point formulas as special cases [5].

Now, if we assume $G_{2k}(x, 0) = 0$ for some $k \geq 2$, the accuracy will increase but the quadrature formulas thus obtained will include values of derivatives of order up to $2k-3$ at the end points of the interval. When those values are easy to calculate, this is not an obstacle. Furthermore, when $f^{(2k-1)}(1) = f^{(2k-1)}(0)$ for $k \geq 1$, we obtain a formula of even higher accuracy.

Thus, suppose $G_4(x, 0) = 0$, that is, let the values of the first derivative be included in the quadrature. The weight produced after imposing this condition is

$$w_c(x) = -\frac{B_4(1/2)}{2(B_4(x) - B_4(1/2))} = \frac{7}{30(1-2x)^2(1+4x-4x^2)}. \tag{2.4}$$

The quadrature now takes the form

$$\begin{aligned} Q_c(x, 1/2, 1-x) &= w_c(x)f(x) + (1 - 2w_c(x))f(1/2) + w_c(x)f(1-x) \\ &= \frac{7f(x) - (480x^4 - 960x^3 + 480x^2 - 16)f(1/2) + 7f(1-x)}{30(1-2x)^2(1+4x-4x^2)}. \end{aligned}$$

The functions G_n and F_n with the weight $w_c(x)$ are denoted by G_n^c and F_n^c and, furthermore, the corresponding function T_{n-1} is denoted by T_{n-1}^c and satisfies

$$\begin{aligned} T_{n-1}^c(x) &= \sum_{k=1}^{\lfloor (n-1)/2 \rfloor} \frac{1}{(2k)!} G_{2k}^c(x, 0)[f^{(2k-1)}(1) - f^{(2k-1)}(0)] \\ &= \frac{10x^2 - 10x + 1}{60(1+4x-4x^2)}[f'(1) - f'(0)] \\ &\quad + \sum_{k=3}^{\lfloor (n-1)/2 \rfloor} \frac{G_{2k}^c(x, 0)}{(2k)!}[f^{(2k-1)}(1) - f^{(2k-1)}(0)]. \end{aligned}$$

To illustrate the effectiveness of these rules, we now give a few examples. First we recall the classical Simpson, dual Simpson and Maclaurin rules,

https://doi.org/10.1017/S1446181111000721 Published online by Cambridge University Press

TABLE 1. Approximate values of $\int_0^1 \sin t\,dt \approx 0.4596976941$ and $\int_0^1 e^t\,dt \approx 1.718281828$.

	$\int_0^1 \sin t\,dt$	$\int_0^1 e^t\,dt$
Simpson	0.4598621899	1.718861152
Dual Simpson	0.4595533000	1.717776532
Maclaurin	0.4596337544	1.718056432
Corrected Simpson	0.4596984786	1.718279074
Corrected dual Simpson	0.4596969336	1.718284495
Corrected Maclaurin	0.4596975257	1.718282423

respectively [6, 7]:

$$\int_0^1 f(t)\,dt \approx \frac{1}{6}f(0) + \frac{2}{3}f(1/2) + \frac{1}{6}f(1),$$
$$\int_0^1 f(t)\,dt \approx \frac{2}{3}f(1/4) - \frac{1}{3}f(1/2) + \frac{2}{3}f(3/4),$$
$$\int_0^1 f(t)\,dt \approx \frac{3}{8}f(1/6) + \frac{1}{4}f(1/2) + \frac{3}{8}f(5/6).$$

Taking $x = 0$, $x = 1/4$ and $x = 1/6$ in (2.1) with the weight $w_c(x)$ given by (2.4), we obtain the corrected Simpson, corrected dual Simpson and corrected Maclaurin rules, respectively:

$$\int_0^1 f(t)\,dt \approx \frac{7}{30}f(0) + \frac{8}{15}f(1/2) + \frac{7}{30}f(1) - \frac{1}{60}[f'(1) - f'(0)],$$
$$\int_0^1 f(t)\,dt \approx \frac{8}{15}f(1/4) - \frac{1}{15}f(1/2) + \frac{8}{15}f(3/4) + \frac{1}{120}[f'(1) - f'(0)],$$
$$\int_0^1 f(t)\,dt \approx \frac{27}{80}f(1/6) + \frac{13}{40}f(1/2) + \frac{27}{80}f(5/6) + \frac{1}{240}[f'(1) - f'(0)].$$

Table 1 gives approximate values of the integrals $\int_0^1 \sin t\,dt$ and $\int_0^1 e^t\,dt$ using the above six quadrature rules.

