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Exact solutions for the initial stage of dam-break flow on a plane hillside or beach

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Inviscid, incompressible liquid is released from rest by a sudden dam break, accelerating under gravity over a uniformly sloping impermeable plane bed. The liquid flows downhill or up a beach. A linearised model is derived from Euler's equations for the early stage of motion, of duration $2\sqrt{H/g}$, where H is the depth scale and g is the acceleration due to gravity. Initial pressure and acceleration fields are calculated in closed form, first for an isosceles right-angled triangle on a slope of 45°. Second, the triangle belongs to a class of finite-domain solutions with a curved front face. Third, an unbounded domain is treated, with a curved face resembling a steep-fronted breaking water wave flowing up a beach. The fluid goes uphill due to a nearshore pressure gradient. In all cases the free-surface-bed contact point is the most accelerated particle, exceeding the acceleration due to gravity. Physical consequences are discussed, and the pressure approximation of shallow water theory is found poor during this early stage, near the steep free surface exposed by a dam break.

Key words: gravity currents, surface gravity waves, wave-structure interactions

1. Introduction

We model a two-dimensional dam-break flow for a region D of incompressible liquid. We assume the fluid in D has been at rest until, at time $t = 0$, it is released by an instantaneous removal of a dam. The fluid is on a plane sloping impermeable bed and flows due to a uniform gravitational field. In this work, D is either a triangle, or a related finite region on a hillside, or D extends far offshore on a sloping beach, in which case the released fluid flows uphill.

In large-scale dam breaks and in breaking-wave settings the depth scale H is at least 10 m, the flow speed increases to no more than $10\,\mathrm{m\,s^{-1}}$, and so for water the Reynolds

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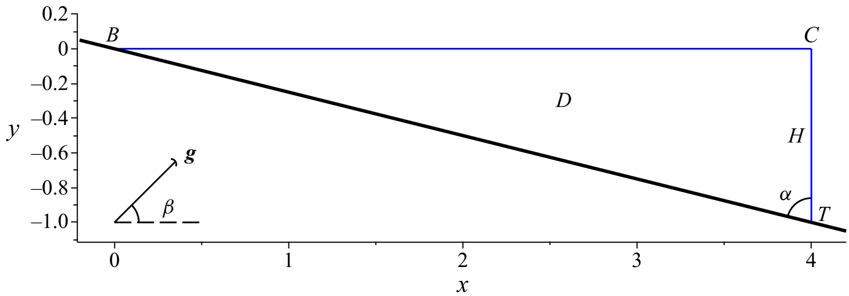

Figure 1. Fluid domain D, on a black sloping bed. Contact angle α at toe point T. Backwater ends at B. Blue lines: free surface BC, CT. Gravity $\boldsymbol{g}$ has angle β (drawn for $\beta = \pi/4$).

number is at least 10^8. Therefore, the fluid is assumed inviscid. Since the fluid starts with zero velocity and hence zero vorticity, the velocity persists as irrotational flow. See Batchelor (1967). In § 4 we use hindsight to show that the influence of viscosity is small and limited to the bed.

The only external force is a uniform gravitational field, $\boldsymbol{g}$, making D deform and accelerate from rest. The only internal force is the gradient of fluid pressure. The pressure field suddenly changes at $t = 0$, from hydrostatic to a new state which accommodates zero pressure on that part of the boundary of D where the removal of a dam exposes a free surface. We calculate the initial pressure, $p(x, y)$ and hence the vector field of fluid acceleration, $\boldsymbol{A}$. Physically, the pressure gradient along the bed is worth calculating because, if large enough, it can start bed-sediment motion.

At $t = 0$ the Eulerian and Lagrangian accelerations coincide. For small times, $O(\sqrt{H/g})$, in a depth H, the fluid velocity $\boldsymbol{u}$ equals $t\boldsymbol{A}$ and the fluid-particle displacement is $\frac{1}{2}t^2\boldsymbol{A}$. Fluid particles with greatest acceleration are where fluid jets emerge. Small-time analysis is a check on computational methods which struggle with the earliest stage of dam break; initial errors can propagate downstream. For instance, the assumptions of shallow water theory can give poor approximation of the pressure field, because at the start of dam break the vertical and horizontal velocity components are similar and the vertical acceleration may dominate the horizontal: both approximately $g = |\boldsymbol{g}|$.

Dam breaks can occur on steep or shallow slopes. Figure 1 shows a vertical cross-section. The domain D is a right-angled triangle BCT. The straight line BC is a free surface, and CT (curved in some later results) is a surface exposed to the atmosphere by the removal of a dam. At toe point, T, fluid includes a contact angle, α, measured between the exposed face and the bed. We will show that the value of α strongly influences the acceleration of the fluid particle at T. For $\alpha : 0 < \alpha < \frac{1}{2}\pi$ we find the acceleration at T in terms of $\boldsymbol{g}$ and α.

