Bull. Aust. Math. Soc. **98** (2018), 177–187
doi:10.1017/S0004972718000473

A GENERAL POSITION PROBLEM IN GRAPH THEORY

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(Received 3 April 2018; accepted 10 May 2018; first published online 18 July 2018)

Abstract

The paper introduces a graph theory variation of the general position problem: given a graph G, determine a largest set S of vertices of G such that no three vertices of S lie on a common geodesic. Such a set is a max-gp-set of G and its size is the gp-number $\mathrm{gp}(G)$ of G. Upper bounds on $\mathrm{gp}(G)$ in terms of different isometric covers are given and used to determine the gp-number of several classes of graphs. Connections between general position sets and packings are investigated and used to give lower bounds on the gp-number. It is also proved that the general position problem is NP-complete.

2010 *Mathematics subject classification*: primary 05C12; secondary 05C70, 68Q25.

Keywords and phrases: general position problem, isometric subgraph, packing, independence number, computational complexity.

1. Introduction

The classical no-three-in-line problem is to find the maximum number of points that can be placed in an $n \times n$ grid so that no three points lie on a line. This celebrated century-old problem posed by Dudeney [5] is still open. For some recent related developments, see [12, 18] and references therein. In [18] the problem is extended to three dimensions, while in [12] it is proved that at most $2\gcd(m, n)$ points can be placed with no three in a line on an $m \times n$ discrete torus. In discrete geometry, the no-three-in-line problem was extended to the general position subset selection problem [7, 15]. Here, for a given set of points in the plane one aims to determine a largest subset of points in general position. In [7] it is proved, among other results, that the problem is NP-hard, while in [15] asymptotic bounds on the function $f(n, \ell)$ are derived, where $f(n, \ell)$ is the maximum integer such that every set of n points in the plane with no more than ℓ collinear contains a subset of $f(n, \ell)$ points with no three collinear.

The above problems motivated us to define a similar problem in graph theory as follows: given a graph G, the *graph theory general position problem* is to find a largest set of vertices $S \subseteq V(G)$ such that no three vertices of S lie on a common geodesic in G. Note that an intrinsic difference between the discrete geometry problem and the

This work was supported and funded by Kuwait University, Research Project No. QI 02/17.

graph theory general position problem is that in the first case for given points x and y there is only one straight line passing through x and y, while in the graph theory problem there can be several geodesics passing through two vertices.

We proceed as follows. In the next section we give necessary definitions, general properties of general position sets and exact values for the gp-number of some classes of graphs. In Section 3 upper bounds on the gp-number in terms of different isometric covers are obtained. It is also proved that the set of simplicial vertices of a block graph forms a maximum general position set. In Section 4 we relate general position sets with the diameter and the k-packing number and derive lower bounds on the gp-number. Then, in Section 5, we prove that the general position problem is NP-complete.

2. Preliminaries and examples

In this section we first define concepts and introduce the notation needed. Then we proceed to give the general position number of some families of graphs and along the way give some related general properties.

All graphs considered in this paper are connected. The *distance* $d_G(u, v)$ between vertices u and v of a graph G is the number of edges on a shortest u, v-path. Shortest paths are also known as *geodesics* or *isometric paths*. The *diameter* $\mathrm{diam}(G)$ of G is the maximum distance between all pairs of vertices of G. A subgraph $H = (V(H), E(H))$ of a graph $G = (V(G), E(G))$ is *isometric* if $d_H(x, y) = d_G(x, y)$ holds for every pair of vertices x, y of H. This is one of the key concepts in metric graph theory (see [2, 16, 17, 19]). A *block* of a graph G is a maximal connected subgraph of G that has no cut-vertex. A graph is a *block graph* if every block of it is complete. A vertex of a graph is *simplicial* if its neighbours induce a complete subgraph. For $n \in \mathbb{N}$, we will use the notation $[n] = \{1, \ldots, n\}$.

