A CONVENIENT METHOD TO OBTAIN STELLAR EIGENFREQUENCIES* **

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Abstract. The differential equations describing stellar oscillations are transformed into an algebraic eigenvalue problem. Frequencies of adiabatic oscillations are obtained as the eigenvalues of a banded real symmetric matrix. We employ the Cowling-approximation, i.e. neglect the Eulerian perturbation of the gravitational potential, and, in order to preserve selfadjointness, require that the Eulerian pressure perturbation vanishes at the outer boundary. For a solar model, comparison of first results with results obtained from a Henyey method shows that the matrix method is convenient, accurate, and fast.

1. Introduction

Stellar oscillations can be described as solutions of linear equations which are obtained from a perturbation of the equations of the internal constitution of stars. The Henyey method (Henyey *et al.*, 1964) which has been employed in most cases to solve these equations, requires an explicit first guess at the eigenfrequency, which then is improved by iteration. However, it often turns out that the desired eigenvalue is not found, inspite of a rather close guess. In the present contribution we therefore use a different method. We transform the system of equations (Section 2A) into an algebraic eigenvalue problem (Section 2B). Castor (1971) has described such a method for radial oscillations. Here we consider the general non-radial case; but unlike Sobouti (1977), who expanded the solutions in terms of complete sets of basis functions, and then obtained the algebraic eigenvalue problem via the Rayleigh–Ritz variational scheme, we employ here only the usual expansion into spherical harmonics and use a difference scheme for the radial part. Since we restrict ourselves to the adiabatic case, our matrix will be real and symmetric. A special consideration will be devoted to the choice of boundary conditions (Section 2C), which must not be in conflict with the self-adjointness, i.e. energy-conservation, of the system.

Standard routines can be employed to obtain all n eigenvalues of a n^2 matrix. Although these also use iteration techniques, they do not in general depend on explicit first guesses, and, in addition, are extremely rapid, in particular if – as in our case – the matrix has a band structure. We report first results in Section 3, and compare them to results obtained with the Henyey method.

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2. The Eigenvalue Problem

A. Differential equations

We consider Lagrangian perturbations $\delta\rho$, $\delta p\ldots$, and Eulerian perturbations ρ_1, $p_1\ldots$, both proportional to $\exp(i\omega t)$. Adiabatic oscillations of small amplitude are described by

$$\frac{\delta T}{T_0} = \nabla_{\mathrm{ad}} \frac{\delta p}{P_0} \tag{1}$$

and by the conservation of momentum and mass,

$$\omega^2 \delta\mathbf{r} = -\frac{\rho_1}{\rho_0^2}\nabla P_0 + \frac{1}{\rho_0}\nabla p_1\,, \tag{2}$$

$$\frac{\delta\rho}{\rho_0} = -\nabla\cdot\delta\mathbf{r} \tag{3}$$

(e.g. Ledoux and Walraven, 1958), where T_0, $P_0\ldots$ are equilibrium quantities, and the Eulerian perturbation, Φ_1, of the gravitational potential has been neglected (Cowling, 1941). The latter approximation was found to change the frequencies of low order radial p-modes by up to 3%, while the effect on radial modes with periods around 5 min was only $\sim 0.2\%$ (Knölker, 1978). We anticipate that the non-radial oscillations, in particular at large l, will be affected still less (e.g. Cox, 1980, p. 248).

Equations (1) and (3) are supplemented by the equation of state

$$P = \frac{\rho\mathscr{R}T}{\mu} + \frac{aT^4}{3} = \frac{\rho\mathscr{R}T}{\beta\mu}\,, \tag{4}$$