Another geometry in which this type of flow is interesting to analyse follows the sudden removal of a barrier, such as a breached seawall, on a plane sloping beach. The nearshore water can move up the beach in the form of a low-amplitude surge. In our examples the offshore (far-field) pressure stays hydrostatic, with the water surface undisturbed. But nearshore, the fluid is accelerated up the beach due to the gradient of a local pressure field.

Stoker (1958) and Whitham (1999) model dam-break flow using shallow water theory for a horizontal bed, with examples for a downstream bed which is either initially dry or has a fixed water depth. Stoker (1958) uses Lagrange particle coordinates to model the early stages of collapse under gravity of a semi-infinite rectangle of fluid onto a dry horizontal bed. He demonstrates a simultaneous fluid motion everywhere and a singularity in velocity at the bed toe. Korobkin & Yilmaz (2009) analysed asymptotically this singularity when $\alpha = \frac{1}{2}\pi$.

The reviews of Simpson (1982) and Huppert (2006) include dam breaks in natural and industrial settings. Recent analyses of dam-break flows use nonlinear shallow water theory, recast via the method of characteristics. Notable papers which model uniform plane sloping beds include Hogg (2006). Applications of his approach have been to right-angled triangular domains. See Fernandez-Feria (2006) and Ancey *et al.* (2008).

Shallow water theory assumes a negligible vertical particle acceleration compared with g, so that the pressure at a field point Q is approximated by the instantaneous head of water from Q up to the free surface. The present paper includes the vertical component of fluid-particle acceleration. Also, in shallow water theory, a restricted dependence of the downslope (s-coordinate) component of velocity to depend only on s and time t, leads to a maximum signal speed of $\sqrt{gH}$, in water of depth H. But potential theory, for incompressible fluid, allows information to be transmitted instantaneously throughout D.

Another motivation for this study is the importance of bed-pressure distributions in modelling bed-sediment movement. The presence of erosion debris, carried downstream by a dam break, is a potentially deadly and destructive hazard. Despite the importance in modelling bed-sediment transport, few studies report pressure fields.

In § 2 we establish from Euler's equations, a linear mixed boundary-value problem for the pressure field. From the pressure, the fluid acceleration is calculated. Section 3 contains the results. In § 3.1 a right-angled triangle flows downhill. In § 3.2 we treat a 45° triangle. Then this is shown to be one of a class of finite-domain flows presented in § 3.3. Another class of unbounded regions of water, some like breaking waves, flow up a beach in § 3.4. In § 4 the analytical results allow us to use hindsight to discuss viscosity, Froude number and assumptions in shallow water theory. Section 4 ends with the conclusions.

2. Modelling assumptions and theory

Inviscid incompressible fluid starts to move from rest at time $t = 0$. The constant fluid density is ρ and H is typical depth. The flow starts irrotational and, from Kelvin's theorem, it stays irrotational as long as domain D is simply connected. Euler's equations are

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\frac{1}{\rho}\nabla p + \boldsymbol{g}, \tag{2.1}$$

where p is initial pressure and $\boldsymbol{g}$ is gravity. The velocity $\boldsymbol{u}$ changes in time on a scale gt. Therefore, compared with $\partial \boldsymbol{u}/\partial t$, the nonlinear convective term $(\boldsymbol{u} \cdot \nabla)\boldsymbol{u}$ is smaller by a factor $O(t^2 g/H)$. It is therefore negligible for small enough times. So we may linearise (2.1) by keeping the first term on the left with which to approximate the acceleration. Hence

$$\frac{\partial \boldsymbol{u}}{\partial t} = -\frac{1}{\rho}\nabla p + \boldsymbol{g}. \tag{2.2}$$

For incompressible fluid $\nabla \cdot \boldsymbol{u} = 0$ and, as gravity is a potential force, $\nabla \cdot \boldsymbol{g} = 0$. Taking the divergence of (2.2) shows that p obeys Laplace's equation. For a two-dimensional flow,

$$p_{xx} + p_{yy} = 0, \quad \text{in } D, \tag{2.3}$$

where subscripts denote partial derivatives and D lies in a vertical (x, y) plane.

For the triangle in figure 1, $x = a = 4$ is the position of a removed dam of height H, and $y = 0$ is a free surface. Since gravity points in any direction, the x and y axes are not necessarily horizontal and vertical. To describe D in any orientation, gravity $\boldsymbol{g}$ has angle β measured from the positive x-axis:

$$\boldsymbol{g} = g(\boldsymbol{i}\cos\beta + \boldsymbol{j}\sin\beta), \tag{2.4}$$

where $g = 9.8\ \mathrm{m\,s^{-2}}$ and $\boldsymbol{i}, \boldsymbol{j}$ are unit vectors in the positive x and y directions, respectively. For $\boldsymbol{g}$ to point vertically down the y-axis, $\beta = -\frac{1}{2}\pi$. Angles α and β are independent.

