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ON THE PRODUCT $\prod_{n\geq 1}(1+q^n x+q^{2n}x^2)$

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Abstract

We study the coefficients A_n in the expansion of the infinite product

$$\prod_{n\geq 1}(1+q^n x+q^{2n}x^2) = A_0 + A_1 x + A_2 x^2 + A_3 x^3 + \cdots + A_n x^n + \cdots .$$

We first derive a recurrence relation for A_n and from it we obtain an explicit expression of A_n. We then prove a convolution identity involving A_n.

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1. Properties of A_n

First, we give some standard notation. Let

$$(a;q)_\infty = (1-a)(1-aq)(1-aq^2)\cdots$$

and for any integer n, $(a;q)_n = (a;q)_\infty/(aq^n;q)_\infty$. Note that $(a;q)_0 = 1$, $(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$, if n is a positive integer, and $1/(q;q)_n = 0$, if n is a negative integer.

THEOREM 1. *Let*

$$G(x) = \prod_{n\geq 1}(1+q^n x+q^{2n}x^2) = A_0 + A_1 x + A_2 x^2 + A_3 x^3 + \cdots + A_n x^n + \cdots .$$

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Then

(1) *A_n is the generating function of the number of partitions into n positive integers each occurring at most twice;*

(2) $A_0 = 1, \quad A_{-1} = 0, \quad$ *and for* $n \geq 1, \quad A_n = q^n(1-q^n)^{-1}(A_{n-1}+A_{n-2})$;

(3) $$A_n = ((q;q)_n)^{-1} \sum_{q^i q^{i-1} \mapsto q^i(1-q^{i-1})} q^n q^{n-1} \cdots q^2 q^1$$

where the summation is over all products (including $q^n q^{n-1}\cdots q^2 q^1$) obtained from substitution of consecutive pairs $q^i q^{i-1}$ by $q^i(1-q^{i-1})$.

(For instance,

$$\begin{aligned} A_4 &= ((q;q)_4)^{-1} \sum_{q^i q^{i-1} \mapsto q^i(1-q^{i-1})} q^4 q^3 q^2 q^1 \\ &= ((q;q)_4)^{-1}[q^4q^3q^2q^1 + q^4(1-q^3)q^2q^1 + q^4q^3(1-q^2)q^1 \\ &\qquad + q^4q^3q^2(1-q^1) + q^4(1-q^3)q^2(1-q^1)].) \end{aligned}$$

PROOF. (1) is obvious.

(2) Since $G(qx) = \prod_{n\geq 1}(1+q^{n+1}x+q^{2n+2}x^2)$, we have $G(x) = (1+qx+q^2x^2)G(qx)$. Comparing the coefficient of x^n, we see that $A_n = q^n(A_n + A_{n-1} + A_{n-2})$. Hence $A_n = q^n(1-q^n)^{-1}(A_{n-1}+A_{n-2})$.

(3) We show the formula by induction, using the recurrence relation in (2). Clearly the formula holds for $n = 2$, and

$$\begin{aligned} A_n &= q^n(1-q^n)^{-1}(A_{n-1}+A_{n-2}) \\ &= q^n(1-q^n)^{-1}\Bigg[((q;q)_{n-1})^{-1} \sum_{q^i q^{i-1} \mapsto q^i(1-q^{i-1})} q^{n-1}q^{n-2}\cdots q^2q^1 \\ &\qquad + ((q;q)_{n-2})^{-1} \sum_{q^i q^{i-1} \mapsto q^i(1-q^{i-1})} q^{n-2}q^{n-3}\cdots q^2q^1\Bigg] \\ &= ((q;q)_n)^{-1}\Bigg[q^n \sum_{q^i q^{i-1} \mapsto q^i(1-q^{i-1})} q^{n-1}q^{n-2}\cdots q^2q^1 \\ &\qquad + q^n(1-q^{n-1}) \sum_{q^i q^{i-1} \mapsto q^i(1-q^{i-1})} q^{n-2}q^{n-3}\cdots q^2q^1\Bigg] \\ &= ((q;q)_n)^{-1} \sum_{q^i q^{i-1} \mapsto q^i(1-q^{i-1})} q^n q^{n-1}\cdots q^2 q^1. \end{aligned}$$

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2. Convolution identity

In Andrews [1, p. 454] it was shown that

$$\prod_{n\geq 1}(1-\alpha q^n x)(1-\beta q^n x)(1-\alpha^{-1}q^{n-1}x^{-1})(1-\beta^{-1}q^{n-1}x^{-1})$$
$$= B_0\sum_{-\infty}^{\infty}\alpha^n\beta^n q^{n(n+1)}x^{2n} - B_1\sum_{-\infty}^{\infty}\alpha^n\beta^n q^{n^2}x^{2n-1},$$

where

$$B_0 = \prod_{n\geq 1}(1+\alpha\beta^{-1}q^{2n-1})(1+\alpha^{-1}\beta q^{2n-1})(1-q^{2n})(1-q^n)^{-2}$$

and

$$B_1 = \beta^{-1}\prod_{n\geq 1}(1+\alpha\beta^{-1}q^{2n})(1+\alpha^{-1}\beta q^{2n-2})(1-q^{2n})(1-q^n)^{-2}.$$

With $\alpha=\omega^2$, $\beta=\omega$ (where $\omega = e^{2\pi i/3}$), this becomes

$$\left(\sum A_n x^n\right)\left(\sum q^{-n}A_n x^{-n}\right)$$
$$= \prod_{n\geq 1}(1+q^n x+q^{2n}x^2)(1+q^{n-1}x^{-1}+q^{2(n-1)}x^{-2})$$
$$= [(-q^3;q^6)_\infty(-q^2;q^2)_\infty/(q;q)_\infty]\sum_{-\infty}^{\infty}q^{n(n+1)}x^{2n}$$
$$+ [(-q^6;q^6)_\infty(-q;q^2)_\infty/(q;q)_\infty]\sum_{-\infty}^{\infty}q^{n^2}x^{2n-1}.$$

Thus, if we define $A_n = 0$ for $n<0$, then we have

THEOREM 2 (convolution identity).

$$\sum_{-\infty}^{\infty}q^{-n}A_nA_{2m+n} = q^{m(m+1)}(-q^3;q^6)_\infty(-q^2;q^2)_\infty/(q;q)_\infty;$$
$$\sum_{-\infty}^{\infty}q^{-n}A_nA_{2m-1+n} = q^{m^2}(-q^6;q^6)_\infty(-q;q^2)_\infty/(q;q)_\infty.$$

This is the analogue of the convolution identity [2, p. 22]

$$\sum_{-\infty}^{\infty}q^{-n}C_nc_{m+n} = q^{m(m+1)/2}(q;q)_\infty$$

for the coefficients C_n defined by $\prod_{n\geq 1}(1+q^n x) = \sum_{-\infty}^{\infty}C_n x^n$.

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