where $\mathscr{R}$ is the gas constant, μ the mean molecular weight, a the radiation density constant, and β the ratio gas pressure/total pressure. We expand the unknown functions in terms of spherical harmonics, $Y_l^m(\theta,\varphi)$, linearize (4), eliminate the horizontal components of the displacement, $\delta\mathbf{r}$, and the perturbations of density and temperature, and introduce a non-dimensional radial displacement, $x = \delta r/r_0$, (Lagrangian) pressure perturbation, $p = \delta p/P_0$, and frequency, $\sigma = \omega(4\pi G\bar{\rho}_0)^{-1/2}$. The resulting two equations for x and p are written in such a way that the equilibrium quantities, c_i, of Baker and Kippenhahn (1962, Equations (27)) can be used (we have, in addition, $c_{20} = c_3\rho_0/\bar{\rho}_0$):

$$\frac{1}{c_4}\frac{\mathrm{d}x}{\mathrm{d}\ln P_0} = \left(3 - \frac{l(l+1)}{c_3\sigma^2}\right)x + \left(\Gamma - c_4\frac{l(l+1)}{c_3\sigma^2}\right)p\,, \tag{5}$$

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$$\frac{\mathrm{d}p}{\mathrm{d}\ln P_0} = -\left(c_3\sigma^2 + 4 - \frac{l(l+1)}{c_3\sigma^2} - c_{20}\right)x + \left(c_4\frac{l(l+1)}{c_3\sigma^2} - 1\right)p\,, \tag{6}$$

where $\Gamma = c_5 - c_6/c_2 = \Gamma_1^{-1}$, and $\Gamma_1 = (\mathrm{d}\ln P_0/\mathrm{d}\ln\rho_0)_{\mathrm{ad}}$.

B. Transformation to an algebraic eigenvalue problem

In order to solve Equations (5) and (6), we have so far used a Henyey type code, originally written by Baker and Kippenhahn (1962), and extended to the non-radial case by Knölker (1978). In the present contribution the equations are however transformed into the problem of finding the eigenvalues of a real symmetric matrix. We first introduce a new variable

$$y = x'/c_4 - 3x - \Gamma p\,, \tag{7}$$

where the prime denotes a derivative with respect to $\ln P_0$. We eliminate p from (5) and (6) and obtain

$$\sigma^2 x = -Ax'' + Bx' + Cx + \frac{1}{c_3\Gamma}\left(1 - \Gamma - \frac{\Gamma'}{\Gamma}\right)y + \frac{y'}{c_3\Gamma}\,, \tag{8}$$

$$\sigma^2 y = \frac{l(l+1)}{c_3\Gamma}(c_4 y - x' + (3c_4 - \Gamma)x)\,, \tag{9}$$

where

$$A = (c_3c_4\Gamma)^{-1}\,,$$

$$B = A(\Gamma'/\Gamma + c_4'/c_4 + 3c_4 - 1)\,,$$

$$C = \frac{1}{c_3\Gamma}(c_{20}\Gamma - 4\Gamma - 3\Gamma'/\Gamma + 3)\,.$$

The next step is the transformation

$$x = au\,, \qquad y = a\sqrt{l(l+1)}\,v\,, \tag{10}$$

where the condition $a'/a = B/2A$ would delete the u'-term in the differential equation for u. Here we use instead $a'/a = (B + A')/2A$. This ensures that the symmetry of our matrix will be conseved at the same time. The resulting equations for u and v can then be written in the form

$$\sigma^2 u = -\tfrac{1}{2}((Au)'' + Au'') + Du + \tfrac{1}{2}((Gv)' + Gv') + Fv\,, \tag{11}$$

$$\sigma^2 v = Ev - \tfrac{1}{2}((Gu)' + Gu') + Fu\,, \tag{12}$$

where

$$D = C + A''/2 + Ba'/a - Aa''/a\,,$$

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$$E = c_4 l(l+1)/(c_3\Gamma)\,,$$

$$F = G(3c_4 - \Gamma - a'/a) + G'/2\,,$$

$$G = \sqrt{l(l+1)}/(c_3\Gamma)\,.$$

We now introduce a 'staggered mesh' (e.g. Williams, 1969): We consider the values v_i of v at equidistant levels of $\ln P_0$, and the values $u_{i+1/2}$ at intermediate levels, and replace the derivatives by centered differences. The unknowns are then arranged into a vector $\mathbf{z} = (v_1, u_{3/2}, v_2, u_{5/2}\ldots)$, and Equations (11) and (12) are combined into the system