Now for boundary conditions on p. The Bond number has a large value, so we neglect surface tension. The inertia of the adjacent air is small compared with that of the water, so

$$p = 0 \quad \text{on a free surface.} \tag{2.5}$$

Let L be the bed, with unit normal $\boldsymbol{n}$ directed into D. For brevity, we write acceleration as $\partial\boldsymbol{u}/\partial t = \boldsymbol{A}$. On L, acceleration is tangent to L, so its normal component is zero. So the dot-product of $\boldsymbol{n}$ with (2.2) leads to a bed-boundary condition on the normal derivative of p:

$$\boldsymbol{n}\cdot\nabla p \equiv \frac{\partial p}{\partial n} = \rho\boldsymbol{n}\cdot\boldsymbol{g}. \tag{2.6}$$

Our theory is a linear mixed boundary-value problem (2.3)–(2.6) whose solution is unique. After finding p we evaluate acceleration from (2.2).

In § 3 we investigate several domains. In §§ 3.1 and 3.2 the forward free surface, CT, is a line segment. In §§ 3.3 and 3.4, CT is a curve found from the solution.

3. Results

3.1. *Downhill dam-break flows*

In figure 1 the triangular cross-section BCT of a prism of liquid is shown. The back point of the tailwater is B, the junction of the free-surface segments is C and the toe is T. At $x = a$, vertical segment CT is the face exposed by a dam break. On the sloping bed we have angle α to describe the line L of the bed $y = -x\cot\alpha$, and L has unit normal

$$\boldsymbol{n} = \boldsymbol{i}\cos\alpha + \boldsymbol{j}\sin\alpha. \tag{3.1}$$

So, in components, with subscripts as partial derivatives, boundary condition (2.6) is

$$p_x\cos\alpha + p_y\sin\alpha = \rho g\cos(\alpha - \beta). \tag{3.2}$$

At T a consequence of condition $p = 0$ on $x = a$ is $p_y = 0$. Hence (3.2) is

$$p_x = \rho g\frac{\cos(\alpha-\beta)}{\cos\alpha}. \tag{3.3}$$

The downhill component of the pressure gradient is $p_x\sin\alpha$. From (2.2) and (3.3), the downhill component of acceleration at T is

$$A_T = -g\frac{\sin\beta}{\cos\alpha}. \tag{3.4}$$

This expression agrees with the t-derivative of (19) of Fernandez-Feria (2006).

If $\sin\beta \neq 0$ expression (3.4) tends to infinity as α increases to $\frac{1}{2}\pi$.

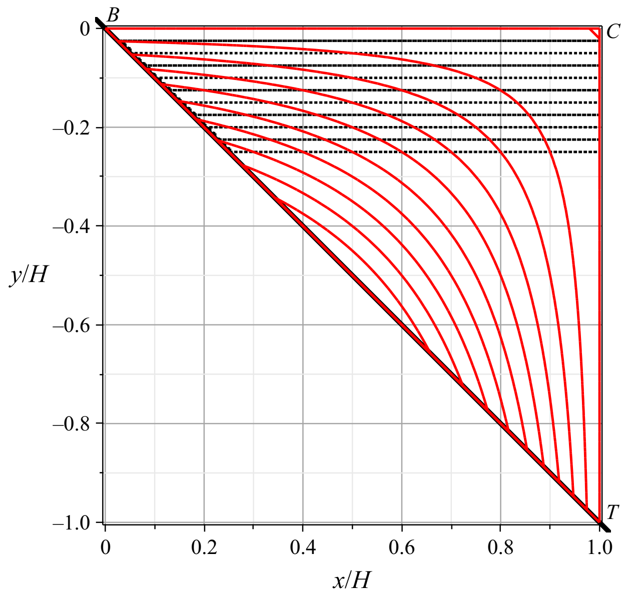

Figure 2. Pressure field for $\alpha = \frac{1}{4}\pi$ and $\beta = -\frac{1}{2}\pi$ at $t = 0$. Red contour values $p/(\rho g H) = 0$, [0.025], 0.225 (lowest, [increment], highest); global maximum is 0.25 at (0.5, −0.5). Black dotted horizontal lines are shallow water theory (hydrostatic pressure) contours for the same set of pressure values.

