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MORE ON CONVERGENCE OF CONTINUOUS FUNCTIONS AND TOPOLOGICAL CONVERGENCE OF SETS

BY

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ABSTRACT. Let $C(X, Y)$ denote the set of continuous functions from a metric space X to a metric space Y. Viewing elements of $C(X, Y)$ as closed subsets of $X \times Y$, we say $\{f_n\}$ converges topologically to f if Li f_n = Ls $f_n = f$. If X is connected, then topological convergence in $C(X, R)$ does not imply pointwise convergence, but if X is locally connected and Y is locally compact, then topological convergence in $C(X, Y)$ is equivalent to uniform convergence on compact subsets of X. Pathological aspects of topological convergence for seemingly nice spaces are also presented, along with a positive Baire category result.

1. **Introduction**. Let $\langle X, d_X\rangle$ be a metric space and let $\{C_n\}$ be a sequence of nonempty subsets of X. The *lower* and *upper closed limits* of $\{C_n\}$ are defined as follows [3]: Li C_n (resp. Ls C_n) is the set of all points x each neighborhood of which meets all but finitely (resp. infinitely) many sets C_n. We say $\{C_n\}$ *converges topologically* to a (possibly empty) set C if Li C_n = Ls $C_n = C$. If $\langle Y, d_Y\rangle$ is another metric space, then we can regard members of $C(X, Y)$, the continuous functions from X to Y, as closed subsets of $X \times Y$. What does convergence of sequences in $C(X, Y)$ in the above sense mean? The relationship between topological convergence in $C(X, Y)$ and uniform convergence is explored in [2]. Here we consider in detail topological convergence versus pointwise convergence. In general both pointwise convergence and topological convergence in $C(X, Y)$ are weaker than Hausdorff metric convergence of graphs (induced by a metric compatible with the product uniformity) which is, in turn, weaker than uniform convergence. However, if $\{f_n\}$ converges to a uniformly continuous function f in the Hausdorff metric, then $\{f_n\}$ actually converges uniformly to f. In particular, if X is compact, then the Hausdorff metric on $C(X, Y)$ is topologically equivalent to the usual metric of uniform convergence [4]; this equivalence has been the basis for a number of papers in constructive approximation theory by B. Sendov, V. Popov, and their associates in Sofia (see, e.g., [5], [7] or [8]).

The relationship between topological convergence and pointwise convergence for general X and Y is a tenuous one. However, if X is locally connected and Y is locally compact, the situation can be described precisely: topological convergence in $C(X, Y)$ means uniform convergence on compact subsets of X.

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In the sequel (1) $S_\epsilon[x]$ will denote the open ball of radius ϵ about a point x in a metric space, (2) diam(A) will denote the diameter of a set A, (3)$\tilde{A}$ will denote the complement of A.

2. **Results**. We first resolve a simple question. Under what conditions on X and/or Y will pointwise convergence in $C(X, Y)$ force topological convergence? Essentially, X must be discrete.

THEOREM 1. *Let $\langle X, d_X\rangle$ and $\langle Y, d_Y\rangle$ be metric spaces. If X is discrete then pointwise convergence in $C(X, Y)$ implies topological convergence. Conversely, if $C([0, 1], Y)$ is nontrivial and pointwise convergence in $C(X, Y)$ implies topological convergence, the X is discrete.*

PROOF: Let X be a metric space without limit points. Suppose $\{f_n\} \subset C(X, Y)$ converges to a continuous function f pointwise. Immediately, we have $f \subset \text{Li} f_n$. To show $\text{Ls} f_n \subset f$, choose (x, y) off f, and let $\delta = \frac{1}{2} \min \{d_X(x, X - \{x\}), d_Y(y, f(x))\}$. Since $\lim_{n\to\infty} f_n(x) = f(x)$, eventually $d_Y(y, f_n(x)) > \delta$. Thus $S_\delta[x] \times S_\delta[y]$ is a neighborhood of (x, y) that meets at most finitely many members of $\{f_n\}$. We conclude $(x, y) \notin \text{Ls} f_n$, whence $\text{Li} f_n = \text{Ls} f_n = f$.