A set S of vertices of a graph G is a *general position set* if no three vertices of S lie on a common geodesic in G. A general position set S of maximum cardinality is called a *max-gp-set* of G. The cardinality of a max-gp-set of G is called the *general position number* (gp-*number* for short) of G and denoted by $\mathrm{gp}(G)$.

As soon as G has two vertices, $\mathrm{gp}(G) \geq 2$. For complete graphs, $\mathrm{gp}(K_n) = n$ for $n \geq 1$. For paths, $\mathrm{gp}(P_n) = 2$ for $n \geq 2$. Consider next the cycle C_n on vertices $v_1, \ldots, v_n$ with natural adjacencies. Let S be an arbitrary general position set of C_n and assume without loss of generality that $v_1 \in S$. Then

$$|S \cap \{v_2, v_3, \ldots, v_{\lceil (n+1)/2 \rceil}\}| \leq 1,$$
$$|S \cap \{v_{\lceil (n+1)/2 \rceil + 1}, v_{\lceil (n+1)/2 \rceil + 2}, \ldots, v_n\}| \leq 1.$$

It follows that $\mathrm{gp}(C_n) \leq 3$. If $n \geq 5$, then it is easy to find a max-gp-set in C_n of order 3. Hence, $\mathrm{gp}(C_n) = 3$ for $n \geq 5$. Note also that $\mathrm{gp}(C_3) = 3$ and $\mathrm{gp}(C_4) = 2$. For $k \geq 2$ and $\ell \geq 2$, let $\Theta(k, \ell)$ be the graph consisting of two vertices A and B which are joined by k internally disjoint paths each of length ℓ. The vertices other than A and B are called internal vertices of $\Theta(k, \ell)$. See Figure 1, where $\Theta(4, 5)$ is drawn. These graphs are known as *theta graphs*.

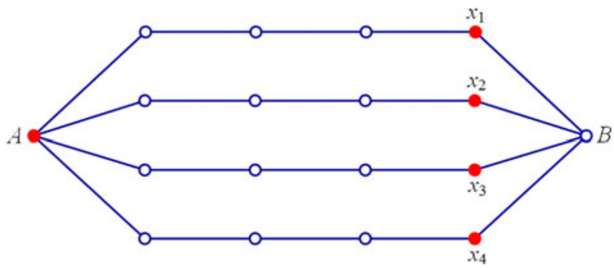

FIGURE 1. The theta graph $\Theta(4, 5)$ and its max-gp-set.

PROPOSITION 2.1. *If* $k \geq 2$ *and* $\ell \geq 3$, *then* $\mathrm{gp}(\Theta(k, \ell)) = k + 1$.

PROOF. Let R be a general position set of $\Theta(k, \ell)$. Let P_i, $i \in [k]$, denote the distinct isometric paths of $\Theta(k, \ell)$ joining A and B. Consider arbitrary paths P_i and P_j of $\Theta(k, \ell)$. Then the union of P_i and P_j induces an isometric cycle C of $\Theta(k, \ell)$ and hence $|R \cap V(C)| \leq 3$. Therefore, if R contains either A or B, then each P_i, $i \in [k]$, contains at most one vertex of R other than A or B, respectively. If R contains neither A nor B, then only one path P_j can contain two vertices from R (clearly, it cannot contain three or more) and all the other paths P_i, where $i \neq j$, can have at most one vertex of R. In either case, $|R| \leq k + 1$. Since R is an arbitrary general position set of $\Theta(k, \ell)$, $\mathrm{gp}(\Theta(k, \ell)) \leq k + 1$.

Let $x_1, \ldots, x_k$ be the vertices of $\Theta(k, \ell)$ that are adjacent to B and introduce the set $S = \{A, x_1, \ldots, x_k\}$. See Figure 1. It is easy to verify that S is a general position set of $\Theta(k, \ell)$. Consequently, $\mathrm{gp}(\Theta(k, \ell)) \geq k + 1$. □

3. Upper bounds on gp(G)

We say that a set of subgraphs $\{H_1, \ldots, H_k\}$ of a graph G is an *isometric cover* of G if each H_i, $i \in [k]$, is isometric in G and $\bigcup_{i=1}^{k} V(H_i) = V(G)$. Every isometric cover of G yields an upper bound on $\mathrm{gp}(G)$ as follows.

