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THE THREE DIMENSIONAL STRUCTURE OF ASTRONOMICAL SOURCES THROUGH OPTIMAL INVERSION

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ABSTRACT We explore the application of optimal inversion techniques to astronomical data with a goal of developing a set of procedures for the determination of the three dimensional structure of astronomical sources. Astronomical data present a particularly difficult problem in inversion because: In any observation, 3 of 6 spatial and velocity dimensions are lost in projection onto the plane of the sky and the line of sight velocity. In any inversion, we would like to solve for a number of physical parameters. Generally, these parameters are closely related in their effect on the single observable, the sky brightness.

The dimensional deficiency leaves us with an unavoidably large degree of ambiguity (non-uniqueness) in any solution, while the inter-related parameters lead to a high probability of correlated errors and hence instability in the presence of to noise.

We show how constraints of symmetry and smoothness source allow us to handle an inversion with an insufficiently sampled data base and mutually dependent solution parameters (mathematically ill-posed and ill-conditioned). The constraints represent a priori information incorporated into the solution; thus very highly constrained inversions are similar to model fitting. In any case the inversion procedure provides us with quantitative statistics on the goodness of fit which may be used to assess the degree of ambiguity in a particular model, and the expected errors and cross-correlated errors on the parameters defining the source structure.

We briefly discuss the background and motivation, and outline the procedure in general terms. We refer to papers published in the Ap. J. where different aspects of the inversion are applied to observational data bases collected at the VLA.

MOTIVATION

There is an invertible relationship between information collected by an observer at different sky positions and information on the internal structure of an astronomical source.
Astronomical data collected by an observer consist of brightness which represents an integral average of emission and absorption along an entire line of sight through a source.

$$I(x) = \int_0^{\infty} S(\tau')e^{-(\tau-\tau')}d\tau$$

Thus each individual line of sight contains information about the structure of the source along the missing third dimension while different lines of sight contain information on the structure across the plane of the sky. To take a simple example, assume the source has a radially symmetric structure. Different lines of sight sample different ranges of regions in the source. Measurements at different sky positions can be used to reconstruct the internal three dimensional density structure. Assume the Rayleigh-Jeans approximation $e^{h\nu/kT} \rightarrow 1 + h\nu/kT$ and the optically thin approximation, $S(\tau')e^{-(\tau-\tau')} \rightarrow S(\tau)$. Then the brightness can be written as a Fredholm Integral of the first kind

$$I(x) = \int_{R_{min}}^{R} (\frac{2a}{\mu}) n^u(r)dr$$

where a incorporates the quantum constants of the line emission process, $n^u(r)$ is the density of the upper state of the line transition and μ is a constant geometric factor relating to the offset of the line of sight from the radially symmetric cloud's center, $\mu = \frac{R}{(R^2-x^2)^{1/2}}$.
The radial density profile may be determined by inversion of this equation. Binning the radial dimension and discretizing the integral,

$$I_i = \sum_j a_{i,j} n_j^u$$

where $a_{i,j} = \frac{2a}{\mu} ds$ and ds is the bin size, leads to a matrix equation with the matrix in upper triangular form.

$$I = [A]n^u \text{ with } [A] = \begin{bmatrix} & a_{i,j} \\ 0 & \end{bmatrix}$$

In simple physical terms, this says that in order to determine the radially symmetric density profile, begin at the outer most radius, compute the density of the outer most radial bin $n^u_{j=R_{max}}$. Next, take the next data point inward, and knowing the density of the outer most zone, compute the density of next inner zone, n^u_{j-1}, etc. See Keto, E. 1990, *Ap. J.*, in press, ***The Spectral Signatures of Collapse and Outflow Around Young Stars***

PROBLEMS TO BE OVERCOME

In general in any optimal inversion of astronomical data there are two difficulties.

Problem 1.
Non-uniqueness or ambiguity arising from:

a. Undersampling – The data do not cover the solution space, but only a projection of the solution space onto a smaller subset. Spatial structure smaller than the sampling interval are undetermined. The effect is similar to a low pass filtering of the structure, or typically in astronomy a convolution $\int f(x - x')s(x)dx$.

b. Geometric Projection – Astronomical data are always projected onto the two dimensional sky plane. In the absence of any other information we have no knowledge of where to place the emitting gas along the line of sight.