Conversely suppose X is not discrete and $C([0, 1]), Y)$ is nontrivial, i.e., there exists $\varphi \in C([0, 1], Y)$ such that $\varphi(0) \neq \varphi(1)$. We construct a sequence $\{f_n\}$ in $C(X, Y)$ convergent pointwise to the function identically equal to $\varphi(0)$ on X that fails to converge topologically to f. Let x_0 be a limit point of X and let $\{x_n\}$ be a sequence in X convergent to x_0 such that for each n, $d_X(x_0, x_{n+1}) < d_X(x_0, x_n)$. Set $\alpha_n = d_X(x_0, x_n)$ and define $f_n \in C(X, Y)$ by

$$f_n(x) = \begin{cases} \varphi\left(\dfrac{1}{\alpha_n} d_X(x, x_0)\right) & \text{if } 0 \leq d_X(x, x_0) \leq \alpha_n \\ \varphi\left(2 - \dfrac{1}{\alpha_n} d_X(x, x_0)\right) & \text{if } \alpha_n < d_X(x, x_0) \leq 2\alpha_n \\ \varphi(0) & \text{if } d_X(x, x_0) > 2\alpha_n \end{cases}$$

Notice for each x in X eventually $f_n(x) = \varphi(0)$; so, $\{f_n\}$ converges to f pointwise. However, $\{(x_n, f_n(x_n))\}$ converges to $(x_0, \varphi(1))$, whence $\{f_n\}$ fails to converge topologically to f.

For general X and Y we can identify well-behaved sequences in $C(X, Y)$ for which pointwise convergence ensures topological convergence.

DEFINITION. $\Omega \subset C(X, Y)$ is called *pointwise equicontinuous* if for each $x \in X$ and each $\epsilon > 0$ there exists $\delta > 0$, perhaps dependent on x, such that whenever $f \in \Omega$ and $d_X(x, w) < \delta$ then $d_Y(f(x), f(w)) < \epsilon$.

THEOREM 2. *Let $\{f_n\}$ be a pointwise equicontinuous sequence in $C(X, Y)$ pointwise convergent to a continuous function f. Then $\{f_n\}$ converges topologically to f.*

PROOF: Since $f \subset \text{Li} f_n$, if topological convergence does not occur, then we must have $\text{Ls} f_n \not\subset f$. Pick $(x, y) \in \text{Ls} f_n - f$ and choose $\epsilon < d_Y(y, f(x))$. By pointwise equi-

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continuity there exists $\delta > 0$ such that for each n, $d_X(w,x) < \delta$ implies $d_Y(f_n(w), f_n(x)) < \epsilon/3$. Choose $N \in Z^+$ so large that $d_Y(f_n(x), f(x)) < \epsilon/3$ whenever $n > N$. Since $(x,y) \in \text{Ls} f_n$, there exists $n > N$ and $w \in S_\delta[x]$ such that $d_Y(f_n(w), y) < \epsilon/3$. Together these facts yield $d_Y(f(x), y) < \epsilon$, a contradiction.

If X is locally connected and Y is locally compact, then the converse of Theorem 2 holds.

THEOREM 3. Let X be a locally connected metric space and Y a locally compact metric space. If $\{f_n\} \subset C(X, Y)$ converges topologically to a continuous function f, then $\{f_n\}$ converges pointwise to f and $\{f_n\}$ is pointwise equicontinuous.