THEOREM 3.1 (Isometric cover lemma). *If* $\{H_1, \ldots, H_k\}$ *is an isometric cover of* G, *then*

$$\mathrm{gp}(G) \leq \sum_{i=1}^{k} \mathrm{gp}(H_i).$$

PROOF. Let R be a max-gp-set of G and let $R_i = R \cap V(H_i)$ for $i \in [k]$. We claim that R_i is a general position set of H_i. Suppose on the contrary that there exist vertices $x, y, z \in V(H_i)$ such that y lies on some x, z-geodesic in H_i, that is, $d_{H_i}(x, z) = d_{H_i}(x, y) + d_{H_i}(y, z)$. Since H_i is isometric in G, this implies that $d_G(x, z) = d_G(x, y) + d_G(y, z)$. But then R is not a general position set of G. This contradiction proves the claim. From the claim it follows that $\mathrm{gp}(H_i) \geq |R_i|$. We conclude that

$$\mathrm{gp}(G) = |R| = \left|\bigcup_{i=1}^{k} R_i\right| \leq \sum_{i=1}^{k} |R_i| \leq \sum_{i=1}^{k} \mathrm{gp}(H_i).$$ □

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The *isometric-path number* [6, 13, 14] of a graph G, denoted by $\mathrm{ip}(G)$, is the minimum number of isometric paths (geodesics) required to cover the vertices of G. We similarly say that the *isometric-cycle number* of G, denoted by $\mathrm{ic}(G)$, is the minimum number of isometric cycles required to cover the vertices of G. If G admits no cover with isometric cycles (for instance, if G is a tree), then we set $\mathrm{ic}(G) = \infty$.

Since $\mathrm{gp}(P_n) \le 2$ for $n \ge 1$ and $\mathrm{gp}(C_n) \le 3$ for $n \ge 3$, the isometric cover lemma has the following corollary.

Corollary 3.2. *If G is a graph, then:*

(i) $\mathrm{gp}(G) \le 2\,\mathrm{ip}(G)$*; and*
(ii) $\mathrm{gp}(G) \le 3\,\mathrm{ic}(G)$.

The bounds of Corollary 3.2 are sharp as demonstrated by paths and complete graphs of even order for the first bound, and cycles for the second bound.

For another upper bound we introduce the following concepts. If v is a vertex of a graph G, then let $\mathrm{ip}(v, G)$ be the minimum number of isometric paths, all of them starting in v, that cover $V(G)$. A vertex of a graph G that lies in at least one max-gp-set of G is called a *max-gp-vertex* of G. By applying the concepts of a breadth-first-search (BFS) tree, one can assert that $ip(v, G)$ is always well defined for connected graphs.

Theorem 3.3. *If R is a general position set of a graph G and $v \in R$, then*

$$|R| \le \mathrm{ip}(v, G) + 1.$$

In particular, if v is a max-gp-vertex, then $\mathrm{gp}(G) \le \mathrm{ip}(v, G) + 1$.

Proof. Let R be a general position set and $v \in R$. Let $k = \mathrm{ip}(v, G)$. Then there exist k geodesics $\{P_{vu_i} : u_i \in V(G), i \in [k]\}$ that cover $V(G)$. Since R is a general position set, $v \in R$ and P_{vu_i} is a geodesic, we have $|R \cap (V(P_{vu_i}) \setminus \{v\})| \le 1$ for $i \in [k]$. It follows that $|R| \le k + 1 = \mathrm{ip}(v, G) + 1$.

If v is a max-gp-vertex, then consider R to be a max-gp-set that contains v. By the above arguments, $\mathrm{gp}(G) = |R| \le \mathrm{ip}(v, G) + 1$. □

If G is a graph and $\mathrm{BFS}(v)$ a breadth-first-search tree of G rooted at v, then let $\ell(v)$ denote the number of leaves of $\mathrm{BFS}(v)$.