$$\sigma^2 \mathbf{z} = \mathsf{N}\mathbf{z}\,, \tag{13}$$

where the special combinations of the derivatives appearing in (11) and (12) guarantee the symmetry of the matrix N. In order to limit the bandwidth of N to 5, we have replaced $(Fv)_{i+1/2}$ by $(F_{i+1}v_{i+1} + F_i v_i)/2$ and u_i by $(u_{i+1/2} + u_{i-1/2})/2$ and all coefficients $A_{i+1/2}, G_{i+1/2}\ldots$ are eliminated by the proper arithmetic means of such quantities at neighbouring levels. Errors arising from these operations are only of order h^2, where $h = (\ln P_0)_{i+1} - (\ln P_0)_i$, consistent with the difference scheme used. – As an alternative we have arranged the unknowns u_i and v_i at *all* levels into an eigenvector. Direct use of (11) and (12), again with a scheme of centered differences, then also yields a problem of type (13) with a symmetric matrix, but in this case the bandwidth is 7.

C. Boundary conditions

We place the outer boundary for our oscillating star at a level where the optical depth is small ($\tau = \tau_B = 2.5 \times 10^{-4}$). The equilibrium atmosphere outside this level is adapted to the atmosphere of Vernazza *et al.* (1976), and P_0 and ρ_0 are finite at $\tau = \tau_B$. The 'zero-'boundary conditions, used by Chandrasekhar (1964) and Unno *et al.* (1979, p. 85), are therefore not satisfied. In order to keep the problem self-adjoint we require that the expression $p_1 + \rho_0\Phi_1$ vanishes at the boundary. Due to the neglect of Φ_1 this means that the Eulerian perturbation, p_1, of the pressure vanishes. In terms of the variables u and v this condition is $Au' = Gv - Au(\Gamma + a'/a - 3c_4)$. Using Equations (10) and (11) we can calculate the expression

$$\int [(11)u^* + (12)v^* - (11)^* u - (12)^* v]\, \mathrm{d}\ln P_0\,,$$

where the asterisk denotes the complex conjugate, and thus show that

$$(\sigma^2 - \sigma^{*2}) \int (uu^* + vv^*)\, \mathrm{d}\ln P_0 = [u(Au' - Gv)^* - u^*(Au' - Gv)]_o^i\,. \tag{14}$$

This expression vanishes at the inner boundary (i), where we require $u = 0$, and at the outer boundary (o), due to the above condition and the reality of $A, G\ldots$.

From Equations (7) and (9) we see that, at the outer boundary,

$$\sigma^2 y = -\frac{l(l+1)}{c_3}(x + c_4 p) = -\frac{l(l+1)c_4}{c_3 P_0}\, p_1 = 0\,.$$

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Hence, if $\sigma^2 \neq 0$, we have $y = 0$ and, therefore, $v = 0$. The variable v_1 can thus be dropped from the list of our unknowns. At the same time, we find from (9) that $x' = (3c_4 - \Gamma)x$ at the outer boundary, and, by (10), $u' = bu$, where $b = 3c_4 - \Gamma - a'/a$. In the form $(u_{3/2} - u_{1/2})/h = b(u_{3/2} + u_{1/2})/2$, this condition is used in order to eliminate $u_{1/2}$ from Equation (11) for $u_{3/2}$. The outer boundary condition is thus incorporated into the first line of our matrix, and the symmetry is preserved.

In the present contribution we exclude from our model the central region of the Sun. We set $u = 0$ at, say, $r = 0.2\, r_\odot$, and count our levels so that $U_{N+1/2} = 0$. We drop this variable from the list of our unknowns. Hence, the eigenvector is $\mathbf{z} = (u_{3/2}, v_2, \ldots u_{N-1/2}, v_N)$, and the size of our matrix is $[2(N-1)]^2$.

