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On the extension of orders in ordered modules

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We introduce the notion of a positively independent set of elements in an ordered module. With this concept we determine a necessary and sufficient condition which insures that on a strictly ordered module over a strictly ordered ring there exists a strict total order refining the given order. This generalizes a previous result of Fuchs, concerning the case of ordered abelian groups.

As an application, let R be a strictly ordered totally ordered ring and let M be the R-module of all mappings from a set I into R, with pointwise order; then this order on M may be refined to a strict total order.

Let R be a (commutative associative) ordered ring (with unit element $1 \neq 0$), let $P_R = \{r \in R \mid r \geq 0\}$ be the cone of positive elements of R (with respect to the given order). That is $P_R + P_R \subseteq P_R$, $P_R \cdot P_R \subseteq P_R$, $P_R \cap (-P_R) = \{0\}$. Moreover we shall assume that $1 \in P_R$. If $P_R \cup (-P_R) = R$ we say that R is totally ordered.

We say that (R, P_R) is *strictly ordered* when: $r, r' \in P_R$, $rr' = 0$ implies $r = 0$ or $r' = 0$. For example, if R is an ordered integral domain then it is strictly ordered. However, the ring Z^I of integral-valued functions on a set, with pointwise order, is not strictly

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ordered (when I has at least two elements).

Let M be an R-module. A subset C of M is called a *cone* when it satisfies the following properties: $C + C \subseteq C$, $P_R \cdot C \subseteq C$. A cone P_M such that $P_M \cap (-P_M) = \{0\}$ defines an order on M, making M into an ordered R-module: $m \geq m'$ whenever $m-m' \in P_M$. If $P_M \cup (-P_M) = M$ we say that M is totally ordered.

We say that (M, P_M) is a *strictly ordered* module over (R, P_R) when:

$$r \in P_R, \quad x \in P_M, \quad rx = 0 \text{ implies } r = 0 \text{ or } x = 0 .$$

Thus (R, P_R) is a strictly ordered ring when it is a strictly ordered module over itself.

For example, if R is a strictly ordered ring, if $M = R^I$ is the R-module of all functions from I to R, with pointwise order, then M is a strictly ordered module over R.

Let (R, P_R) be an ordered ring, let (M, P_M) be an ordered module over (R, P_R).

We say that the set $\{x_1, \ldots, x_n\}$ of elements of M is *positively independent* when the following holds:

$$\text{if } r_i \in P_R \text{ and } \sum_{i=1}^{n} r_i x_i \in P_M \text{ then each } r_i = 0 .$$

For example if $x \in P_M$ then $\{x\}$ is not positively independent.

a) *The following conditions are equivalent:*

1) *for every* $x \notin P_M$ *the set* $\{x\}$ *is positively independent;*

2) *if* $0 \neq r \in P_R$ *and* $rx \in P_M$ *then* $x \in P_M$.

The proof is immediate.

We shall now indicate a generalization of Theorem 1, p. 113 in [1]:

THEOREM. *Let* (M, P_M) *be a strictly ordered module over the*

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strictly ordered ring (R, P_R) . *The following statements are equivalent:*

1) *There exists a total order* T_M *on* M *such that* $P_M \subseteq T_M$ *and* (M, T_M) *is a strictly ordered module over* (R, P_R) ;

2) *if* $a_1, \ldots, a_n$ *are non-zero elements of* M , *there exist* $\varepsilon_1, \ldots, \varepsilon_n \in \{1, -1\}$ *such that the set* $\{\varepsilon_1 a_1, \ldots, \varepsilon_n a_n\}$ *is positively independent in* (M, P_M) .

Proof: $1 \to 2$. Let $a_1, \ldots, a_n$ be non-zero elements of M ; then either $a_i \in T_M$ or $-a_i \in T_M$. Let $\varepsilon_i \in \{1, -1\}$ be such that $-\varepsilon_i a_i \in T_M$. Then $\{\varepsilon_1 a_1, \ldots, \varepsilon_n a_n\}$ is positively independent. For if $r_i \in P_R$ and $\sum_{i=1}^{n} r_i \varepsilon_i a_i \in P_M \subseteq T_M$, since $r_i \varepsilon_i a_i \in -T_M$ then $r_i(-\varepsilon_i a_i) = 0$ for every $i = 1, \ldots, n$. But (M, T_M) is strictly ordered, hence $r_i = 0$ for every $i = 1, \ldots, n$.

$2 \to 1$. To prove this implication, we shall need a lemma. For every element $a \in M$ we denote by $C(a)$ the intersection of all cones of M containing a ; $C(a)$ is clearly a cone, namely $C(a) = P_R a$.

LEMMA. *Let* (M, P_M) *be a strictly ordered module over* (R, P_R) *satisfying condition* (2). *If* $a \in M$ *then either* $P_M + C(a)$ *or* $P_M + C(-a)$ *defines a strict order on* M , *satisfying condition* (2).