3.2. *The case $\alpha = \frac{1}{4}\pi$; an isosceles triangle*

We now present the exact solution for $\alpha = \frac{1}{4}\pi$, for which $a = H$. We report and discuss the pressure and acceleration throughout the triangle. A solution of (2.3) which satisfies the free-surface conditions is $p = C(x - H)y$, where C is a constant determined by the bed condition (3.2). Hence

$$p(x, y) = \sqrt{2}\cos\left(\frac{1}{4}\pi - \beta\right)\frac{\rho g}{H}(H - x)y \quad \text{for } 0 \le x \le H \text{ and } -x \le y \le 0. \tag{3.5}$$

The corresponding acceleration in D is

$$\begin{aligned}\boldsymbol{A}(x, y) = g\Bigg(&\boldsymbol{i}\left[\cos\beta + \frac{y}{H}\sqrt{2}\cos\left(\frac{1}{4}\pi - \beta\right)\right] \\ &+ \boldsymbol{j}\left[\sin\beta + \left(\frac{x}{H} - 1\right)\sqrt{2}\cos\left(\frac{1}{4}\pi - \beta\right)\right]\Bigg).\end{aligned} \tag{3.6}$$

If $\beta = -\pi/2$, then (3.5) and (3.6) give the pressure and acceleration fields as simply

$$p(x, y) = -\frac{\rho g}{H}y(H - x) \quad \text{and} \quad \boldsymbol{A}(x, y) = -g\left(\boldsymbol{i}\frac{y}{H} + \boldsymbol{j}\frac{x}{H}\right). \tag{3.7a,b}$$

From (3.7*a*,*b*) pressure contours are red in figure 2. Dotted horizontal lines show the (hydrostatic) pressure distribution from shallow water theory, for the same set of contour values. The difference between the two theories is discussed in § 4.

Since the velocity $\boldsymbol{u} = t\boldsymbol{A}$, we show the acceleration as instantaneous streamlines in figure 3. Also shown is the displacement of the free-surface segments BC and CT. They remain straight, and turn in opposite senses as they descend. At B : (0, 0), the acceleration

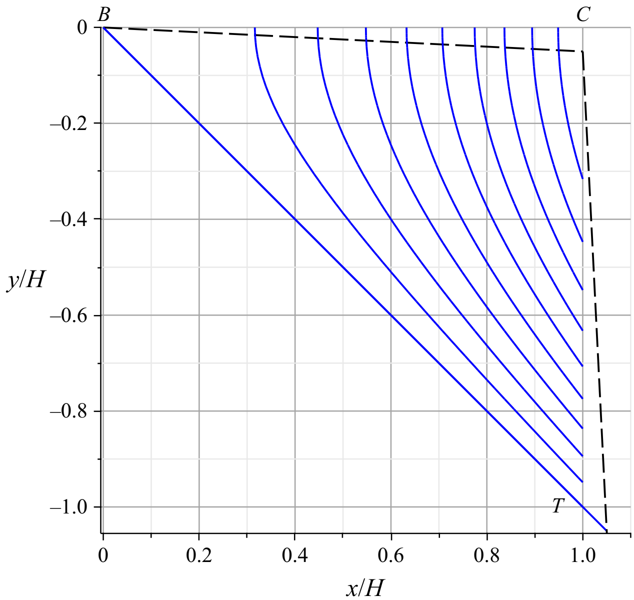

Figure 3. Blue streamlines in a dam-break flow at $t = 0$ for $\alpha = \frac{1}{4}\pi$ and $\beta = -\frac{1}{2}\pi$, including the bed streamline. Stream function values plotted are $\psi/[g^{1/2}H^{3/2}] = 0, [-0.1], -1$. Dashed lines: free-surface position at small time $t : 0 < t\sqrt{g/H} << 1$.

is $\boldsymbol{A}_B = \boldsymbol{0}$, so the fluid particle at the back of the tailwater does not move. At $C : (H, 0)$, the fluid is in free fall: $\boldsymbol{A}_C = -g\boldsymbol{j}$. And at $T : (H, -H)$, the acceleration is maximal:

$$\boldsymbol{A}_T = \sqrt{2}g\left(\frac{\boldsymbol{i}-\boldsymbol{j}}{\sqrt{2}}\right), \tag{3.8}$$

where the brackets contain a unit vector pointing downhill. Here T has twice the acceleration of a frictionless point mass, which descends with acceleration $g/\sqrt{2}$. The magnitude of (3.8) agrees with (3.4) for $\alpha = \frac{1}{4}\pi$ and $\beta = -\frac{1}{2}\pi$. On the bed L: $y = -x$. Hence the pressure distribution, $p = p_b(x)$, along the bed is

$$p_b(x) = \rho g\left(x - \frac{x^2}{H}\right) \quad \text{for } 0 \le x \le H. \tag{3.9}$$

This bed pressure has a maximum, $p_{bm} = \frac{1}{4}\rho g H$, half-way down the wetted slope at $x = \frac{1}{2}H$. Now p_{bm} is one-half the hydrostatic pressure, illustrating the dramatic pressure drop everywhere in D at $t = 0$. The pressure gradient along the bed has two extreme values: first at B directed uphill and keeping B still; second at T directed downhill and doubling the free-fall downslope acceleration, to give the total in (3.8).