After the evaluation of $\mathbf{z}$, all operations described in this section can be reversed in order to recover the variables x and p at all levels. We normalize the solution such that $x = 1$ at the outer boundary.

The condition $u' = bu$ can also be incorporated into the direct difference scheme. An asymmetry is then obtained in the first line of the matrix of bandwidth 7, and a Jacobian transformation, leaving the eigenvalues unchanged, must be applied to the vector $\mathbf{z}$ in order to restore the symmetry (e.g. Acton, 1970, p. 316).

Instead of the condition $p_1 = 0$ at the outer boundary, Ando and Osaki (1975) have employed the condition of complete reflexion of the stellar oscillations at the surface. This is possible only for those frequencies which fall into the range of evanescent atmospheric waves. In the case of radial pulsations ($l = 0$), and frequencies small compared to the atmospheric cut-off, the condition has the form (Baker and Kippenhahn, 1965)

$$p = -(4 + 3\sigma^2)x\,,$$

or

$$\sigma^2 u = -Au' + A\left(3c_4 - 4\Gamma c_4 - \frac{a'}{a}\right)u\,. \tag{15}$$

The surface term on the right-hand side of (14) is then $-(\sigma^2 - \sigma^{*2})uu^*$. In combination with the left-hand side this means that the eigenvalue must be real. This must not, however, be understood as a proof of selfadjointness since, by assumption, we have restricted ourselves to the frequencies of evanescent waves in the atmosphere.

Equation (15) can be incorporated into the algebraic system (13). The ensuing asymmetry of the matrix N again can be removed by a Jacobian transformation.

3. Results

We report here only a few results in order to demonstrate how the method outlined in the preceding section works. For the equilibrium we use a model computed with a stellar envelope program similar to that of Baker and Temesváry (1966), with $T_{\text{eff}} = 5770$ K, and extending in radius down to $r = 0.2\, r_\odot$. Böhm-Vitense's (1958) formulation of the

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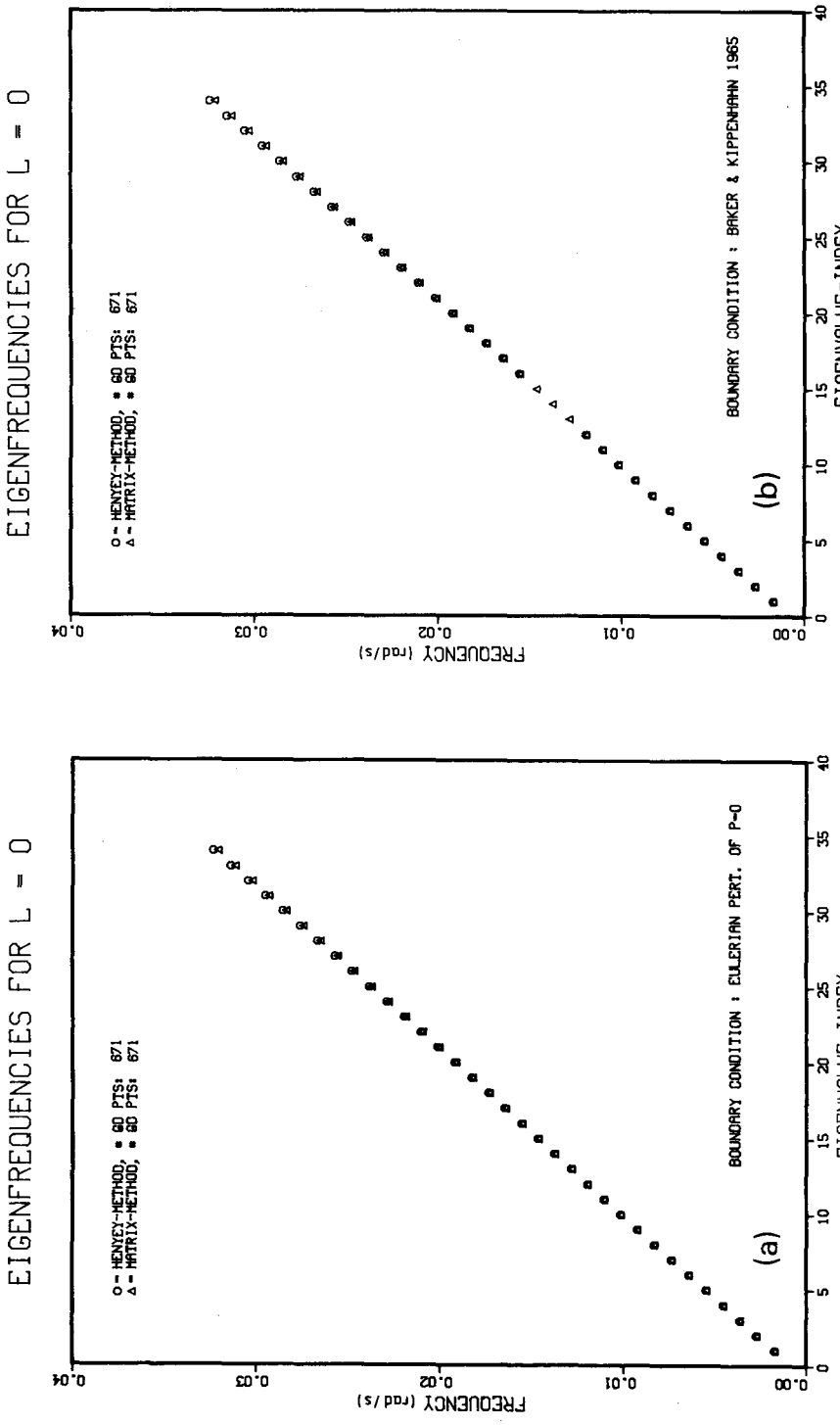