Corollary 3.4. *If G is a graph, then*

$$\mathrm{gp}(G) \le 1 + \min\{\ell(v) : v \text{ is a max-gp-vertex of } G\}.$$

Proof. Let v be a max-gp-vertex of G and let S be a max-gp-set containing v. Then $\mathrm{ip}(v, G) \le \ell(v)$ and hence $\mathrm{gp}(G) \le \mathrm{ip}(v, G) + 1 \le \ell(v) + 1$. Since the argument holds for any max-gp-vertex, the assertion follows. □

Corollary 3.4 is particularly useful for vertex-transitive graphs because in that case it suffices to consider a single BFS tree. For a simple example, consider the cycle C_n, $n \ge 3$. Then $\ell(v) = 2$ for any vertex v of C_n and hence $\mathrm{gp}(C_n) \le 3$ holds by Corollary 3.4.

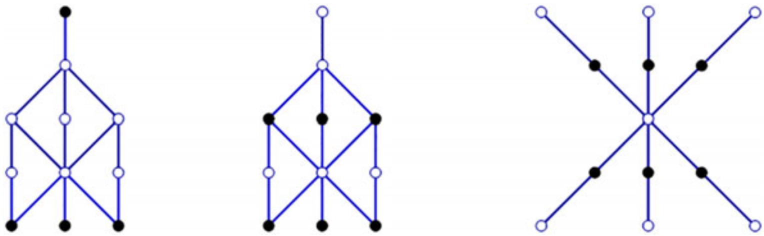

FIGURE 2. The solid vertices form general position sets. (a) A general position set containing the unique simplicial vertex. (b) A general position set without the simplicial vertex. (c) A general position set with none of the six simplicial vertices.

To show that in Corollary 3.4 the minimum cannot be taken over all vertices, consider the following example. Let $n \geq 2$ and let G_n be the graph on the vertex set $X_n \cup Y_n \cup Z_n \cup \{w\}$, where $X_n = \{x_1, \ldots, x_n\}$, $Y_n = \{y_1, \ldots, y_n\}$ and $Z_n = \{z_1, \ldots, z_n\}$. The vertices from X_n induce a complete subgraph. In addition, x_i is adjacent to y_i and z_i for $i \in [n]$, while w is adjacent to all vertices from Z_n. Then the BFS tree rooted in w has n leaves, that is, $\ell(w) = n$. On the other hand, if $u, v \in Y_n \cup Z_n$, $u \neq v$, then $d_{G_n}(u, v) \in \{2, 3\}$. It follows that $Y_n \cup Z_n$ is a general position set of G_n and therefore $\mathrm{gp}(G_n) \geq 2n$.

We now turn our attention to simplicial vertices.

LEMMA 3.5. *If S is the set of simplicial vertices of a graph G, then S is a general position set.*

PROOF. Assume on the contrary that there exist different vertices $u, v, w \in S$ such that $d_G(u, w) = d_G(u, v) + d_G(v, w)$ and let P be a u, w-geodesic that contains v. Let v' and v'' be the neighbours of v on P, where v' lies in the u, v-subpath of P and v'' in the v, w-subpath of P. (Note that it is possible that $v' = u$ or $v'' = w$.) Since v is a simplicial vertex, $v'v'' \in E(G)$. But then P is not a geodesic, which is a contradiction. □

So, simplicial vertices form general position sets. The examples in Figure 2 illustrate that there is no general correlation between sets of simplicial vertices and max-gp-sets. However, in specific classes of graphs, the set of simplicial vertices forms a max-gp-set. We have already noticed that this holds for complete graphs. By applying Theorem 3.3, we can generalise this observation to all block graphs.

THEOREM 3.6. *Let S be the set of simplicial vertices of a block graph G. Then S is a max-gp-set and hence* $\mathrm{gp}(G) = |S|$.