Lemma 2.1 below is the key step for obtaining the rest of the results in this paper. It concerns the sign of the functions $G^c_{2k-1}(x, t)$ in the variable t on $(0, 1/2)$ for $k \geq 3$. It suffices to consider these functions for $t \in (0, 1/2)$ because of (2.2). Also, routine calculation reveals that $G^c_1(x, t)$ and $G^c_3(x, t)$ do not have constant sign in t on $(0, 1/2)$. The proof of Lemma 2.1 is by induction, and the basis is that the function $G^c_5(x, t)$ has constant sign in t for $x \in [0, 1/2 - \sqrt{15}/10] \cup [1/6, 1/2)$; note here that $1/2 - \sqrt{15}/10 \approx 0.11270$. That is, for $x \in (1/2 - \sqrt{15}/10, 1/6)$, the function $G^c_5(x, t)$

https://doi.org/10.1017/S1446181111000721 Published online by Cambridge University Press

has at least one zero in t on $(0, 1/2)$. We now give a short proof of this fact. We have

$$G_5^c(x,0) = \frac{\partial G_5^c}{\partial t}(x,0) = \frac{\partial^2 G_5^c}{\partial t^2}(x,0) = G_5^c(x,1/2) = 0, \tag{2.5}$$

and after some calculation we see that

$$x \in (1/2 - \sqrt{15}/10, 1/6) \quad \text{if and only if} \quad \frac{\partial^3 G_5^c}{\partial t^3}(x,0) > 0 \text{ and } \frac{\partial G_5^c}{\partial t}(x,1/2) > 0.$$

From $\partial G_5^c/\partial t(x, 1/2) > 0$, we conclude that $\partial G_5^c/\partial t(x,t) > 0$ for t sufficiently close to $t = 1/2$ and with $t < 1/2$, and then from $G_5^c(x, 1/2) = 0$ in (2.5) it follows that $G_5^c(x,t) < 0$ for those values of t. On the other hand, from $\partial^3 G_5^c/\partial t^3(x,0) > 0$ we conclude that $\partial^3 G_5^c/\partial t^3(x,t) > 0$ for t sufficiently close to $t = 0$ and with $t > 0$. Then, similarly, using (2.5) we conclude that $G_5^c(x,t) > 0$ for those values of t. It follows that for $x \in (1/2 - \sqrt{15}/10, 1/6)$, $G_5^c(x,t)$ has at least one zero in t on $(0, 1/2)$.

We now proceed to the lemma.

Lemma 2.1. *For $x \in [0, 1/2 - \sqrt{15}/10] \cup [1/6, 1/2)$ and $k \geq 3$, the function $G_{2k-1}^c(x,t)$ has no zeros in the variable t on $(0, 1/2)$. The sign of $G_{2k-1}^c(x,t)$ is determined by*

$$(-1)^k G_{2k-1}^c(x,t) > 0 \quad \text{for } x \in [0, 1/2 - \sqrt{15}/10],$$
$$(-1)^{k+1} G_{2k-1}^c(x,t) > 0 \quad \text{for } x \in [1/6, 1/2).$$

Proof. First, we claim that $G_5^c(x,t)$ has constant sign for $x \in [0, 1/2 - \sqrt{15}/10] \cup [1/6, 1/2)$. We show that it is increasing in x for $x \in [0, 1/2)$ and, after considering its behaviour at the end points, the claim follows. For $0 \leq t \leq x < 1/2$,

$$\frac{\partial G_5^c}{\partial x}(x,t) = \frac{t^3}{3} \cdot \frac{14(1-2x)}{(4x^2 - 4x - 1)^2} > 0.$$

So $G_5^c(x,t)$ is increasing in x for $0 \leq t \leq x < 1/2$. For $0 \leq x \leq t \leq 1/2$,

$$\frac{\partial G_5^c}{\partial x}(x,t) = \frac{14x(1-2t)}{3(1-2x)^3(4x^2-4x-1)^2} \cdot g(x,t),$$

where

$$g(x,t) = 4t^3(x-1) + t^2(-8x^3 + 4x^2 + 3) + tx(8x^2 - 4x - 3) + x^2 + 2x^3 - 4x^4.$$

The zeros of $\partial g/\partial t(x,t)$ are $t_1 = 1/2$ and $t_2 = (8x^3 - 4x^2 - 3x)/(6(x-1))$, and $t_2 < x$. Also, a simple calculation shows that $g(x,0) > 0$ and $g(x,1/2) > 0$, so $g(x,t) > 0$. Thus, it follows that $G_5^c(x,t)$ is increasing in x.