PROOF: Suppose for some x, $\{f_n(x)\}$ fails to converge to $f(x)$. Then there exists $\epsilon > 0$ and a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that for each k, $d_Y(f_{n_k}(x), f(x)) \geq \epsilon$. Also, since $\text{Li} f_n = f$ there exists a sequence $\{(w_k, f_{n_k}(w_k)\}$ convergent to $(x, f(x))$. Since X is locally connected, by passing to a subsequence we can assume that x and w_k lie in a common connected subset C_k of X where $\lim_{k\to\infty} \text{diam}(C_k) = 0$. Choose $\epsilon^* < \epsilon$ such that $E = \{y: d_Y(y, f(x)) = \epsilon^*\}$ is compact. For all k sufficiently large, $d_Y(f_{n_k}(w_k), f(x)) < \epsilon^*$ so that $f_{n_k}(C_k)$ meets both $\{y: d_Y(y, f(x)) < \epsilon^*\}$ and $\{y: d_Y(y, f(x)) > \epsilon^*\}$. Since each set $f_{n_k}(C_k)$ is connected, all but finitely many sets $f_{n_k}(C_k)$ meet E. Since E is compact, $\text{Ls}(f_{n_k}(C_k) \cap E)$ is nonempty. Choosing a point y_0 in this set, the condition $\lim_{k\to\infty} \text{diam}(C_k) = 0$ yields $(x, y_0) \in \text{Ls} f_n - f$, a contradiction.

Suppose now that $\{f_n\}$ is not equicontinuous at some x in X. Then there exists $\epsilon > 0$, a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ and a sequence $\{z_k\}$ convergent to x such that for each k, $d_Y(f_{n_k}(z_k), f_{n_k}(x)) \geq \epsilon$. However, we know that $\lim_{k\to\infty} f_{n_k}(x) = f(x)$; so, without loss of generality we can assume that for all k, $d_Y(f_{n_k}(z_k), f(x)) \geq \epsilon$. Arguing as in the first part of the proof with $(z_k, f_{n_k}(z_k))$ replacing $(x, f_{n_k}(x))$ for each k, we reach a contradiction in exactly the same manner.

Example 3 of [2] shows that the local compactness assumption for Y cannot be dropped, even if X is compact. Example 2 of [2] shows that the local connectedness assumption for X cannot be dropped, again, even if X is compact. However, this last example is not completely satisfying in that X is not connected. Now the main result of [2] says that if X is compact and connected and Y is locally compact, then topological convergence in $C(X, Y)$ ensures not only pointwise convergence but also uniform convergence. If X is merely connected and Y is locally compact, we cannot expect topological convergence to force uniform convergence (see Example 1 of [1]). But does it force pointwise convergence or pointwise equicontinuity? The answer is negative, even if $Y = R$.

EXAMPLE 1. In the plane for each $n \in Z^+$ let $E_n = \{(x, n+1): 1/(n+1) \leq x \leq 1/n\}$; also define A and B as follows:

$$A = \{(0, y): y \geq 0\}$$

$$B = \left\{(x, y): \text{for some } n \in Z^+,\ x = \frac{1}{n} \text{ and } y \geq 0\right\}$$

Clearly $X = A \cup B \cup \bigcup_{n=1}^{\infty} E_n$ is a closed connected subset of the plane that fails to be locally connected. We define $f_n: X \to R$ by

$$f_n(x,y) = \begin{cases} 0 & \text{if } x > \frac{1}{n} \text{ or } y > n \\ n & \text{if } x \leq \frac{1}{n} \text{ and } y < n-1 \\ -ny + n^2 & \text{if } x \leq \frac{1}{n} \text{ and } n-1 \leq y \leq n \end{cases}$$