Fig. 1. Eigenfrequencies, ω, of the first 34 radial pulsations, computed with the Henyey and matrix methods: (a) with zero Eulerian pressure perturbation, (b) with complete reflexion at the outer boundary.

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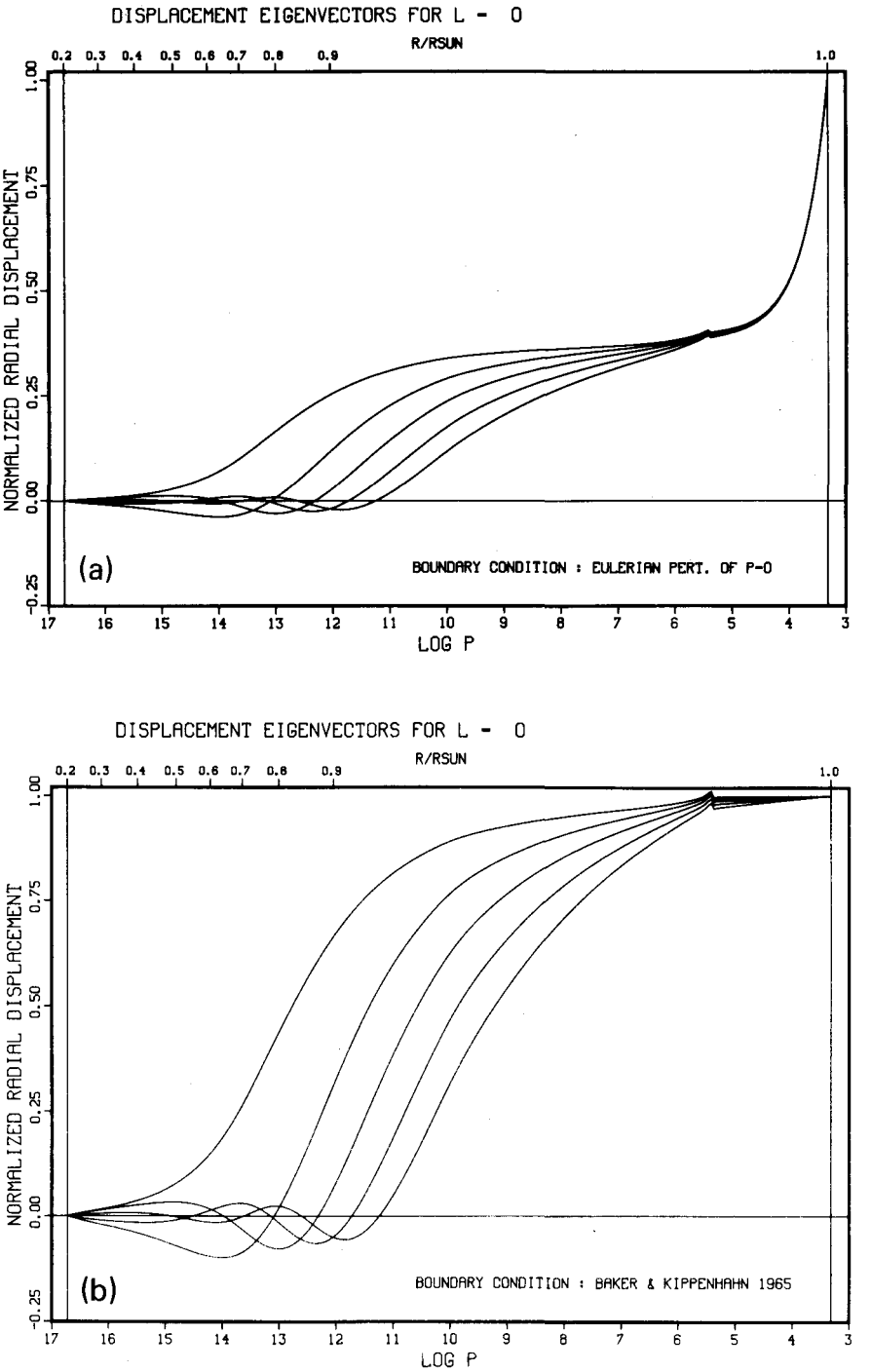

Fig. 2. Displacement eigenfunctions, $\delta r/r_0$, for the first five radial pulsations: (a) zero Eulerian pressure perturbation, (b) complete reflexion at the outer boundary.