Staying with the isosceles triangle, we summarise results for two other values of orientation angle. If $\beta = -3\pi/4$, then D is an isosceles triangle lying on a horizontal plate BT. The triangle collapses symmetrically onto the plate as a thinning layer. The apex at C is in free fall, and contact points B and T separate horizontally with acceleration g. The exactly reversed flow occurs if $\beta = \frac{1}{4}\pi$ but in practice the fluid falls off the horizontal plate.

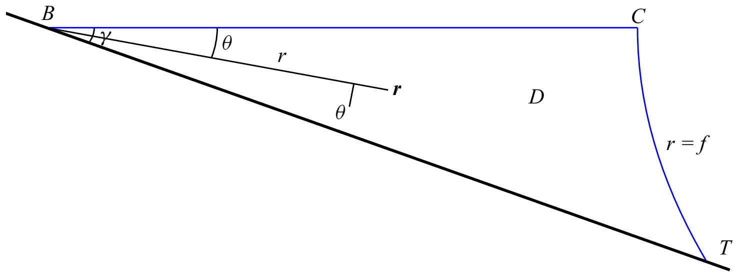

Figure 4. Sketch of fluid domain on a beach (black line); polar coordinates r, θ centred at origin B, with unit vectors' directions indicated. Gravity $\boldsymbol{g}$ is vertically down. Free-surface sections are BC along the x-axis, and CT at the forward face. The shape, $r = f(\theta)$, of CT is found as part of the solution.

3.3. *Dam-break flows on a beach: downhill flow*

Throughout this section, gravity acts along the negative y-axis ($\beta = -\frac{1}{2}\pi$). See figure 4. Part of the x-axis coincides with the upper horizontal free surface of D. The rest of D lies inside a wedge $0 \le \theta \le \gamma$, with respect to r, θ, plane polar coordinates centred on point B. Here θ increases clockwise from zero on the horizontal positive x-axis, up to $\theta = \gamma$ at the beach. The beach angle γ is such that $0 < \gamma \le \frac{1}{2}\pi$.

The arc CT is $r = f(\theta)$, a barrier released at $t = 0$ allowing the fluid to move. Since CT becomes a free surface, p obeys (2.5), and CT is consequently found in terms of $f(\theta)$ and a chosen length scale $|BC| = a$. First, we consider finite D such that $0 \le r \le f(\theta)$. Consider the following two-term solution of Laplace's equation (2.3):

$$p(r, \theta) = \rho g(r \sin\theta - K r^q \sin(q\theta)), \quad \text{where } 0 \le r \le f(\theta), \ \ 0 \le \theta \le \gamma \tag{3.10}$$

where K is a constant and $q = \pi/(2\gamma)$ is the only physically relevant eigenvalue that lets p satisfy bed condition (2.6). Both terms satisfy $p = 0$ on $\theta = 0$. The first term of (3.10) accommodates the inhomogeneous part of condition (2.6). Arc CT is found by satisfying (2.5). Point C lies on $\theta = 0$ at $r = a = H \cot\gamma > 0$. Hence, constant $K = q^{-1} a^{1-q}$ and so

$$f(\theta) = a \left(\frac{q \sin\theta}{\sin(q\theta)} \right)^{1/(q-1)} \quad \text{for } 0 \le \theta \le \gamma. \tag{3.11}$$

Now $f(\theta)$ is a monotone increasing function, with zero derivative at $\theta = 0$. Therefore at C the sections of free surface are orthogonal, and at T the contact angle α is acute. Examples of arcs CT, on their hill slopes, are drawn in figure 5(*a*); they are all nearly vertical.

The corresponding pressure distribution (3.10) is

$$p(r, \theta) = \rho g a \left(\frac{r}{a} \sin\theta - \left(\frac{r}{a}\right)^q \frac{\sin(q\theta)}{q} \right), \quad 0 \le r \le f(\theta), \ \ 0 \le \theta \le \gamma. \tag{3.12}$$

From (2.2), in terms of unit vectors $\hat{\boldsymbol{r}}$ radial, and $\hat{\boldsymbol{\theta}}$ clockwise transverse, the acceleration is

$$\boldsymbol{A}(r, \theta) = g \left(\frac{r}{a}\right)^{q-1} \left(\hat{\boldsymbol{r}} \sin(q\theta) + \hat{\boldsymbol{\theta}} \cos(q\theta) \right). \tag{3.13}$$

If $\gamma = \frac{1}{4}\pi$, then $q = 2$. Then (3.12) is identical to (3.6) for the triangle in § 3.2. If $\gamma > \frac{1}{4}\pi$ then CT has an overhang, similar to the recurve of a dam. See figure 5(*a*) and $\gamma = 60°$.