Notice that the graph of f_n restricted to each of the rays $x = 0$, $x = 1/n$, $x = 1/(n+1)$, $x = 1/(n+2)$, . . . consists of a horizontal segment at height n, a horizontal ray at height zero, and a segment joining them. The rest of the graph lies in the xy plane. It is easy to check that each such f_n is continuous. Now suppose f denotes the zero function. Since $f_n(0,0) = n$, the sequence $\{f_n\}$ does not converge pointwise to f. Also $\{f_n\}$ is not pointwise equicontinuous at the origin. However, we claim $\{f_n\}$ does converge topologically to f. First, it is obvious that $f \subset \text{Li}\, f_n$ and that whenever $x > 0$ and $(x, y, z) \in \text{Ls}\, f_n$, then $z = 0$. No problems can occur on the y-axis, either: if $y_0 \geq 0$ and $n - 2 \geq y_0$, then whenever $|y - y_0| < 1$ for all x we have either $f_n(x, y) = n$ or $f(x, y) = 0$. Thus, $(0, y_0, z) \in \text{Ls}\, f_n$ implies $z = 0$. We have shown $\text{Ls}\, f_n \subset f \subset \text{Li}\, f_n$.

For arbitrary metric spaces X and Y if $\{f_n\}$ is a sequence in $C(X,Y)$ convergent pointwise to a continuous function f, then $\{f_n\}$ converges uniformly on compact subsets of X if and only if $\{f_n\}$ is pointwise equicontinuous. This observation, in conjunction with a standard diagonalization argument [6], is all there is to the following version of the Ascoli Theorem: Let X be a separable metric space, let Y be an arbitrary metric space, and let $\Omega \subset C(X,Y)$. Then each sequence $\{f_n\}$ in Ω has a subsequence convergent uniformly on compact subsets of X to some continuous function if and only if (i) for each x, $\Omega_x = \{g(x): x \in \Omega\}$ has compact closure in Y, (ii) Ω is pointwise equicontinuous. Theorems 2 and 3 now say that for arbitrary X and Y uniform convergence on compact subsets implies topological convergence, whereas if X is locally connected and Y is locally compact, then these notions of convergence in $C(X,Y)$ agree. The Ascoli Theorem translates as follows.

THEOREM 4. *Suppose X is a separable metric space and Y is an arbitrary metric space. Let $\Omega \subset C(X, Y)$, and consider the following statements.*

(1) Each sequence in Ω has a subsequence convergent topologically to a continuous function.

(2) Ω is pointwise equicontinuous and for each x, $\Omega_x = \{g(x): g \in \Omega\}$ has compact closure.

Condition (2) always implies condition (1), and if X is locally connected and Y is locally compact, then condition (1) implies condition (2).

Previous examples show that condition (1) need not imply either subcondition of (2) if X is connected and $Y = R$, or $X = [0, 1]$ and Y is a Hilbert space. However, separability of X is required to obtain condition (1) from condition (2).

EXAMPLE 2. If X is an arbitrary metric space, then by Zorn's Lemma there exists for each $\epsilon > 0$ a maximal subset A_ϵ of X such that whenever $\{w, z\} \subset A_\epsilon$ then $d_X(w, z) \geq \epsilon$. It follows that $X \subset \cup\{S_\epsilon[x]: x \in A_\epsilon\}$, so that if X is nonseparable some A_ϵ must be uncountable. Suppose such an A_ϵ has cardinal number at least c (which would be guaranteed for nonseparable X by the continuum hypothesis). Let $K = \{g: g: Z^+ \to \{0, 1\}$ and $g(n) = 1$ infinitely often$\}$, and let $\varphi: K \to A_\epsilon$ be an injection. For each g in K set $E(g) = \{n: g(n) = 1\}$. For each $n \in Z^+$ let $W_n \subset X$ be described as follows: $x \in W_n$ if there exists g_x in K such that $d_X(x, \varphi(g_x)) < \epsilon/3$ and n has an odd number of predecessors in $E(g_x)$. By the construction of A_ϵ the assignment $x \to g_x$ on W_n is well defined. We now define $f_n: X \to [0, 1]$ by the formula

$$f_n(x) = \begin{cases} \dfrac{3}{\epsilon} d_X(x, \varphi(g_x)) & \text{if } x \in W_n \\ 1 & \text{if } x \notin W_n \end{cases}$$

