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A DECOMPOSITION PROPERTY OF WEAK $L_{\hat{1}}$

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Let $(f_i)_{i=1}^n$ be a set of disjointly supported, positive functions in the Banach envelope of weak L_1. We prove that each f_i can be written as e_i+g_i, where e_i and g_i are disjointly supported and satisfy these additional properties: the e_i's are isometrically the l_∞^n basis in the envelope norm; the envelope norm of a linear combination of the g_i's is equal to the envelope norm of the corresponding combination of the f_i's.

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In Lotz–Peck–Talagrand **[2, 3]**, it was proved that the Banach envelope of Weak $L_1(0,1)$ (which we denote by Weak $L_{\hat{1}}$) is universal as a Banach lattice for separable Banach lattices. These two papers also show that a number of non-separable Banach lattices are lattice-isometric to sublattices of Weak $L_{\hat{1}}$.

In this note we show a connection between l_∞^n and the structure of Weak $L_{\hat{1}}$ as a Banach lattice. Before we state the result, recall that the envelope semi-norm on Weak L_1, which we denote by $\|\cdot\|_w$, is given by

$$\|f\|_w = \varlimsup_{\substack{q/p\to\infty \\ 0<p<q}} \left(\int_{p<|f|<q} |f|\right) \Big/ \ln(q/p)$$

for f in Weak L_1; see Cwickel–Fefferman **[1]**. And then Weak $L_{\hat{1}}$ is the completion of (Weak L_1, $\|\cdot\|_w$) modulo the elements x of the completion with $\|x\|_w=0$. Under the usual quotient order, Weak $L_{\hat{1}}$ is a Banach lattice.

Our result now is:

Theorem. *For $1\leqq i\leqq n$, let $f_i>0$ be elements of Weak L_1 such that the f_i have pairwise disjoint supports, and such that $\|f_i\|_w=1$ for each i. Then there exist functions e_i, $0<e_i\leqq f_i$ so that*

(*i*) $\|\sum_{i=1}^n a_ie_i\|_w=\sup|a_i|$ *for all* $(a_i)_{i=1}^n$;

(*ii*) *if we set* $g_i=f_i-e_i$, *then*

$$\left\|\sum_{i=1}^n a_if_i\right\|_w = \left\|\sum_{i=1}^n a_ig_i\right\|_w \quad \textit{for all} \quad (a_i)_{i=1}^n.$$

Proof. First, some notation: for a positive function f in Weak L_1 and $0<p<q$, denote by f_p^q the product of f by the indicator function of the set $\{p<f<q\}$. Also, $\|\cdot\|_w$ is a lattice semi-norm so in calculating the norm of a linear combination of disjointly supported functions, we can assume all the coefficients are non-negative.

Turning now to the proof, whose arguments are very similar to those of [2] and [3], we first show how to achieve condition (i) of the conclusion. We then show how to achieve condition (ii) by strengthening the arguments of the first part of the proof.

To start, choose a positive summable decreasing sequence (ε_m) and positive sequence $k_m \to \infty$ so that

$$q/p > k_m \quad \text{implies} \quad \int (f_i)_p^q / \ln(q/p) < 1+\varepsilon_m \quad \text{for each } i. \tag{1}$$

Now choose, by induction, positive sequences $(p_{i,m})$, $(q_{i,m})$, $1 \leqq i \leqq n$, $m=1,2,\ldots$ satisfying

$$q_{i,m}/p_{i,m} > k_m; \tag{2}$$

$$1-\varepsilon_m < \int (f_i)_{p_{i,m}}^{q_{i,m}} / \ln(q_{i,m}/p_{i,m}); \tag{3}$$

$$0 < p_{i,m} < q_{i,m} < p_{i+1,m} \cdots, \quad 1 \leqq i \leqq n-1; \tag{4}$$

$$q_{n,m} < p_{1,m+1}; \; p_{i,m} \to \infty \quad \text{for each } i;$$

$$\ln(q_{i,m}/p_{i,m})/\ln(p_{i+1,m}/q_{i,m}) < \varepsilon_m, \quad 1 \leqq i \leqq n-1; \tag{5}$$

$$\ln(q_{n,m}/p_{n,m})/\ln(p_{1,m+1}/q_{n,m}) < \varepsilon_m.$$

Note that it is possible to choose $p_{i,m}$ arbitrarily large since $\|f\|_w = 0$ for every bounded f in Weak L_1; (1) and (3) are possible simply because $\|f_i\|_w = 1$ for each i.

