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PRODUCTS AND CARDINAL INVARIANTS OF MINIMAL TOPOLOGICAL GROUPS

BY

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ABSTRACT. It is a question of Arhangel'skiĭ [1] (Problem 2) whether the identity $\psi(G) = \chi(G)$ holds for every minimal Hausdorff topological group $G = \langle G, u \rangle$. (Here, as usual, $\psi(G)$, the pseudocharacter of G, is the least cardinal number κ for which there is Ⓐ $\subset u$ such that $|$Ⓐ$| = \kappa$ and $\cap$Ⓐ $= \{e\}$, and $\chi(G)$, the character of G,is the least cardinality of a local base at e for $\langle G, u \rangle$.) That $\langle G, u \rangle$ is minimal means that, if v is a Hausdorff topological group topology for G and $v \subset u$, then $v = u$.

In this paper, we give some conditions on G sufficient to ensure a positive response to Arhangel'skiĭ's question, and we offer an example which responds negatively to a question on minimal groups posed some years ago (cf. [6] (p. 107) and [4] (p. 259)).

1. **Terminology and Notation**. The smallest infinite cardinal is denoted ω. For an infinite cardinal α, the symbol α^+ denotes the least cardinal β such that $\beta > \alpha$, and $cf(\alpha)$ is the least cardinal κ for which there is a set $\{\alpha_\xi : \xi < \kappa\}$ of cardinals such that $\alpha_\xi < \alpha$ for all $\xi < \kappa$, and $\Sigma_{\xi<\kappa}\alpha_\xi = \alpha$. The cardinal α is called regular if $cf(\alpha) = \alpha$, singular otherwise. For all $\alpha \geqslant \omega$, we say that a topological group $G = \langle G, u \rangle$ is α-totally bounded if, for every non-empty $U \in u$, there is $S \subset G$ with $|S| < \alpha$ such that $G = US$. (Those groups which many authors call totally bounded are, in our terminology, exactly the ω-totally bounded groups.)

2. **On the relation $\psi = \chi$**. Our first lemma is closely related to [2] (1, Proposition 1), [10] (4.5) and [11] (§3, Proposition 1 and §20, Theorem 3). We outline a proof in some detail since (a) none of these sources fits our context exactly, and (b) the distinctions are crucial and unexpectedly subtle; in this latter connection, we note in particular that in 2.1(e) the inclusion Ⓥ $\subset u$ may fail.

LEMMA 2.1. *Let G be a group and* Ⓥ *a family of non-empty subsets of G such that*
(i) *if* Ⓕ $\subset$ Ⓥ *and* $|$Ⓕ$| < \omega$ *then* $\cap$Ⓕ $\in$ Ⓥ*;*
(ii) *for all* $V \in$ Ⓥ, *there is* $U \in$ Ⓥ *such that* $U^2 \subset V$*;*
(iii) *if* $V \in$ Ⓥ, *then* $V = V^{-1}$*; and*
(iv) *for all* $V \in$ Ⓥ *and* $x \in G$, *there is* $U \in$ Ⓥ *such that* $xUx^{-1} \subset V$.

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For $V \in \textcircled{V}$, define $\tilde{V} = \{x \in V$: there is $U \in \textcircled{V}$ such that $xU \subset V\}$, and define $v = \{\bigcup_{i \in I} x_i \tilde{V}_i : x_i \in G,\ V_i \in \textcircled{V}\}$. Then

(a) *each $V \in \textcircled{V}$ satisfies $e \in \tilde{V}$;*

(b) *$\langle G, v \rangle$ is a topological group;*

(c) *$\{\tilde{V} : V \in \textcircled{V}\}$ is a local v − base at e;*

(d) *if $\bigcap \textcircled{V} = \{e\}$, then v is a Hausdorff topology; and*

(e) *if u is a topology for G and $\textcircled{V} \subset u$, then $v \subset u$.*

PROOF: (a) follows from (iii) and (ii). Next we note (#) if $x \in G$ and $V \in \textcircled{V}$ and $e \in x\tilde{V}$, then there is $U \in \textcircled{V}$ such that $x^{-1}U \subset V$ (and hence $x^{-1}\tilde{U} \subset \tilde{V}$, so $e \in \tilde{U} \subset x\tilde{V}$).