Proof of the Lemma. The lemma is trivial when $a = 0$, so we may suppose that $a \neq 0$. We assume that M contains non-zero elements $a_1, \ldots, a_n$ and non-zero elements $b_1, \ldots, b_m$ such that for all $\varepsilon_i, \eta_j \in \{1, -1\}$ the sets $\{a, \varepsilon_1 a_1, \ldots, \varepsilon_n a_n\}$, $\{-a, \eta_1 b_1, \ldots, \eta_m b_m\}$ are not positively independent.

Then for all $\delta, \varepsilon_i, \eta_j \in \{1, -1\}$ the sets $\{\delta a, \varepsilon_1 a_1, \ldots, \varepsilon_n a_n, \eta_1 b_1, \ldots, \eta_m b_m\}$ are not positively independent. This contradicts condition (2).

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Hence, there are two possibilities:

1) for all non-zero elements $a_1, \ldots, a_n \in M$ there exist $\varepsilon_i \in \{1, -1\}$ such that $\{a, \varepsilon_1 a_1, \ldots, \varepsilon_n a_n\}$ are positively independent; in particular $\{a\}$ is positively independent and $P_M \cap C(a) = 0$.

2) for all non-zero elements $a_1, \ldots, a_n \in M$ there exist $\varepsilon_i \in \{1, -1\}$ such that $\{-a, \varepsilon_1 a_1, \ldots, \varepsilon_n a_n\}$ are positively independent; in particular $P_M \cap C(-a) = 0$.

In case (1) let $P'_M = P_M + C(-a)$;

in case (2) let $P'_M = P_M + C(a)$.

Then clearly $P'_M + P'_M \subseteq P'_M$ and $P_R P'_M \subseteq P'_M$. Now we show condition (2) for P'_M (for example in the first case). Let $a_1, \ldots, a_n$ be non-zero elements of M, let $\varepsilon_i \in \{1, -1\}$ be such that $\{a, \varepsilon_1 a_1, \ldots, \varepsilon_n a_n\}$ are positively independent (relatively to P_M). We show that if $r_i \in P_R$ and $\sum_1^n r_i \varepsilon_i a_i \in P'_M$ then $r_i = 0$, $\forall i = 1, \ldots, m$. For $\sum_1^n r_i \varepsilon_i a_i = x - ra$ with $x \in P_M$, $r \in P_R$; thus $ra + \sum r_i \varepsilon_i a_i \in P_M$, hence $r = r_i = 0$, $\forall i = 1, \ldots, n$.

From this follows $P'_M \cap (-P'_M) = 0$. Because, if $0 \neq x \in P'_M \cap (-P'_M)$ then $x, -x \in P'_M$, so the sets $\{x\}$, $\{-x\}$ are not positively independent, against (2).

Hence P'_M defines an order on M which makes it strictly ordered over (R, P_R). In fact, let $0 \neq r \in P_R$, $x-sa \in P'_M$, with $x \in P_M$, $s \in P_R$ and assume $r(x-sa) = 0$, so $rx = rsa \in P_M \cap C(a) = 0$; since the order P_M is strict then $x = 0$; since $\{a\}$ is positively independent then $rs = 0$; but (R, P_R) is a strictly ordered ring, hence $s = 0$; so $x-sa = 0$.

Thus we have established the lemma.

Continuation of the proof of the Theorem. We consider all subsets Q of M satisfying

a) $P_M \subseteq Q$,

b) $Q + Q \subseteq Q$,

c) $P_R Q \subseteq Q$,

d) $Q \cap (-Q) = 0$,

e) if $r \in P_R$, $x \in Q$ and $rx = 0$ then either $r = 0$ or $x = 0$,

f) condition (2) is satisfied by Q .

The family $\mathcal{Q}$ of such subsets contains P_M . If $(Q_i)_{i=1,2,\ldots}$ is any strictly increasing chain of subsets in $\mathcal{Q}$, let $Q = \bigcup_1^\infty Q_i$; then $Q \in \mathcal{Q}$. Everything but (f) is immediate. Now we check (f). Let $a_1, \ldots, a_n \in M$; for every i there exists $\varepsilon_j^i \in \{1, -1\}$ $(j = 1,\ldots,n)$ such that $\varepsilon_1^i a_1, \ldots, \varepsilon_n^i a_n$ are positively independent (with respect to Q_i). Since there are only finitely many n-tuples of elements $1, -1$, then there exists an infinite chain $Q_{i_1} \subset Q_{i_2} \subset \ldots \subset Q_{i_m} \subset \ldots$ such

$$\left(\varepsilon_1^{i_1}, \ldots, \varepsilon_n^{i_1}\right) = \left(\varepsilon_1^{i_2}, \ldots, \varepsilon_n^{i_2}\right) = \ldots = \left(\varepsilon_1^{i_m}, \ldots, \varepsilon_n^{i_m}\right) = \ldots .$$

Let $\delta_j = \varepsilon_j^{i_m}$ for $m = 1, 2, \ldots$, $j = 1, \ldots, n$.

Then $\delta_1 a_1, \ldots, \delta_n a_n$ are positively independent over Q ; for if $r_j \in P_R$ and $\sum_1^n r_j \delta_j a_j \in Q$ then there exists m such that $\sum_1^n r_j \delta_j a_j \in Q_{i_m}$, hence $r_j = 0$ for $j = 1, \ldots, n$.