PROOF. Let S be the set of simplicial vertices of a block graph G and let R be a general position set of G. Let w be an arbitrary vertex of R. Since G is a block graph, w is either a simplicial vertex or a cut-vertex. Hence, we distinguish two cases.

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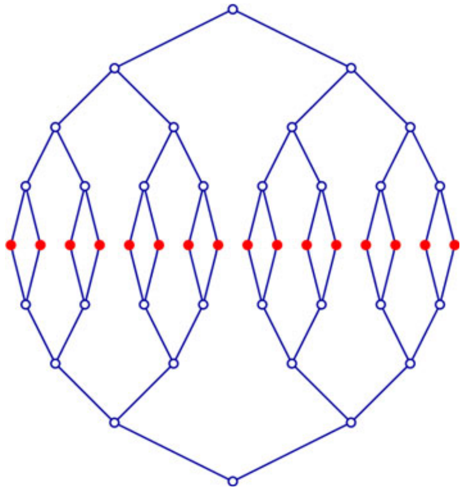

Figure 3. The glued binary tree $GT(4)$.

Case 1: $w \in S$.

Consider $\Psi_w = \{P_{wv} : v \neq w, v \in S, P_{wv}$ is a w, v-geodesic$\}$. It is known [13] that Ψ_w is an isometric path cover of G. Hence, Theorem 3.3 implies that $|R| \leq |\Psi_w| + 1 = |S|$.

Case 2: $w \notin S$, that is, w is a cut-vertex.

Let $\Psi_w = \{P_{wv} : v \in S, P_{wv}$ is a w, v-geodesic$\}$. Then again Ψ_w is an isometric path cover of G and hence as before $|R| \leq |\Psi_w| + 1 = |S| + 1$. Let now v_1 and v_2 be simplicial vertices of G that are in different connected components of $G - w$. Let P be the concatenation of the geodesics P_{wv_1} and P_{wv_2}. It is easy to see that P is a geodesic in G. Since $|R \cap V(P)| \leq 2$, one of P_{wv_1} and P_{wv_2} intersects R only in w. Hence, $|R| \leq (|S| + 1) - 1 = |S|$.

We have thus proved that in both cases $|R| \leq |S|$, so that $\mathrm{gp}(G) \leq |S|$. Lemma 3.5 completes the argument. □

Corollary 3.7. *If L is the set of leaves of a tree T, then* $\mathrm{gp}(T) = |L|$.

Consider next the *glued binary tree* $GT(r)$, $r \geq 2$, which is obtained from two copies of the complete binary trees of depth r by pairwise identifying their leaves. The construction should be clear from Figure 3, where the glued binary tree $GT(4)$ is shown. The vertices obtained by identification are shown as solid dots; we will call them *quasi-leaves* of the glued binary tree.

Proposition 3.8. *If $r \geq 2$, then* $\mathrm{gp}(GT(r)) = 2^r$.

Proof. Let R be a max-gp-set of $GT(r)$ and let S be the set containing the quasi-leaves of $GT(r)$. We now consider two cases.

Case 1: $R \cap S \neq \emptyset$.

Let u be an arbitrary vertex from $R \cap S$. Then it is easy to construct geodesics P_{uv} from u to all other vertices v of S in such a way $\{P_{uv} : v \in S\}$ is an isometric path cover of $GT(r)$. Therefore, $\mathrm{gp}(GT(r)) \leq 1 + (|S| - 1) = |S|$ by Theorem 3.3.

Case 2: $R \cap S = \emptyset$.

Let u be a vertex of R that is closest to a quasi-leaf among the vertices of R and let w be a quasi-leaf that is closest to u among all quasi-leaves. Then $R' = (R \setminus \{u\}) \cup \{w\}$ is a general position set. Indeed, suppose that this is not the case. Then a triple U of vertices from R' exists such that they lie on the same geodesic. Clearly, $w \in U$ for otherwise R would not be a general position set. But then $(U \setminus \{w\}) \cup \{u\}$ is a triple of vertices of R lying on a common geodesic. This contradiction proves that R' is a general position set. Since $|R'| = |R|$, the set R' is actually a max-gp-set. But now we are in Case 1 and hence conclude again that $\mathrm{gp}(GT(r)) \leq |S|$.