From (#), it follows that, if p, $x_i \in G$, and $V_i \in \textcircled{V}$ ($i = 1, 2$) and if $p \in x_1\tilde{V}_1 \cap x_2\tilde{V}_2$, then there are $U_i \in \textcircled{V}$ such that $e \in \tilde{U}_i \subset p^{-1}x_i\tilde{V}_i$. Then, with $U = U_1 \cap U_2 \in \textcircled{V}$, we have

$$p \in p\tilde{U} = p(\tilde{U}_1 \cap \tilde{U}_2) = p\tilde{U}_1 \cap p\tilde{U}_2 \subset x_1\tilde{V}_1 \cap x_2\tilde{V}_2.$$

It follows that v is a topology for G and that $\{x\tilde{V} : V \in \textcircled{V}\}$ is a base for v; indeed, from (#) we have (c). One completes the proof of (b) by showing that the functions $(a, b) \to ab$ and $a \to a^{-1}$ are v-continuous. We omit the detailed argument, since it is available as indicated in [2], [11], [10]. (We note that if U, $V \in \textcircled{V}$ and $U^2 \subset V$, then $(\tilde{U})^2 \subset \tilde{V}$; and if $x \in G$ and U, $V \in \textcircled{V}$ and $xUx^{-1} \subset V$, then $x\tilde{U}x^{-1} \subset \tilde{V}$.)

From (ii) and the inclusion $\tilde{V} \subset V$ for $V \in \textcircled{V}$, (d) follows immediately.

We prove (e). Let $x \in \tilde{V} \subset V \in \textcircled{V} \subset u$ and choose U, $W \in \textcircled{V}$ so that $xU \subset V$ and $W^2 \subset U$. Then $(xW)W \subset V$ and hence $xW \subset \tilde{V}$, so $\tilde{V}$ is u-open.

The centre of a group G is denoted $Z(G)$.

LEMMA 2.2. *Let $\alpha \geq \omega$ and let $\langle G, u \rangle$ be a Hausdorff topological group such that $\psi(G, u) < cf(\alpha)$ and $G/Z(G)$ is α-totally bounded. Then there is a Hausdorff topological group topology v for G such that $v \subset u$ and $\chi(G, v) < \alpha$.*

PROOF: There is $\textcircled{A} \subset u$ such that $|\textcircled{A}| = \psi(G, u)$ and $\bigcap \textcircled{A} = \{e\}$. If $|\textcircled{A}| < \omega$ then $\chi(G, u) = \psi(G, u) = 1$, and it is enough to take $v = u$. We assume then that $|\textcircled{A}| \geq \omega$ and we choose $\textcircled{B} \subset u$ so that $|\textcircled{B}| = |\textcircled{A}|$, $\textcircled{A} \subset \textcircled{B}$, and $\textcircled{B}$ satisfies (the analogues of) properties (i), (ii) and (iii) of Lemma 2.1.

Let $q : G \to G/Z(G)$ be the usual quotient map. Since $G/Z(G)$ is α-totally bounded, and q is open, for each $B \in \textcircled{B}$ there is $S_B \subset G/Z(G)$ such that $|S_B| < \alpha$ and $G/Z(G) = q(B)S_B$. We set $S = \bigcup\{S_B : B \in \textcircled{B}\}$. Then $|S| < \alpha$, and $G/Z(G) = q(B)S$, for all $B \in \textcircled{B}$.