Note that each f_n is actually Lipschitz with Lipschitz constant $3/\epsilon$; so, condition (2) of Theorem 4 is satisfied. However, no subsequence of $\{f_n\}$ can converge topologically, because each subsequence is of the form $\{f_{h(n)}\}$ where h is an order isomorphism from Z^+ onto $E(g)$ for some $g \in K$, and by construction

$$\text{Li}\, f_{h(n)} \cap (\{\varphi(g)\} \times [0, 1]\} = \emptyset$$

3. **Points of convergence of a topologically convergent sequence**. We first exhibit a sequence in $C(X, R)$ for a certain compact metric space X that is topologically convergent to a continuous function but which converges nowhere pointwise.

EXAMPLE 3. Let X denote the usual Cantor set in $[0, 1]$ and let $f: X \to R$ denote the zero function. Since X is a nowhere dense subset of $[0, 1]$ for each $j \in Z^+$ we can select j points $\{a_{j1}, a_{j2}, \ldots, a_{jj}\}$ in $[0, 1] - X$ satisfying

$$(1) \qquad a_{j1} < a_{j2} < \ldots < a_{jj}$$

$$(2) \qquad \bigcup_{i=1}^{j} S_{1/j}[a_{ji}] \supset [0, 1]$$

Set $I(j, 1) = [0, a_{j1}]$, $I(j, 2) = [a_{j1}, a_{j2}]$, $\ldots$, $I(j, j+1) = [a_{jj}, 1]$ and let

$$\varphi: Z^+ \to \{(J, k): j \in Z^+, K \in Z^+ \text{ and } k \leq j + 1\}$$

lexicographically order the codomain. Denote $\varphi(n)$ by (j_n, k_n) and define $f_n: X \to R$ by

$$f_n(x) = \begin{cases} j_n & \text{if } x \in I(j_n, k_n) \\ 0 & \text{otherwise} \end{cases}$$

Since the endpoints of each interval $I(j_n, k_n)$ lie in $\{0, 1\} \cup ([0, 1] - X)$, each f_n is continuous. Since at each $x \in X$, $\{f_n(x)\}$ exceeds one frequently, $\{f_n\}$ converges nowhere pointwise to f. We claim, however, that $\text{Li}\, f_n = \text{Ls}\, f_n = f$. First, since lim diam