Now define

$$e_i = \sum_{m=1}^{\infty} (f_i)_{p_{i,m}}^{q_{i,m}} \leqq f_i.$$

From (1)–(3) it is clear that $\|e_i\|_w = 1$ for each i. To prove that $\|\sum_{i=1}^n a_i e_i\|_w = \sup |a_i|$, it is enough to prove that

$$\left\| \sum_{i=1}^{n} e_i \right\|_w = 1,$$

since $\|\cdot\|_w$ is a lattice semi-norm.

Suppose now that $0<p<q$, with

$$0 < q_{i-1,j} \leqq p \leqq p_{i,j} < q_{i,j} < p_{i+1,j} \cdots < p_{k,m} < q_{k,m} \leqq q < p_{k+1,m}.$$

Then

$$\int\left(\sum_{l=1}^{m} e_l\right)_p^q \Big/ \ln(q/p) \leqq [\int(f_i)_{p_{i,j}}^{q_{i,j}}/\ln(q_{i,j}/p_{i,j})][\ln(q_{i,j}/p_{i,j})\ln(q/p)]\dots$$

$$+[\int(f_k)_{p_{k,m}}^{q_{k,m}}/\ln(q_{k,m}/p_{k,m})][\ln(q_{k,m}/p_{k,m})/\ln(q/p)]$$

$$<1+\varepsilon_j\text{ (from (4))}$$

$$\to 1 \quad \text{as} \quad q/p\to\infty,\ p\to\infty. \tag{6}$$

If, say,

$$p_{i,j}<p<q_{i,j}\leqq q<p_{i+1,j}$$

or

$$q_{i,j}\leqq p\leqq p_{i+1,j}<q<q_{i+1,j},$$

it is easy to see that

$$\int\left(\sum_{l=1}^{n} e_l\right)_p^q \Big/ \ln(q/p)\leqq 1+\varepsilon_j \quad \text{if} \quad q/p>k_j.$$

If

$$p_{i,j}<p<q_{i,j}<p_{i+1,j}\dots<p_{k,m}<q<q_{k,m},$$

then

$$\int\left(\sum_{l=1}^{n} e_l\right)_p^q \Big/ \ln(q/p)\leqq[\int(f_i)_{p_{i,j}}^{q_{i,j}}/\ln(q_{i,j}/p_{i,j})][\ln(q_{i,j}/p_{i,j})/\ln(q/p)]+\dots$$

$$+\int(f_k)_p^q/\ln(q/p) \quad \text{(for the last term)}$$

$$\leqq n\sum_{l=j}^{m}(1+\varepsilon_l)\varepsilon_l+1+\varepsilon_m\text{ (by (5))}$$

$$\to 1 \quad \text{as} \quad j\to\infty,\ q/p\to\infty,$$

using the fact that (ε_l) is summable.

There are two more cases, depending on the positions of p and q relative to $(p_{i,j})$ and

$(q_{i,j})$. Their treatment is similar to that of the cases already discussed, and we leave these cases to the reader.

This shows part (i) of the conclusion.

To achieve part (ii), we have to choose $p_{i,m}, q_{i,m}$ so that the intervals $(p_{i,m}, q_{i,m})$ occur sparsely enough on the line. The way to make this precise is to redo one of the arguments in [3]; we start by letting $(a_{i,m})$ be an indexing of all the n-tuples of positive rational numbers, $m=1,2,\ldots$, with each such n-tuple occurring infinitely often in the indexing.

For each m, there are $0<r<s$ with $s/r>k_m$ and such that

$$\int \left(\sum_{i=1}^{n} a_{i,m} f_i\right)_r^s \Big/ \ln(s/r) \geqq (1-\varepsilon_m) \left\|\sum_{i=1}^{n} a_{i,m} f_i\right\|_w, \tag{7}$$

simply from the definition of $\|\cdot\|_w$. We may choose r as large as we please. Now,

$$\left(\sum_{i=1}^{n} a_{i,m} f_i\right)_r^s = \sum_{i=1}^{n} a_{i,m} (f_i)_{r/a_{i,m}}^{s/a_{i,m}}.$$

So if we have chosen $p_{i,m}, q_{i,m}$ at the m'th step in the construction for part (i), we choose $0<r_m<s_m$ so that

(a) $s_m/r_m > k_m$; (8)

(b) inequality (7) above is satisfied with r and s replaced by r_m and s_m;

(c) $r_m/a_{i,m} > q_{i,m}$.