Thus $\mathcal{Q}$ is inductive and by Zorn's Lemma, there exists a maximal element $T_M \in \mathcal{Q}$.

Now, let $a \in M$. By the lemma, either $T_M + C(a)$ or $T_M + C(-a)$ defines an order satisfying condition (2) which is strict. By the

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maximality of T_M we must have $a \in T_M$ or $-a \in T_M$, showing that T_M is a total order on M. □

We shall turn to the special case where (R, P_R) is a strictly ordered ring, $M = R^I$ is the R-module of all functions from I to R and $P_M = \left\{f \in M \mid f(x) \in P_R \text{ for every } x \in I\right\}$.

Let us consider the following condition:

3) *If* $f_1, \ldots, f_n$ *are non-zero elements of* $M = R^I$ *there exists* k, $1 \leq k \leq n$, *elements* $x_1, \ldots, x_k \in I$, *a partition of* $\{1, \ldots, n\}$ *into disjoint non-empty subsets* $S_1, \ldots, S_k$ *and* $\varepsilon_1, \ldots, \varepsilon_n \in \{-1, 1\}$ *such that*

$$\begin{cases} \varepsilon_i f_i(x_j) < 0 & \text{when } i \in S_j , \\ f_i(x_j) = 0 & \text{when } i \in S_{j+1} \cup \ldots \cup S_k . \end{cases}$$

We prove:

b) *If* (R, P_R) *is a strictly ordered ring then condition* (3) *implies condition* (2) *of the theorem.*

Proof. Let $f_1, \ldots, f_n$ be non-zero elements of M. We choose $x_1, \ldots, x_k$, $S_1, \ldots, S_k$ and $\varepsilon_1, \ldots, \varepsilon_n$ as indicated in the hypothesis, and we proceed to show that if $r_1, \ldots, r_n \in P_R$ and $\sum_{i=1}^{n} r_i \varepsilon_i f_i \in P_M$ then each $r_i = 0$. We have $\sum_{i=1}^{n} r_i \varepsilon_i f_i(x_1) \in P_R$, but $f_i(x_1) = 0$ when $i \in S_2 \cup \ldots \cup S_k$, hence $\sum_{i \in S_1} r_i \varepsilon_i f_i(x_1) \in P_R$. From $\varepsilon_i f_i(x_1) < 0$ when $i \in S_1$ we deduce that

$\sum_{i \in S_1} r_i \varepsilon_i f_i(x_1) \in P_R \cap (-P_R) = \{0\}$. Thus $r_i \varepsilon_i f_i(x_1) = 0$ for every $i \in S_1$. Since (P, R_P) is strictly ordered and $r_i\left(-\varepsilon_i f_i(x_1)\right) = 0$ with $-\varepsilon_i f_i(x_i) > 0$ we deduce that $r_i = 0$ for $i \in S_1$.

So we have $\sum_{i \notin S_1} r_i \varepsilon_i f_i \in P_M$ and we may proceed by induction showing successively that $r_i = 0$ for every $i \in S_j$ and $j = 1, \ldots, k$, hence that $r_i = 0$ for every $i = 1, \ldots, n$. □

Now we prove:

c) *If* (R, P_R) *is a totally ordered ring then condition* (3) *is satisfied by* $M = R^I$ *with pointwise order.*

Proof. Let $f_1, \ldots, f_n$ be non-zero elements of $M = R^I$, let $x_1 \in I$ be such that $f_1(x_1) \neq 0$ and $S_1 = \{i \mid 1 \leq i \leq n , f_i(x_1) \neq 0\}$. Since (R, P_R) is totally ordered, for every $i \in S_1$ there exists $\varepsilon_i \in \{-1, 1\}$ such that $\varepsilon_i f_i(x_1) < 0$. If $S_1 = \{1, \ldots, n\}$ then condition (3) is satisfied with $k = 1$.

If $S_1 \neq \{1, \ldots, n\}$ let n_2 be the smallest integer such that $n_2 \notin S_1$ (thus $1 < n_2 \leq n$); since $f_{n_2} \neq 0$ and $f_{n_2}(x_1) = 0$ there exists $x_2 \in I$, $x_2 \neq x_1$ such that $f_{n_2}(x_2) \neq 0$; let $S_2 = \{i \mid i \notin S_1 , f_i(x_2) \neq 0\}$. Since (P, R_P) is totally ordered, for every $i \in S_2$ there exists $\varepsilon_i \in \{-1, 1\}$ such that $\varepsilon_i f_i(x_2) < 0$. If $S_1 \cup S_2 \neq \{1, \ldots, n\}$ we may proceed in this way, and after a finite number of steps we establish the validity of condition (3). □

We have therefore shown:

d) *Let* (R, P_R) *be a strictly ordered, totally ordered ring; let* $M = R^I$ *be the ordered* R*-module with pointwise order. Then there exists a total order* T_M *on* M *such that* $P_M \subseteq T_M$ *and* (M, T_M) *is a strictly ordered module over* (R, P_R).

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