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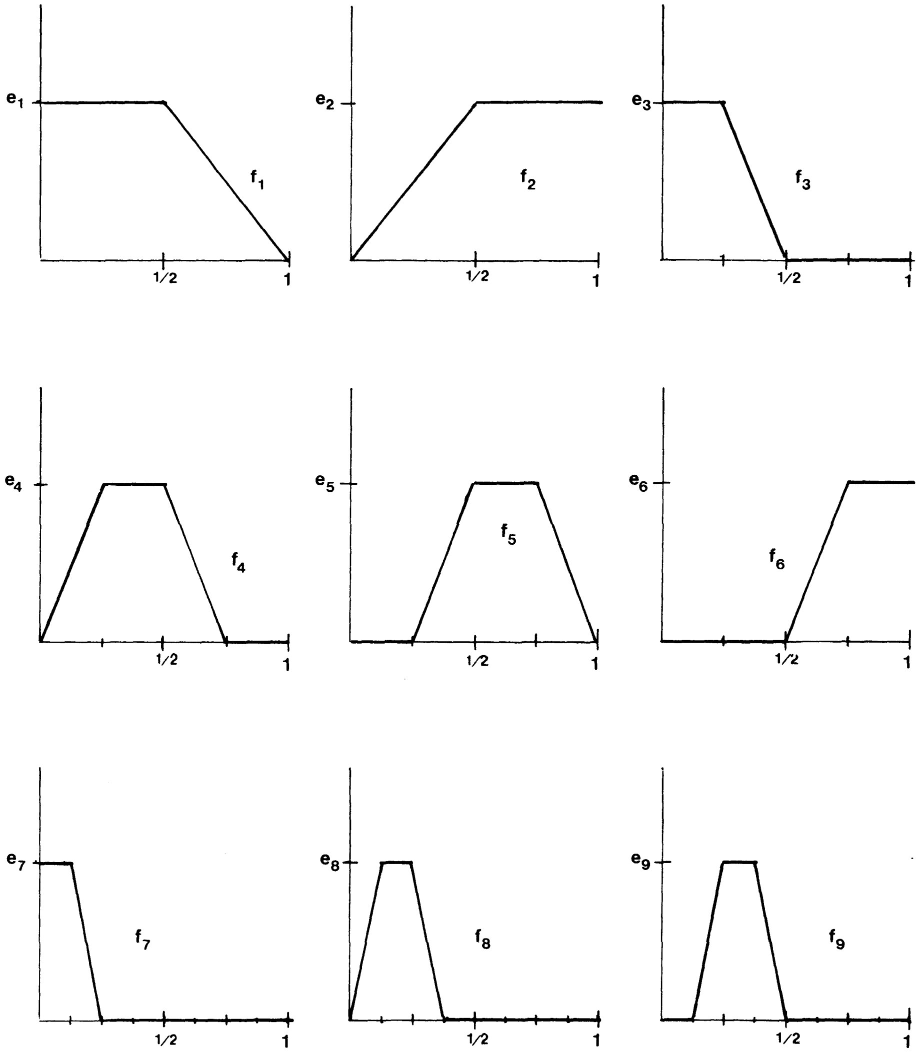

FIGURE 1.

$(\{x: f_n(x) \neq 0\}) = 0$, we have $f \subset \text{Li}\, f_n$. Also, for each $\alpha > 0$ the graphs of $\{f_n\}$ eventually all fail to meet $X \times (0, \alpha)$, and it follows that $\text{Ls}\, f_n \subset f$. This establishes the topological convergence of $\{f_n\}$ to f.

We can also exhibit an example of nowhere pointwise convergence in the context of Example 3 of [2], i.e., $X = [0, 1]$ and $Y =$ Hilbert space of square summable sequences. Heeding the advice of Professor G. Piranian that "one filthy picture is worth a thousand dirty words", in lieu of an analytic argument, we sketch the graphs of the first nine terms of a sequence $\{f_n\}$ convergent topologically to the zero function but convergent nowhere pointwise (see Figure 1). As in Example 3 of [2], $\{e_i: i \in Z^+\}$ denotes the standard orthonormal basis in the Hilbert space.

We next present a Baire category result that says that if X is complete and Y is arbitrary and $\{f_n\}$ converges topologically to f, then $f(x)$ is a subsequential limit of $\{f_n(x)\}$ at most points.

THEOREM 5. *Let X be a complete metric space and let Y be any metric space. Let* $\{f_n\}$ *be a sequence in* $C(X, Y)$ *topologically convergent to f in* $C(X, Y)$. *Then there is a dense* G_δ *subset E of X such that for each x in E,* $f(x)$ *is a subsequential limit of* $\{f_n(x)\}$.

PROOF: For each $n \in Z^+$ and $\epsilon > 0$ we form the closed set

$$A(n, \epsilon) = \bigcap_{j=n}^{\infty} \{x \colon d_Y(f_j(x), f(x)) \geq \epsilon\}.$$