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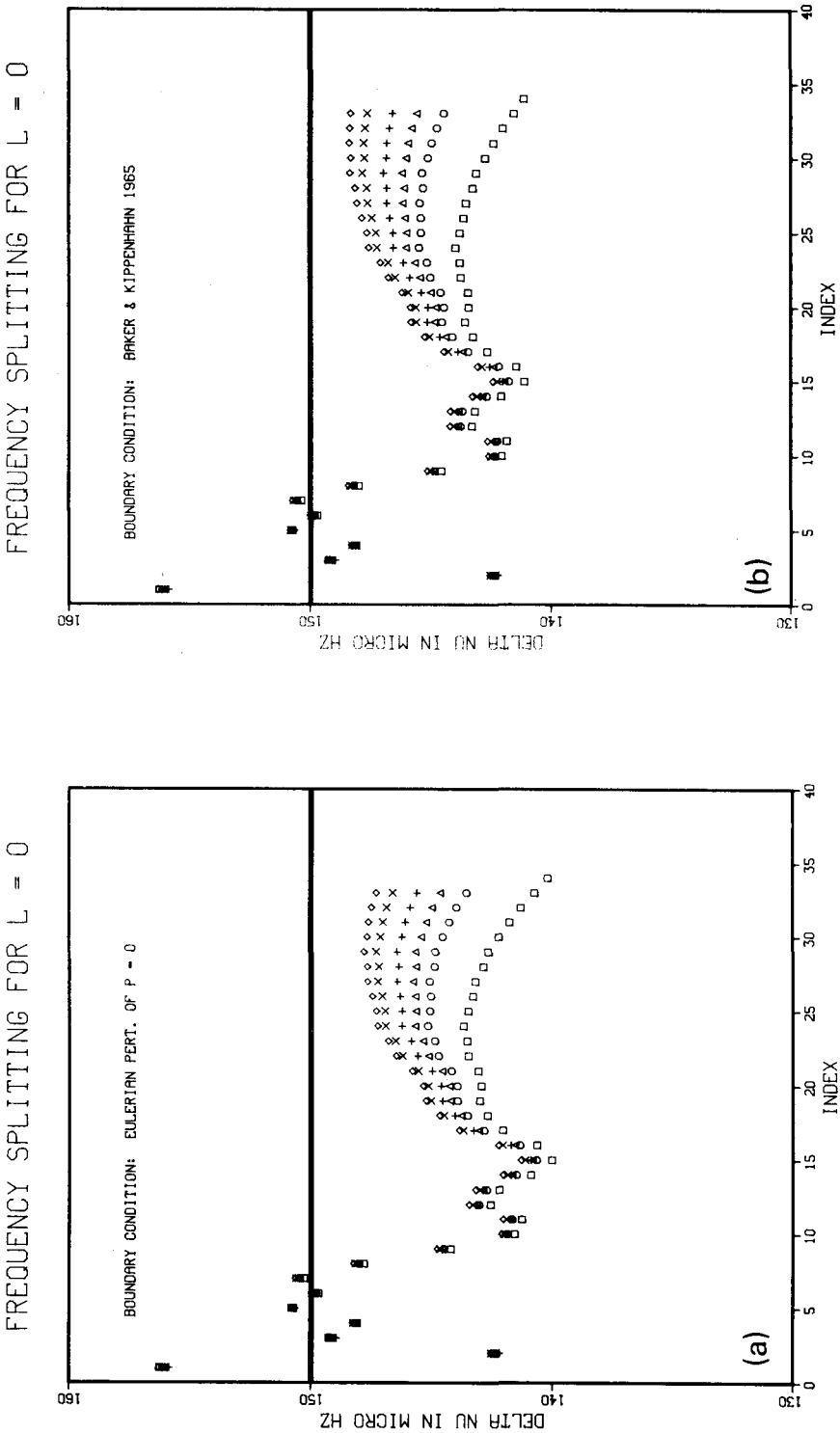

Fig. 3. Frequency differences, $\Delta\nu$, between two consecutive radial pulsation eigenfrequencies: (a) zero Eulerian pressure perturbation, (b) complete reflexion at the outer boundary. The various symbols mark the cases of 269, 336, 384, 448, 537, and 671 grid points, from below in this order; the horizontal line at 150 μHz is the asymptotic value $(2\int c^{-1}\,\mathrm{d}r)^{-1}$.

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mixing length theory is used, with α = mixing length/scale height = 1. For the opacity we use the tables of Cox and Tabor (1976), and the abundances of H, He, and heavier elements are specified by $X = 0.7$, $Y = 0.28$, and $Z = 0.02$.

Eigenfrequencies, ω, of radial pulsations ($l = 0$), are shown in Figures 1a, b, up to the 34th mode. The two choices of boundary conditions discussed above virtually lead to the same frequencies, although the corresponding eigenvectors, depicted in Figures 2a, b, exhibit substantial differences near the outer boundary. The results obtained from the Henyey and matrix methods differ at high frequencies only. The reason is probably the intrinsic error of the finite difference scheme: The matrix method, applied to $N = 336$ and $N = 671$, produces frequency differences of the same order as those shown for the two methods in Figure 1. The frequencies found by the matrix method have been used as first guesses for the Henyey-frequencies shown in Figure 1. In three cases (Figure 1b, modes 13 to 15) the iteration was still unsuccessful. This illustrates our point that the first guesses for the Henyey method must be very close indeed.