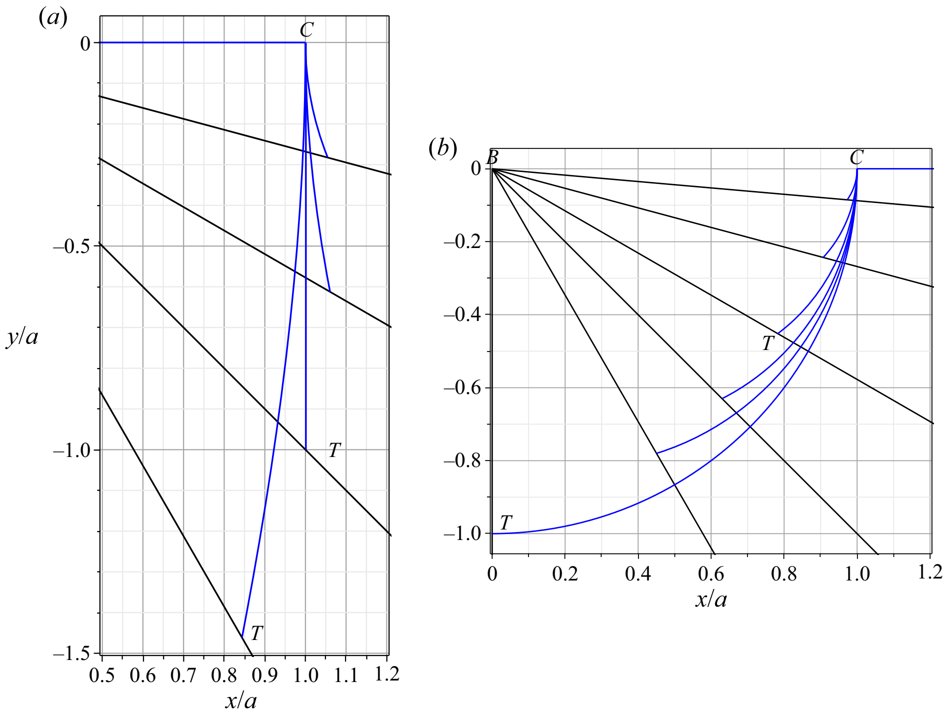

Figure 5. Blue free-surface positions; black beds. (*a*) As figure 4, finite domain to the left of arc CT : $\gamma = 15^\circ, 30^\circ, 45^\circ, 60^\circ$. (*b*) Infinite domains right of CT : $\gamma = 5^\circ, 15^\circ, 30^\circ, 45^\circ, 60^\circ, 90^\circ$ (last is circular arc).

If $\gamma = \pi/6$, then $q = 3$ and (3.12) is $p = \rho g H^{-2} y(x^2 - \frac{1}{3}y^2 - H^2)$. And CT is a hyperbola meeting the bed with $\alpha = 49^\circ$ – less than the 60° between a vertical line and this bed.

As $\gamma \to 0$, with $H = a \tan\gamma$ fixed, CT remains curved and $\alpha = \arctan(\pi/2) = 57.5^\circ$.

3.4. *Barrier-break flows on a beach: unbounded wave and uphill flow*

We next consider region $r \geq F(\theta)$ in the wedge $0 \leq \theta \leq \gamma$, where $r = F(\theta)$ is a new shape for CT. Now D has a far-field hydrostatic pressure, $p_h = \rho g r \sin\theta$, which is unchanged by the initial dam break along CT. We want p to tend to p_h as $r \to \infty$, so we add a second term which vanishes in the far field and accommodates the exposed face. An expression which does all this and obeys conditions (2.3), (2.5), (2.6) is

$$p(r,\theta) = \rho g a \left(\frac{r}{a}\sin\theta - \left(\frac{a}{r}\right)^q \frac{\sin(q\theta)}{q} \right), \quad r \geq F(\theta), \ \ 0 \leq \theta \leq \gamma, \tag{3.14}$$

where $q = \pi/(2\gamma)$ and $a > 0$ is distance $|OC|$. The pressure is zero on arc CT described by

$$r = F(\theta) = a \left(\frac{\sin(q\theta)}{q \sin\theta} \right)^{1/(q+1)} \quad \text{for } 0 \leq \theta \leq \gamma. \tag{3.15}$$

Examples of arcs CT and their corresponding beach slopes are drawn in figure 5(*b*).

Point T has position $r = a(q\sin\gamma)^{-1/(q+1)}$ on $\theta = \gamma$. Now $0 < \gamma < \frac{1}{2}\pi$, so $q > 1$. Also, $F(\theta)$ is monotone decreasing and has zero derivative at $\theta = 0$. Hence, the free surface has a right-angle at C and at T the contact angle α is acute. See figure 5(*b*).