We claim that each such set is nowhere dense. Let $x_0 \in A(n, \epsilon)$ and $\lambda > 0$ be arbitrary. Choose $\lambda^* < \lambda$ for which $d_X(x, x_0) < \lambda^*$ implies $d_Y(f(x), f(x_0)) < \epsilon/2$. Since Ls $f_n \supset f$ we can find $j > n$ and $x \in X$ such that both $d_X(x, x_0) < \lambda^*$ and $d_Y(f(x_0), f_j(x)) < \epsilon/2$. It follows that $d_Y(f(x), f_j(x)) < \epsilon$, establishing the claim. Next for each pair of positive integers n and k let $B(n, k) =$ the complement of $A(n, 1/k)$, a dense open set. By the Baire Category Theorem

$$E = \bigcap_{k=1}^{\infty} \bigcap_{n=1}^{\infty} B(n, k)$$

is a dense G_δ set. If $x \in E$ then for each n and k let $j(n, k)$ be the smallest integer exceeding n for which $f_{j(n,k)}(x)$ has distance less than $1/k$ from $f(x)$. If we set $n_1 = j(1, 1)$ and for each $k > 1$ we let $n_k = j(n_{k-1}, k)$, we have $\lim_{k \to \infty} f_{n_k}(x) = f(x)$.

A second look at the proof of Theorem 5 reveals that we really did not need the full strength of $\text{Li} f_n = \text{Ls} f_n = f$, but only $f \subset \text{Ls} f_n$. With this weaker assumption, our result has a rather nice interpretation: if we can approach each (x, y) in f along some trajectory of the form $\{(x_k, f_{n_k}(x_k))\}$, then we can approach most points along a vertical trajectory. Under the stronger assumption $\text{Li} f_n = \text{Ls} f_n = f$, must there actually exist a subsequence of $\{f_n\}$ convergent to f on some dense G_δ subset of X? The answer is negative, even if X is compact.

EXAMPLE 4. Let μ denote Lebesgue measure on the line, and let X be a Cantor set of positive measure in $[0, 1]$ as constructed in [6]. Note that for each x_0 in X and each $\epsilon > 0$, $\mu(\{x \colon x \in X \text{ and } |x - x_0| < \epsilon\}) > 0$. For each $n \in Z^+$ let V_n be the union of a finite collection of disjoint open intervals $\{W_{ni} \colon i = 1, 2, 3, \ldots, k_n\}$ in $(0, 1)$ such that

(1) For each n and $i \leq k_n$ the endpoints of W_{ni} lie in $[0, 1] - X$.

(2) For each n and $i \leq k_n$, $W_{ni} \cap X \neq \emptyset$.

(3) For each $x \in X$ and $n \in Z^+$ there exists $y \in V_n$ such that $|x - y| < 1/n$.

(4) For each n, $\mu(V_n) < 1/n$.

By condition (1) each set $V_n \cap X$ is clopen in X. Define for each n, $f_n \colon X \to R$ by

$$f_n(x) = \begin{cases} n & \text{if } x \notin V_n \\ 0 & \text{if } x \in V_n \end{cases}$$

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Since $V_n \cap X$ is clopen in X, each f_n is continuous, and conditions (2), (3) and (4) imply that $\{f_n\}$ converges topologically to the zero function. Let $\{f_{n_k}\}$ be an arbitrary subsequence of $\{f_n\}$. For each k write A_k for $V_{n_k} \cap X$, denote the closed set $\bigcap_{n=k}^{\infty} A_n$ by B_k. Clearly,

$$\{x: \lim_{k\to\infty} f_{n_k}(x) = 0\} = \bigcup_{k=1}^{\infty} B_k.$$

Now whenever $n \geq k$ we have $B_k \subset A_n$; so, condition (4) implies that for all k, $\mu(B_k) = 0$. Since $\mathrm{int}_X(B_k) \neq \emptyset$ would imply $\mu(B_k) > 0$, we conclude that each B_k is nowhere dense in X. Thus $\{x: \lim_{k\to\infty} f_{n_k}(x) = 0\}$ is a set of first category in X, and since X is complete $\{x: \lim_{k\to\infty} f_{n_k}(x) = 0\}$ contains no dense G_δ set.

We remark in closing that a somewhat simpler counterexample can be constructed for $X = [0, 1]$ and $Y =$ the Hilbert space of square summable sequences. We leave this to the imagination of the reader.

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