Since $G_5^c(1/2 - \sqrt{15}/10, t) < 0$ [5] and $G_5^c(1/6, t) > 0$ [3] for $t \in (0, 1/2)$, we have $G_5^c(x,t) < 0$ for $x \in [0, 1/2 - \sqrt{15}/10]$ and $G_5^c(x,t) > 0$ for $x \in [1/6, 1/2)$. So $G_5^c(x,t)$

https://doi.org/10.1017/S1446181111000721 Published online by Cambridge University Press

has constant sign for $x \in [0, 1/2 - \sqrt{15}/10]$ and $x \in [1/6, 1/2)$, and our statement is therefore valid for $k = 3$. For $k \geq 4$, the statement follows by induction.

Now, since $G_5^c(x, t) > 0$ for $t \in (0, 1/2)$, it follows from (2.3) that $G_7^c(x, t)$ is convex in t on $(0, 1/2)$. Since we now know that $G_7^c(x, t)$ has no zeros in t on $(0, 1/2)$, and since $G_{2k-1}^c(x, 0) = G_{2k-1}^c(x, 1/2) = 0$ for $k \geq 1$, we conclude that $G_7^c(x, t) < 0$ for $t \in (0, 1/2)$. The claim that two successive functions $G_{2k-1}^c(x, t)$ have opposite signs in t on $(0, 1/2)$ for a fixed x now follows by induction. □

Remark 2.2. Lemma 2.1 yields that for $k \geq 3$ and $x \in [0, 1/2 - \sqrt{15}/10]$, the function $(-1)^{k+1} F_{2k}^c(x, t)$ is strictly increasing in t on $(0, 1/2)$ and strictly decreasing in t on $(1/2, 1)$. Since $F_{2k}^c(x, 0) = F_{2k}^c(x, 1) = 0$, the function $(-1)^{k+1} F_{2k}^c(x, t)$ has constant sign in t on $(0, 1)$ and attains its maximum at $t = 1/2$. An analogous statement, but with the opposite sign, is valid for $x \in [1/6, 1/2)$.

Denote

$$R_{2k}(x, f) = \frac{1}{(2k)!} \int_0^1 F_{2k}^c(x, t)\, d f^{(2k-1)}(t).$$

Theorem 2.3. *Let $f : [0, 1] \to \mathbb{R}$ be such that $f^{(2k)}$ is continuous on $[0, 1]$ for some $k \geq 3$ and let $x \in [0, 1/2 - \sqrt{15}/10] \cup [1/6, 1/2)$. Then there exists $\xi \in [0, 1]$ such that*

$$R_{2k}(x, f) = -\frac{G_{2k}^c(x, 0)}{(2k)!} \cdot f^{(2k)}(\xi), \tag{2.6}$$

where

$$G_{2k}^c(x, 0) = \frac{7[B_{2k}(x) + (1 - 2^{1-2k}) B_{2k}]}{15(1 - 2x)^2 (1 + 4x - 4x^2)} - (1 - 2^{1-2k}) B_{2k}. \tag{2.7}$$

If, in addition, $f^{(2k)}$ has constant sign on $[0, 1]$, then there exists $\theta \in [0, 1]$ such that

$$R_{2k}(x, f) = \frac{\theta}{(2k)!} F_{2k}^c(x, 1/2)[f^{(2k-1)}(1) - f^{(2k-1)}(0)], \tag{2.8}$$

where

$$F_{2k}^c(x, 1/2) = \frac{7[B_{2k}(1/2 - x) - B_{2k}(x) - (2 - 2^{1-2k}) B_{2k}]}{15(1 - 2x)^2 (1 + 4x - 4x^2)} + (2 - 2^{1-2k}) B_{2k}. \tag{2.9}$$

Proof. Remark 2.2 shows that $F_{2k}^c(x, t)$ has constant sign under the assumptions of the theorem, so (2.6) follows from the mean value theorem for integrals. To prove (2.8), first suppose $f^{(2k)}(t) \geq 0$ and let $x \in [0, 1/2 - \sqrt{15}/10]$. Then we have

$$0 \leq \int_0^1 (-1)^{k+1} F_{2k}^c(x, t) f^{(2k)}(t)\, dt \leq (-1)^{k+1} F_{2k}^c(x, 1/2) \cdot \int_0^1 f^{(2k)}(t)\, dt,$$

which implies that there exists $\theta \in [0, 1]$ such that

$$(2k)! \cdot R_{2k}(x, f) = \theta \cdot F_{2k}^c(x, 1/2)[f^{(2k-1)}(1) - f^{(2k-1)}(0)].$$