Thus the inversion problem is ill-posed. In the example above, this problem is overcome by assuming a bin size for the radial structure and assuming that the structure is radially symmetric.

Problem 2.
The physical parameters we seek to determine are often inter-related and therefore form a set of basis vectors for the solution space which are not linearly independent. For example, in astronomy the gas density and temperature both contribute to the line brightness. An error in one may be compensated by an error in the other to produce a line brightness nearly identical to that produced by the correct temperature and density, but nevertheless incorrect. Thus the solution may be sensitive to correlated error and noise. Thus the inversion problem is ill-conditioned.
In the example above where we determine the radial density profile, the densities in the different bins are dependent on one another. Thus if our data fluctuates up and down in brightness across the source, the derived density profile may contain zones of "negative density".

PROCEDURE

To effect an inversion of astronomical data we need an assumption of source structure and an assumption of structural smoothness. In our simple example, a better solution would be:

1. Assume radial structure as before.
2. Assume a degree of smoothness.

Mathematically, this could be accomplished by minimizing the squared difference of predicted and observed brightness subject to a constraint on smoothness.

$$\chi^2 = \sum_{i=1}^{N}(I_i - i_i^*)^2 + \lambda \sum_{j=2}^{M-1} \left(\frac{n_{j+1}^u - 2n_j^u + n_{j-1}^u}{r_{j+1} - r_{j-1}}\right)^2$$

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This solution for the radial densities would represent the best fit radially symmetric model subject to the constraint. The Lagrange multiplier, λ, may be set by requiring the solution to be smooth on a level consistent with the noise in the data. See Jeffrey, W. 1988 *Ap. J.*, **327**, 987

In cases where the astronomical data are not of sufficient quality to define the physical properties of a source to such detail, an alternative approach is to parameterize the source properties and solve instead for the parameters.

For example, instead of solving for a binned radial density profile, the radial profile may be parameterized as a power law, $n^u(r) = n_0^u(r/r_0)^\alpha$, or some other function based on an *a prior* expectation. In this case we would solve for the two parameters n_0^u and α. The smoothing is implicit in the parameterization and we need only minimize $\chi^2 = \sum(I_i - I_i^*)^2$.

See Keto, E. 1990, *Ap. J.*, **350**, 772, "The Collapse of Self-Gravitating Condensations in DR21", and *Ap. J.*, **355**, 190, "Radiative Transfer Modeling of Spectral Line Data: Accretion onto G10.6–0.4".

The trade off is a fewer number of parameters for a more highly constrained model. We may address the question of whether the adopted model is correct by asking whether the adopted model achieves a significantly better solution than another model. The significance of a model fit may be quantified as the probability that if we "observed the model", the statistical errors of our measurement would by chance result in a fit as "poor" as achieved. The probability is computed as an incomplete gamma function.

$$P = 1 - \Gamma\left(\frac{\nu}{2}, \frac{\chi^2}{2}\right)$$

where ν is the number of independent data points less the number of parameters in the fit, and χ^2 is as above, the squared difference of the model and the data.

Using other standard statistical techniques, the expected errors on each of the modeled parameters may be determined from the covariance matrix or by Monte Carlo methods. The degree of correlation of the parameters which is related to the degree of dependence of the parameters as basis vectors in the inversion may be determined using parametric rank-order correlation statistics.

PROGNOSIS

Advancements in instrumentation such as the VLA have provided astronomical data of exceptional quality. Advances in computational power now make imaging and analysis techniques such as optimal inversion feasible for even relatively complex problems in remote sensing such as three dimensional radiative transfer modeling of astronomical sources.

Barry Clark: Are the numerical goodness-of-fit measures available for the three examples you showed?
E. Keto: No. This research is observationally motivated; as we collect new data we apply new techniques of inversion to the data and improve our understanding of the inversion process. Two of the examples, G10.6-0.4 and DR21, illustrate the feasibility of the numerical inversion technique by showing that the modeled maps and the data are sensitive to changes in model temperature, density, and velocities of a scientifically interesting level. The third example on NGC7538 derives from a limited inversion of the temperature and density structure and includes an error analysis on the derived parameter, but not the overall fit. We have recently finished a complete inversion of data in G34.3+011 including the goodness-of-fit. This will be published soon.

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