We have thus proved that $\mathrm{gp}(GT(r)) \leq |S|$. Since it is easy to see that S is a general position set of $GT(r)$, we also have $\mathrm{gp}(GT(r)) \geq |S|$. We are done because $|S| = 2^r$. □

4. Lower bounds on gp(G)

In this section we consider lower bounds on the general position number. We already have a lower bound based on Lemma 3.5: if S is the set of simplicial vertices of a graph G, then $\mathrm{gp}(G) \geq |S|$.

Additional lower bounds given here are in terms of the diameter of a graph and the k-packing number that is defined as follows. A set S of vertices of a graph G is a *k-packing* if $d(u, v) > k$ holds for every pair of different $u, v \in S$. The *k-packing number* $\alpha_k(G)$ of G is the cardinality of a maximum k-packing set [11]. For additional results on k-packing, see [4, 10]. Moreover, k-packings are the key ingredients for the concept of the S-packing chromatic number (see [1, 3, 9] and references therein). The 1-packings are precisely independent sets and so the independence number $\alpha(G)$ is just $\alpha_1(G)$.

A general position set need not be an independent set and *vice versa*. But we do have the following connection.

Proposition 4.1. *Let G be a graph and $k \geq 1$. Then* $\mathrm{diam}(G) \leq 2k + 1$ *if and only if every k-packing of G is a general position set.*

Proof. Suppose that S is a k-packing of G that is not a general position set. Then S contains vertices x, y, z such that y lies on an xz-geodesic P_{xz}. Since S is a k-packing, we have $d(x, y) \geq k + 1$ and $d(y, z) \geq k + 1$. Since P_{xz} is a geodesic, it follows that $d(x, z) \geq 2k + 2$. So, $\mathrm{diam}(G) \geq 2k + 2$.

Conversely, suppose that $\mathrm{diam}(G) \geq 2k + 2$. Let x and z be vertices with $d(x, z) = 2k + 2$. In addition, let P_{xz} be an xz-geodesic and let y be a vertex of P_{xz} such that $d(x, y) = d(y, z) = k + 1$. Then $\{x, y, z\}$ is a k-packing that is not a general position set. □

Proposition 4.1 provides a lower bound on gp-sets of a graph.

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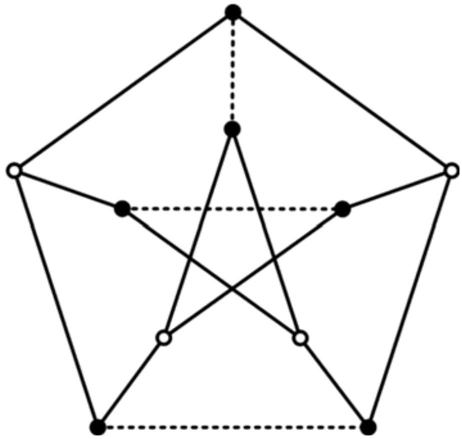

FIGURE 4. $\mathrm{gp}(P) \geq 6$.

COROLLARY 4.2. *If G is a graph with* $\mathrm{diam}(G) \leq 2k + 1$, *then* $\mathrm{gp}(G) \geq \alpha_k(G)$.

Since 1-packing sets of a graph are precisely its independent sets, Proposition 4.1 for $k = 1$ asserts the following corollary.

COROLLARY 4.3. *Let G be a graph and $k \geq 1$. Then* $\mathrm{diam}(G) \leq 3$ *if and only if every independent set of G is a general position set.*

In general, however, there is no connection between the independence number $\alpha(G)$ of G and $\mathrm{gp}(G)$. For instance, $\mathrm{gp}(K_n) = n$ and $\alpha(K_n) = 1$, while, on the other hand, $\mathrm{gp}(P_n) = 2$ and $\alpha(P_n) = \lceil n/2 \rceil$.