When $x \in [1/6, 1/2)$ or $f^{(2k)}(t) \leq 0$, the statement follows similarly. □

https://doi.org/10.1017/S1446181111000721 Published online by Cambridge University Press

When (2.6) is applied for $k = 3$ to the remainder in (2.1) for $n = 6$ with the weight $w_c(x)$ from (2.4), the following formula is produced:

$$\int_0^1 f(t)\,dt - Q_c(x, 1/2, 1-x) + \frac{10x^2 - 10x + 1}{60(-4x^2 + 4x + 1)}[f'(1) - f'(0)] \\ = \frac{98x^4 - 196x^3 + 102x^2 - 4x - 1}{604800(4x^2 - 4x - 1)} \cdot f^{(6)}(\xi). \tag{2.10}$$

As mentioned earlier, for $x = 0$ the formula (2.10) recaptures the corrected Simpson formula, for $x = 1/4$ the corrected dual Simpson formula and for $x = 1/6$ the corrected Maclaurin formula. All related results from our previous papers on these formulas [3, 4, 9] follow as special cases of the results in this paper. Furthermore, for $x = 1/2 - \sqrt{15}/10$, which is equivalent to $10x^2 - 10x + 1 = 0$, (2.10) becomes the Gauss three-point formula and, for $x = 1/2 - (225 - 30\sqrt{30})^{1/2}/30$, namely $w_c(x) = 1/2$, the corrected Gauss two-point formula. These formulas were also studied previously [5].

Notice that $x = 1/2 - \sqrt{15}/10$ is the unique solution of the equation $G_2^c(x, 0) = 0$. In fact, the nodes and the coefficients of the Gauss three-point formula comprise the unique solution of the system

$$G_2^c(x, 0) = G_4^c(x, 0) = 0.$$

This is the system one would use to obtain from (2.1) the quadrature formula which is not corrected (does not include first derivatives) and has the maximum accuracy.

The next theorem gives both sharp and best possible estimates for the error in these quadrature formulas. Recall that a sharp inequality is obtained when there is a nontrivial function for which equality holds, while the best possible inequality is obtained when one can prove that there is no smaller constant for which the inequality holds.

Theorem 2.4. *Let $p, q \in \mathbb{R}$ satisfy $1 \le p, q \le \infty$ and $1/p + 1/q = 1$. If $f : [0, 1] \to \mathbb{R}$ is such that $f^{(n)} \in L_p[0, 1]$ for some $n \ge 1$, then*

$$\left| \int_0^1 f(t)\,dt - Q_c(x, 1/2, 1-x) + T_{n-1}^c(x) \right| \le K(n, q, x) \cdot \|f^{(n)}\|_p, \tag{2.11}$$

where

$$K(n, q, x) = \frac{1}{n!}\left[\int_0^1 |F_n^c(x, t)|^q\,dt\right]^{1/q}.$$

This inequality is sharp for $1 < p \le \infty$ and the best possible for $p = 1$.

Proof. The inequality (2.11) follows immediately after applying the Hölder inequality to the remainder in (2.1) with weight $w_c(x)$. To prove that the inequality is sharp, take

$$f^{(n)}(t) = \begin{cases} \operatorname{sgn}(F_n^c(x, t)) \cdot |F_n^c(x, t)|^{1/(p-1)} & \text{for } 1 < p < \infty, \\ \operatorname{sgn}(F_n^c(x, t)) & \text{for } p = \infty \end{cases}$$

in (2.11). The proof that (2.11) is the best possible for $p = 1$ is analogous to the corresponding part of the proof of Theorem 2 of Pečarić et al. [10]. □

https://doi.org/10.1017/S1446181111000721 Published online by Cambridge University Press

For $x \in [0, 1/2 - \sqrt{15}/10] \cup [1/6, 1/2)$, $k \geq 3$, using Lemma 2.1 and Remark 2.2, we calculate the following constants for $p = 1$ and $p = \infty$ from Theorem 2.4:

$$K(2k, 1, x) = (-1)^{k+1} \frac{2}{(2k)!} \int_0^{1/2} F_{2k}^c(x, t)\, dt = \frac{1}{(2k)!} |G_{2k}^c(x, 0)|,$$
$$K(2k, \infty, x) = \sup_{t \in [0,1]} \frac{1}{(2k)!} |F_{2k}^c(x, t)| = \frac{1}{(2k)!} |F_{2k}^c(x, 1/2)| = \frac{1}{2} K(2k - 1, 1, x),$$