Now return to the construction for part (i), making sure that $p_{i,m+1}$ is chosen so that $s_m/a_{i,m} < p_{i,m+1}$. Thus the intervals $(p_{i,m}, q_{i,m})$, $(r_m/a_{i,m}, s_m/a_{i,m})$, and $(p_{i,m+1}, q_{i,m+1})$ are pairwise disjoint. This dovetailing construction leaves undisturbed part (i) of the conclusion of the theorem. With $g_i = f_i - e_i$, (8) also yields that

$$\int \left(\sum_{i=1}^{n} a_{i,m} g_i\right)_{r_m}^{s_m} \Big/ \ln(s_m/r_m) \geqq (1-\varepsilon_m) \left\|\sum_{i=1}^{n} a_{i,m} f_i\right\|_w. \tag{9}$$

Every n-tuple of positive rationals occurs infinitely often in the indexing, so (9) implies that if $(a_i)_{i=1}^n$ is any n-tuple of positive rationals, then

$$\left\|\sum_{i=1}^{n} a_i g_i\right\|_w \geqq \left\|\sum_{i=1}^{n} a_i f_i\right\|_w. \tag{10}$$

Finally, by approximating n-tuples of non-negative reals by n-tuples of positive rationals, we see that (10) is true if $(a_i)_{i=1}^n$ is any n-tuple of non-negative real numbers. Since $0 \leqq g_i \leqq f_i$ and the f_i are disjointly supported, the inequality

$$\left\|\sum_{i=1}^{n} a_i g_i\right\|_w \leqq \left\|\sum_{i=1}^{n} a_i f_i\right\|_w$$

is automatic for any n-tuple $(a_i)_{i=1}^n$ of real numbers. The proof of the theorem is complete.

Remark. Let X be an n-dimensional Banach lattice which is spanned by n positive disjoint elements x_i, with $\|x_i\|=1$ for each i.

Let Y be the Banach lattice $X \oplus l_\infty^n$, with norm given by

$$\|(x,y)\|_Y = \sup(\|x\|_X, \|y\|_{l_\infty^n}).$$

Let (e_i), $1 \leqq i \leqq n$, be the usual basis for l_∞^n, and for $1 \leqq i \leqq n$ set $z_i=(x_i,0)+(0,e_i)$, $w_i=(x_i,0)$. It is easy to check that

$$\left\|\sum_{i=1}^{n} \alpha_i z_i\right\|_Y = \left\|\sum_{i=1}^{n} \alpha_i w_i\right\|_Y = \left\|\sum_{i=1}^{n} \alpha_i x_i\right\|_X \tag{11}$$

for all $\alpha_1,\ldots,\alpha_n$. And it is obvious from the construction that

$$\left\|\sum_{i=1}^{n} \alpha_i (z_i - w_i)\right\|_Y = \sup|\alpha_i|. \tag{12}$$

In Y, let W be the span of the w_i and Z be the span of the z_i. From the above, it is clear that both projections $w+z \to w$ and $w+z \to z$ of $Y=W+Z$ on to W and Z have norm at least $\|\sum_{i=1}^n x_i\|_X$.

Our theorem can be used to give a construction of a $2n$-dimensional Banach lattice Y' with the above properties, but which is isometrically different from the Y above, in general. The construction goes as follows: let X be the given Banach lattice spanned by the positive disjoint elements x_i, $1 \leqq i \leqq n$, $\|x_i\|=1$ for each i.

By [2], [3], we may regard X as a sublattice of (weak L_1, $\| \|_w$), with the x_i corresponding to non-negative, norm-one functions f_i with pairwise disjoint supports. On applying the theorem, we have $f_i=e_i+g_i$, with the conclusions of the theorem. Set $Y'=\operatorname{span}\{e_i,f_j\} 1 \leqq i,j \leqq n$, and define $|||\cdot|||$ on Y' by

$$|||\sum(\alpha_i f_i + \beta_i g_i)||| = \left\|\sum_{i=1}^{n} (\alpha_i f_i - \beta_i g_i)\right\|_w.$$

Then with f_i corresponding to z_i and g_i corresponding to w_i, properties (11) and (12) are immediate for (f_i), (g_i).

Suppose now that X is l_1^n. The canonical way of embedding X into (weak L_1, $\| \|_w$) is to let each basis vector x_i correspond to a function f_i so that for each i, the decreasing rearrangement of f_i at t is $1/t$, for sufficiently small t. If $n \geqq 2$, then for m sufficiently large,

$$\begin{aligned}\int (e_1+g_2)_{p_{1,m}}^{q_{1,m}} &= \int (e_1+f_2-e_2)_{p_{1,m}}^{q_{1,m}} \\ &= \int (e_1+f_2)_{p_{1,m}}^{q_{1,m}} \\ &= 2\ln(q_{1,m}/p_{1,m}), \end{aligned} \tag{13}$$

since $e_2 \chi_{p_{1,m}} < e_2 < q_{1,m} = 0$. Thus $\|e_1+g_2\|_w = 2$, showing that there is no isometry of Y' onto Y which takes f_i to z_i and g_i to w_i.

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