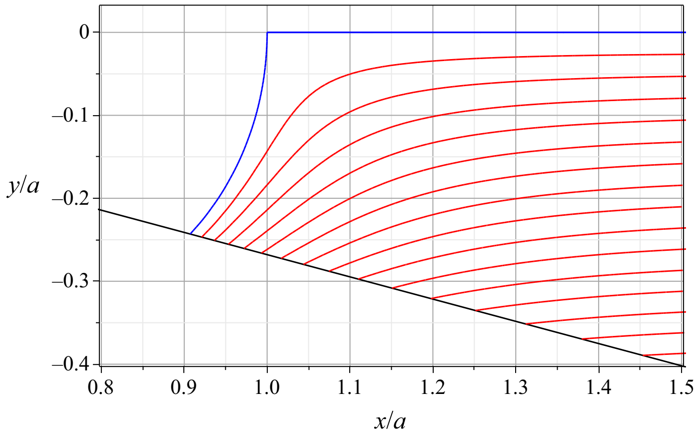

Figure 6. Pressure contours for beach angle $\gamma = 15^{\circ}$. Blue contour: free surface $p = 0$. Contours: $p/(\rho g a) = 0$, [0.025], 0.375; maximum at lower right. Hydrostatic pressure is in the far field as $x \to \infty$.

As $\gamma \to 0$, with $H = a \tan \gamma$ fixed, we find $\alpha = \arctan(\pi/2) = 57.5^{\circ}$. So, in this limit, D is not a semi-infinite rectangular strip. At the other extreme, $\gamma = \frac{1}{2}\pi$, we have CT a quarter-circle, radius a, centre O, with $\alpha = \frac{1}{2}\pi$. See figure 5(*b*).

The pressure distribution (3.14) is plotted in figure 6, for beach angle $\gamma = \pi/12$ (or 15°).

The acceleration found from (3.14) is written in plane polar unit vectors: radial $\hat{\boldsymbol{r}}$ and clockwise transverse $\hat{\boldsymbol{\theta}}$, as follows

$$\boldsymbol{A}(r, \theta) = g \left(\frac{a}{r}\right)^{q+1} \left(-\hat{\boldsymbol{r}} \sin(q\theta) + \hat{\boldsymbol{\theta}} \cos(q\theta)\right). \tag{3.16}$$

For gently sloping natural beaches, $q >> 1$, and then for $r > 2a$, acceleration $|\boldsymbol{A}| << g$.

The acceleration field in figure 7 is for $\gamma = \pi/12$ radians (15°). Particle C is in free fall. On the bed, $\boldsymbol{A}$ is parallel to the bed and it achieves its greatest magnitude at T:

$$\boldsymbol{A}_T = -\hat{\boldsymbol{r}} \frac{\pi \sin \gamma}{2\gamma} g. \tag{3.17}$$

As expected, $\boldsymbol{A}_T$ is directed up the beach. As $\gamma \to 0$, the magnitude of $\boldsymbol{A}_T$ increases to a maximum of $\frac{1}{2}\pi g$, with $\alpha = \arctan(\pi/2) = 57.5^{\circ}$, noted above.

4. Discussion and conclusions

On release of a region of fluid from rest under gravity, the initial acceleration is directly proportional to g. It also depends on gravity's direction, β, and the size of the contact angle, α. The point T has an acceleration which increases with α up to a singularity at $\alpha = \frac{1}{2}\pi$ (except the vertical wall noted in § 3.4). Fluid near T forms a high-acceleration jet.

The Froude number, Fr, is a way to characterise the flow and obtain a time limit on the theory. We define $Fr = \bar{u}/\sqrt{gh}$, where $\bar{u}$ is the mean of the horizontal velocity $u = tA_x$, averaged over local depth h. From results in § 3.2 with $\beta = -\frac{1}{2}\pi$, and (3.7*a,b*), we have $u(x, y) = -tgy/H$ over the water column $y \in [-x, 0]$. The depth-averaged mean is $\bar{u}(x) = \frac{1}{2}tgx/H$. Hence $Fr(x) = \frac{1}{2}t\sqrt{gx}/H$. We now see that Fr is zero at B, at $x = 0$ in figure 3. Also, $Fr(x)$ increases with x to a maximum, $Fr_m = \frac{1}{2}t\sqrt{g/H}$, at $x = H$. Now Fr_m