The displacement eigenvectors shown in Figures 2a, b are normalized to 1 at the outer boundary. This obscures somewhat the close similarity of the cases *a* and *b* in the deeper part: all the radial nodes lie at the same levels. The small hump at $\log P_0 = 5.4$ is caused by a numerical differentiation of the temperature gradient ∇ which we need in order to evaluate the coefficient D of Equation (11) above; ∇ has a sharp maximum at this level, cf. e.g. Figure 2 of Ando and Osaki (1975). We hope to smooth the hump through the use of analytical derivatives for all coefficients needed in the program.

The convergence of the eigenfrequencies with increasing number, N, of grid points is demonstrated in Figures 3a, b. The frequency splitting, $\Delta\nu$, between two neighbouring modes converges rapidly for the lower modes, and more slowly for the higher ones. In the range arround the 5-min oscillations (modes 20 to 25, say) the difference in $\Delta\nu$ between the cases $N = 537$ and $N = 671$ is less than 0.5 μHz. We made however no effort to adjust our equilibrium model so that the observed value for $\Delta\nu$ around 136 μHz (Grec *et al.*, 1980) would be obtained. Our $\Delta\nu$ values lie consistently higher, as does the asymptotic value $(2\int c^{-1}\,\mathrm{d}r)^{-1}$, represented by the horizontal line at 150 μHz in Figures 3a, b.

First results of a non-radial case, $l = 100$, are shown in Figure 4. The frequencies of the first 13 *p*-modes were computed using both the matrix and the Henyey methods. For comparison, the frequencies computed by Ando and Osaki (1975) agree very well with our results. In the intermediate range (modes 5 to 8), where we obtain slightly different frequencies from the two methods, their results lie almost exactly in the middle between our two values.

We hope to report results for more *l*-values, and also to show eigenvectors for non-radial oscillations in the near future. In particular, we hope that our method permits a convenient access to the eigenfrequencies of *g*-modes as well. To this end, we plan to replace our present equilibrium envelope by a complete stellar model.

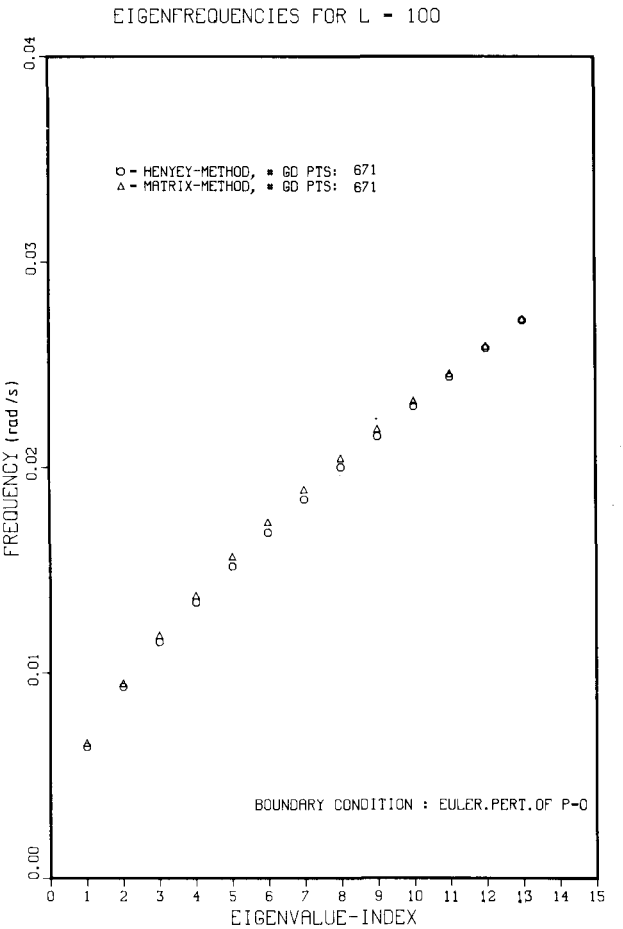

Fig. 4. Eigenfrequencies, ω, of the first 13 non-radial pulsations for $l = 100$, computed with the Henyey and matrix methods, with the condition of zero Eulerian pressure perturbation at the outer boundary.

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