There is $A \subset G$ such that $|A| = |S|$ and $S = \{aZ(G) : a \in A\}$. Denoting by H the subgroup of G generated by A, we have $|H| < \alpha$ and $G = BHZ(G)$ for all $B \in \textcircled{B}$. We set $\textcircled{U} = \{hBh^{-1} : h \in H, B \in \textcircled{B}\}$, and $\textcircled{V} = \{\bigcap \textcircled{F} : \textcircled{F} \subset \textcircled{U}, |\textcircled{F}| < \omega\}$. It is clear that $\textcircled{V} \subset u$, that $|\textcircled{V}| = |\textcircled{U}| < \alpha$, and that $\textcircled{V}$ satisfies (i) of Lemma 2.1; (ii) and (iii) of 2.1 also hold for $\textcircled{V}$, since they hold for $\textcircled{B}$ and hence for $\textcircled{U}$. To verify

(iv) for Ⓥ it is enough to show that if $h \in H, B \in$ Ⓑ and $x \in G$, then there are $k \in H$ and $C \in$ Ⓑ such that $xkCk^{-1}x^{-1} \subset hBh^{-1}$. There is $C \in$ Ⓑ such that $C^3 \subset B$, and since $G = BHZ(G)$ there are $k \in H$, $z \in Z(G)$ such that $h^{-1}x \in Ck^{-1}z$. We then have $xkCk^{-1}x^{-1} \subset (hCk^{-1}z)(kCk^{-1})(z^{-1}kCh^{-1}) = hC^3h^{-1} \subset hBh^{-1}$, as required.

An appeal to Lemma 2.1 now completes the proof: the Hausdorff topology v given there satisfies $v \subset u$, and $\chi(G, v) \leqslant |\{\tilde{V}: V \in$ Ⓥ$\}| \leqslant |$Ⓥ$| < \alpha$.

THEOREM 2.3. *Let $\alpha \geqslant \omega$ and let G be a minimal Hausdorff topological group such that $\psi(G) < cf(\alpha)$ and $G/Z(G)$ is α-totally bounded. Then $\chi(G) < \alpha$.*

COROLLARY 2.4. *Let G be a minimal Hausdorff topological group and set $\alpha = \psi(G)$.*
(a) *If $G/Z(G)$ is α^+-totally bounded, then $\psi(G) = \chi(G)$;*
(b) *if G is Abelian, then $\psi(G) = \chi(G)$.*

PROOF: Statement (a) follows from 2.3 by substituting for α the regular cardinal α^+, and (b) is a consequence of (a).

COROLLARY 2.5. *If G is a minimal Hausdorff topological group and G is ω^+-totally bounded, then $\chi(G) = \psi(G)$.*

PROOF: Set $\alpha = \psi(G)$. If $\alpha < \omega$ the statement is clear, and if $\alpha \geqslant \omega$ then G (and hence $G/Z(G)$) is α^+-totally bounded, so that 2.4(a) applies.

REMARK 2.6. (a) Our results above on the problem of Arhangel'skiĭ were obtained in collaboration at Wesleyan University during the academic year 1979–80 (see [4]) and announced at the Spring Topology Conference at Blacksburg in March, 1981. More recently, we have observed in the literature related results, also not definitive and achieved approximately simultaneously, by Guran [9]. Our Corollary 2.5 appears explicitly in [9], and Theorem 6 of [9] is essentially identical with our Lemma 2.1 except that in [9] the family Ⓥ is required to be a base at e for u, and u is assumed a topological group topology for G. For other related work, see Brown [3] (Remark 1).

(b) The minimality hypothesis in 2.3–2.5 cannot legitimately be omitted, even for totally bounded (that is, ω-totally bounded) groups G. For an example, let α be an infinite cardinal such that $\alpha > \log(\alpha)$ (e.g., $\alpha = 2^\beta$ for some $\beta \geqslant \omega$) and, using the Hewitt–Marczewski–Pondiczery theorem [5] (Theorem 2.3.15), let G be a dense subgroup of the group $\{-1, +1\}^\alpha$ (or of T^α) such that $|G| = \log(\alpha)$. Like every subgroup of a compact group, G is totally bounded, but $\psi(G) \leqslant |G| = \log(\alpha) < \alpha = \chi(G)$.