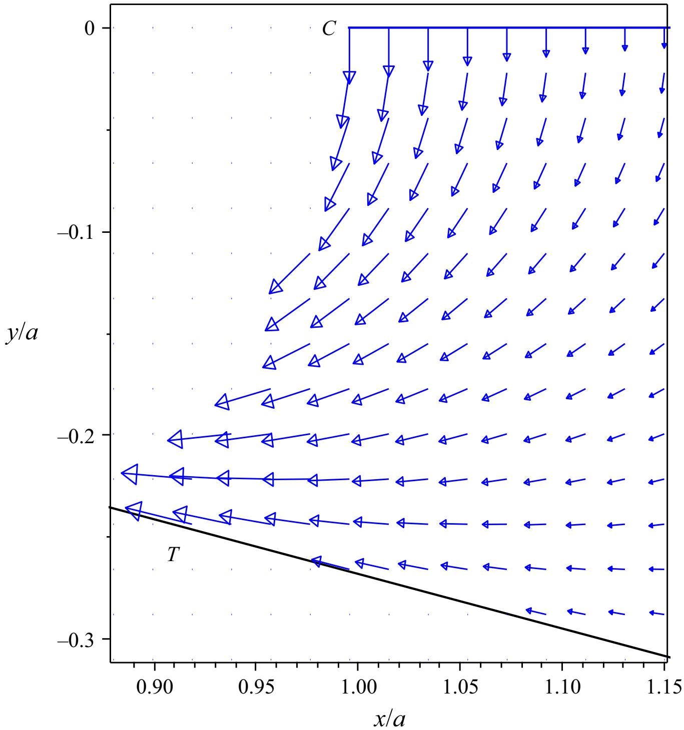

Figure 7. As figure 6. Acceleration field near the front face CT. Blue horizontal line is the free surface falling for all $x/a > 1$. Point C is in free fall, g. Maximum $|\boldsymbol{A}|$ is 1.55 g up the beach at T.

remains less than unity while $t < 2\sqrt{H/g}$. These considerations set a time limit of $t = T_m = 2\sqrt{H/g}$ on the theory. At laboratory scale $H = 1$ m, and $T_m \approx 0.7$ s. At full-scale, $H = 10$ m and $T_m \approx 2$ s. Results in § 3 show that $|\boldsymbol{A}|/g = O(1)$, so that when $t = T_m$ the flow has the right magnitude of velocity, $|\boldsymbol{u}| = T_m|\boldsymbol{A}|$, at the end of the initial stage of dam break, for shallow water theory to take over at the next stage, when $|\boldsymbol{u}/\sqrt{gH}| = O(1)$. After T_m, particle displacements may be $O(H)$, free-surface self-intersection may start and discontinuities in free-surface elevation or slope may appear.

We next show boundary layers are thin and limited to the bed up to $t = T_m$. A half-space of fluid of kinematic viscosity ν, is accelerated from rest by a uniform force parallel to a fixed flat plate. Drazin & Riley (2006) show that after time t the plate's boundary layer has thickness $\delta = 0.6\sqrt{\nu t}$. When $t = T_m$, we have $\delta \approx 0.6\nu^{1/2}(H/g)^{1/4}$. For water and $H = 1$ m, we find it is thin: $\delta \approx 1$ mm. There is too little time for the boundary layer to form, separate and advect its vorticity into D. Even if all of time T_m is spent on advection, the boundary-layer displacement is a fraction of $\frac{1}{2}gT_m^2$ – much shorter than H.

The exact results of § 3.2 let us assess assumptions in approximate theories. Shallow water theory assumes the vertical component of fluid–particle acceleration is much less than g. Hence, shallow water theory approximates pressure as $p \approx \rho g d$ at all points at depth d below a free surface. So shallow-water-theory pressure contours are the shape of free surface but displaced vertically beneath it. Figure 2 (and 6) show the very different results from shallow water theory, especially near the steep free surface. The above investigation of Fr lets us find the ratio R of the vertical acceleration at the upper free surface compared with g, that is $R = |\boldsymbol{A} \cdot \boldsymbol{j}|/g = x/H$. Towards B, R is small so shallow water theory is consistent; near the front face $R \approx 1$ so shallow water theory is inadequate.

In all cases, contact point T has acceleration at least g. In a frame of reference moving with C, the shape of D in § 3.4, and the flow depicted in figures 6 and 7, resembles a steep-fronted wave before it collapses and ascends the beach. All this underlines the violence of dam-break flow to hikers caught out on a hillside, and to pedestrians overcome by tsunami.

The conclusions are as follows. A linear theory of initial acceleration, in the presence of gravity, was used to model dam-break-type flows, both downhill and uphill. Expressions in elementary functions are found for the fields of pressure and acceleration, $\boldsymbol{A}$. The theory includes the vertical component of acceleration, neglected in shallow water theory. Our model predicts fluid–particle displacements of $\frac{1}{2}t^2\boldsymbol{A}$, valid up to time $2\sqrt{H/g}$ when shallow water theory can take over. Emerging jets along the bed occur, led by the contact point. One implication of our results is the influence on sediment of forces due to the sudden pressure gradient associated with abruptly excited water.

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