where $G_{2k}^c(x, 0)$ and $F_{2k}^c(x, 1/2)$ are as in (2.7) and (2.9). In view of this, let us now consider the inequality (2.11) for $n = 5, 6$ and $p = \infty$:

$$\left| \int_0^1 f(t)\, dt - Q_c(x, 1/2, 1 - x) + \frac{10x^2 - 10x + 1}{60(-4x^2 + 4x + 1)} [f'(1) - f'(0)] \right| \leq K(n, 1, x) \cdot \|f^{(n)}\|_\infty \quad \text{for } n = 5, 6,$$

where

$$K(5, 1, x) = \frac{1}{115200} \left| \frac{112x^3 - 88x^2 + 4x + 1}{4x^2 - 4x - 1} \right|,$$
$$K(6, 1, x) = \frac{1}{604800} \left| \frac{98x^4 - 196x^3 + 102x^2 - 4x - 1}{4x^2 - 4x - 1} \right|.$$

In order to determine the admissible x for which the error estimate (2.11) is smallest, we have to minimize $K(5, 1, x)$ and $K(6, 1, x)$. It is not difficult to verify that both are decreasing on $[0, 1/2 - \sqrt{15}/10]$ and increasing on $[1/6, 1/2)$ and attain their minimal values at $x = 1/6$:

$$K(5, 1, 1/6) \approx 1.44676 \times 10^{-6} \quad \text{and} \quad K(6, 1, 1/6) \approx 3.55949 \times 10^{-7}.$$

On the other hand,

$$\sup\{K(n, 1, x) : x \in [0, 1/2 - \sqrt{15}/10] \cup [1/6, 1/2)\} = K(n, 1, 1/2) \quad \text{for } n = 5, 6,$$

where

$$K(5, 1, 1/2) \approx 2.17014 \times 10^{-5} \quad \text{and} \quad K(6, 1, 1/2) \approx 3.41022 \times 10^{-6}.$$

The same is valid for $n = 6$ and $p = 1$, since $K(5, 1, x) = 2K(6, \infty, x)$.

Therefore, the node which provides the smallest error in these three cases is $x = 1/6$. This means that the corrected Maclaurin formula is the optimal corrected three-point quadrature formula, which is a rather interesting fact since it was previously shown [5] that amongst the three-point quadrature formulas accurate for all polynomials of degree at most three, the classical Maclaurin formula is the optimal one.

Remark 2.5. Although only values of x in $[0, 1/2)$ were considered here, results for $x = 1/2$ can easily be obtained by considering the limit process $x \to 1/2$.

Acknowledgements

This research was supported by the Croatian Ministry of Science, Education and Sports under the Research Grants 058-1170889-1050 (Iva Franjić and Ivan Perić) and 117-1170889-0888 (Josip Pečarić).

References

[1] M. Abramowitz and I. A. Stegun (eds), *Handbook of mathematical functions: with formulas, graphs and mathematical tables* (Dover Publications, New York, 1992).
[2] Lj. Dedić, M. Matić and J. Pečarić, "On generalizations of Ostrowski inequality via some Euler-type identities", *Math. Inequal. Appl.* **3** (2000) 337–353.
[3] I. Franjić and J. Pečarić, "Corrected Euler–Maclaurin's formulae", *Rend. Circ. Mat. Palermo* **54** (2005) 259–272; doi:10.1007/BF02874640.
[4] I. Franjić and J. Pečarić, "On corrected dual Euler–Simpson formulae", *Soochow J. Math.* **32** (2006) 575–587.
[5] I. Franjić, I. Perić and J. Pečarić, "Quadrature formulae of Gauss type based on Euler identities", *Math. Comput. Modelling* **45** (2007) 355–370; doi:10.1016/j.mcm.2006.05.009.
[6] F. B. Hildebrand, *Introduction to numerical analysis* (McGraw-Hill, New York, 1956).
[7] V. I. Krylov, *Approximate calculation of integrals* (Macmillan, New York–London, 1962).
[8] C. Lanczos, *Applied analysis* (Prentice-Hall, Englewood Cliffs, NJ, 1956).
[9] J. Pečarić and I. Franjić, "Generalisation of a corrected Simpson's formula", *ANZIAM J.* **47** (2006) 367–385; doi:10.1017/S1446181100009895.
[10] J. Pečarić, I. Perić and A. Vukelić, "Sharp integral inequalities based on general Euler two-point formulae", *ANZIAM J.* **46** (2005) 555–574; doi:10.1017/S1446181100009676.
[11] N. Ujević and A. J. Roberts, "A corrected quadrature formula and applications", *ANZIAM J.* **45**(E) (2004) E41–E56.