3. **On Čech-complete groups**. Here we give a further condition, quite independent of minimality, sufficient to yield the equality $\chi(G) = \psi(G)$. A Tychonoff space X is said to be Čech-complete if it is homeomorphic to a dense G_δ in some compact Hausdorff space; a condition easily shown equivalent is that X be a G_δ in βX. We say that X is locally Čech-complete at $x \in X$ if x has an open Čech-complete neighbourhood.

It is well known (see, for example, [10] (8.4)), that every Hausdorff topological group is a completely regular Hausdorff topological space, i.e., a Tychonoff space. In the following proof, we use these two elementary propositions, both valid in the context of Tychonoff spaces (and indeed in wider contexts):

(1) If X is dense in Y and $x \in X$, then $\chi(x,X) = \chi(x,Y)$;

(2) If X is locally compact at the point $x \in X$, then $\psi(x,X) = \chi(x,X)$.

THEOREM 3.1. *Every locally Čech-complete topological group G satisfies $\chi(G) = \psi(G)$.*

PROOF: The statement is clear in case $\psi(G) < \omega$, so we assume $\psi(G) \geqslant \omega$. Choosing an open Čech-complete neighbourhood U of e, we have from (1) and (2) that $\chi(G) = \chi(e,U) = \chi(e,\beta U) = \psi(e,\beta U) = \omega + \psi(e,U) = \psi(e,U) = \psi(G)$.

REMARK 3.2. (a) The argument just given shows that, in effect, if G is locally α-Čech-complete (in the sense that some open neighbourhood U of e in G is the intersection of at most α open subsets of βU), and if $\psi(G) \geqslant \alpha$, then $\chi(G) = \psi(G)$.

(b) An alternative proof of 3.1 in the case $\psi(G) = \omega$ results upon juxtaposing these four sources: [3] (Theorem 1), [5] (Exercise 4.1.I), [12] and [1] (Theorem 3). Indeed, G is paracompact and Čech-complete, hence is metrizable since the diagonal of G is a G_δ in GxG.

4. **On powers of minimal groups**. Here we furnish a negative answer to the following question, raised in [7]: if G is a topological group such that G^n is minimal for every positive integer n, must G^α be minimal for every cardinal number α?

Our construction uses the following special case of a theorem of Stephenson [13]: in order that a dense subgroup G of a compact, Abelian group K be minimal, it is necessary and sufficient that every closed, non-trivial subgroup N of K satisfy $N \cap G \neq \{e\}$.

As in [7] we denote by U the torsion subgroup of the circle group T, we denote by $P = \{2, 3, 5, 7, \ldots\}$ the set of positive primes, and for every integer $r > 1$ we set $G_r = \{x \in U : p \in P \text{ implies } p^r \nmid \text{ord}(x)\}$.

THEOREM 4.1. (a) *The groups G_r^m ($r > 1, 0 \leqslant m < \omega$) are minimal;* (b) *the groups G_r^ω ($r > 1$) are not minimal.*

PROOF: (a) For m fixed, we have the dense inclusions $G_2^m \subset G_r^m \subset U^m \subset T^m$, so by Stephenson's criterion cited above it is enough to show G_2^m is minimal. Since U^m is known to be minimal [6], [7] (and hence has non-trivial intersection with each non-trivial closed subgroup of T^m), it is enough to show that every non-trivial closed subgroup N of U^m satisfies $N \cap G_2^m \neq \{e\}$.

Let $\{(1)\} \neq x = (x_1, x_2, \ldots, x_m) \in N \subset U^m$ and for $1 \leqslant i \leqslant m$ let $\text{ord}(x_i) = \Pi_{j=1}^{n_i} p_{ij}^{e_{ij}}$ with each $p_{ij} \in P$ and $e_{ij} \geqslant 0$. (Since $x \neq (1)$, there are i, j such that $e_{ij} > 0$.) Then, with $f_{ij} = \max\{0, e_{ij} - 1\}$ and $q = \text{lcm}\{\Pi_{j=1}^{n_i} p_{ij}^{f_{ij}} : 1 \leqslant i \leqslant m\}$, we have $(1) \neq x^q \in N \cap G_2^m$, as required.

(b) Fix $r > 1$, choose $p \in P$, and for $1 \leqslant i < \omega$ choose $x_i \in U$ such that

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ord $(x_i) = p^i$. Let N be the closure in T^ω of the cyclic subgroup generated by $x = (x_i: 1 \leqslant i < \omega) \in T^\omega$. To show G_r^ω is not minimal, it is enough to show $N \cap G_r^\omega = \{(1)\}$.

Let $y = (y_i: 1 \leqslant i < \omega) \in N \cap G_r^\omega$ and (using the fact that the metric space T^ω is first countable) choose a sequence $\{n_k: k < \omega\}$ of positive integers such that $x^{n_k} \rightarrow y$. For $1 \leqslant i < \omega$ the set $\{x_i^n: n < \omega\}$ is a finite (discrete) subgroup of T, so for each i the sequence $\{x_i^{n_k}: k < \omega\}$ is eventually constant.

We show $y_i = 1 (1 \leqslant i < \omega)$. Fix such i, define $j = i + r - 1$, and choose $n = n_k$ such that x_i^n and y_i and $x_j^n = y_j$. From $y_j \in G_r$, it follows that

$$p^r \nmid \text{ord}\,(y_j) = \text{ord}\,(x_j^n) = p^j/gcd(n, p^j),$$

and hence $p^i|n$. Then, from ord $(x_i) = p^i$ follows $y_i = 1$, as required.

REMARK 4.2. (a) Let G be a minimal group which is dense in a compact Abelian group K, and let H be a (relatively) closed subgroup of G. From Stephenson's criterion, it follows that H itself is minimal. (For, let $K' = Cl_K H$ and let N be a closed, non-trivial subgroup of K'. We need $N \cap H \neq \{e\}$. We have $H = K' \cap G$ and hence $N \cap H = N \cap (K' \cap G) = N \cap G \neq \{e\}$, as required.)

Now let $\alpha \geqslant \omega$, and continue the notation of 4.1. Since G_r^ω is (topologically isomorphic to) a closed subgroup of G_r^α, we have that G_r^α is not minimal.

(b) A condition strongly analogous to that called "Stephenson's Criterion" in this paper has a long history in functional analysis. In fact, it was L. J. Sulley [15], working in the context of open mapping thorems for Abelian topological groups, who first observed that G_2 satisfies the criterion. Slightly later and without knowledge of Sulley's work [15], Stephenson [13] achieved his own formulation after constructing weak topological group topologies on the real line.

(c) The result announed in [8] is less startling (though perhaps conceptually simpler) than an example obtained by Stojanov [14] subsequently: there is a totally bounded, Abelian group G such that G^ω is not minimal and each G^m $(0 \leq m < \omega)$ is even totally minimal in the sense that each of its (Hausdorff) quotients is minimal.

(d) [Note added December, 1984] We announced the existence of groups G as in Theorem 4.1 above in [4]. It was remarked to us in conversation by V. Eberhardt in September, 1984 in Primorsko, Bulgaria that the fact that the groups G_r^m $(r > 1, 0 \leq m < \omega)$ are minimal is immediate from the fact that the product of a minimal torsion group and a minimal group is itself minimal; this result appears in his work with U. Schwanengel (Rev. Roumaine Math. Pures Appl. 27 (1982), 957–964, Example 2).

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