Another lower bound on $\mathrm{gp}(G)$ involves the distance between the edges of a graph, which is defined as follows. If $e = uv$ and $f = xy$ are edges of a graph G, then

$$d(e, f) = \min\{d(u, x), d(u, y), d(v, x), d(v, y)\}.$$

PROPOSITION 4.4. *Let G be a graph with* $\mathrm{diam}(G) = k \geq 2$. *If F is a set of edges of G such that $d(e, f) = k$ for every $e, f \in F$, $e \neq f$, then* $\mathrm{gp}(G) \geq 2|F|$.

PROOF. We claim that the set S consisting of the end-vertices of the edges from F is a general position set. If $x \in S$, then let f_x be the edge of F containing x. Let x, y, z be an arbitrary triple of vertices from S and suppose that y lies on an x, z-geodesic P. Clearly, $f_x \neq f_z$. Since $d(f_x, f_z) = k$ and $\mathrm{diam}(G) = k$, we must necessarily have $d(x, z) = k$. Suppose without loss of generality that $f_y \neq f_x$. But, then, as P is a geodesic, we have $d(y, x) \leq k - 1$ and hence $d(f_x, f_y) \leq k - 1$, which is a contradiction. □

As an application of the above proposition, consider the Petersen graph P. In Figure 4, three dotted edges of P pairwise at distance 2 are shown. Hence, $\mathrm{gp}(P) \geq 6$ by Proposition 4.4. Since $V(P)$ can be covered with two disjoint isometric cycles, $\mathrm{gp}(P) \geq 6$ by Corollary 3.2(ii). Thus, $\mathrm{gp}(P) = 6$.

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Additional examples demonstrating sharpness of Proposition 4.4 where k is large can be constructed as follows. Start with the star $K_{1,n}$ and subdivide each edge of it the same number of times. Then to each of the n leaves attach a private triangle by identifying a vertex of the triangle with the leaf. The n edges of these triangles whose end-vertices are of degree 2 are edges that satisfy the assumption(s) of the proposition.

5. Computational complexity of the problem

The (graph) general position problem is the following:

General Position Problem	
Input:	A graph G and an integer k.
Question:	Is $\mathrm{gp}(G) \geq k$?

The general position subset selection problem from discrete geometry which is a main motivation of this paper has been proved as NP-hard [7, 15]. We next prove a parallel result for the General Position Problem.

Theorem 5.1. *The* General Position Problem *is NP-complete.*

Proof. Note first that the General Position Problem is in NP. A set S of vertices of a graph G is a general position set of G if and only if for each pair of vertices x and z of S, we have $d(x,z) \neq d(x,y) + d(y,z)$ for every y in S. This task can be done in polynomial time. In the rest of the proof, we give a reduction of the NP-complete Maximum Independent Set Problem to the General Position Problem. The former problem is one of the classical NP-complete problems [8].

Given a graph $G = (V, E)$, we construct a graph $\widetilde{G} = (\widetilde{V}, \widetilde{E})$ as follows. Its vertex set is $\widetilde{V} = V \cup V' \cup V''$, where $V' = \{v' : v \in V\}$ and $V'' = \{v'' : v \in V\}$. The set of edges is $\widetilde{E} = E \cup E' \cup E'' \cup E'''$, where E' is the set of all possible edges between the vertices of V', while $E'' = \{vv' : v \in V\}$ and $E''' = \{v'v'' : v \in V\}$. Thus, the graph $\widetilde{G}$ can be considered as composed of three parts: the original graph G, the complete graph induced by V' and the independence set induced by V''. These three parts are connected by the matching E'' between V and V' and the matching E''' between V' and V''.

We first claim that $X \subseteq V$ is an independent set of G if and only if $X \cup V_2$ is a general position set of $\widetilde{G}$. Suppose first that $X \subseteq V$ is an independent set of G. Then, clearly, $X \cup V''$ is an independent set of $\widetilde{G}$. Since $\mathrm{diam}(\widetilde{G}) = 3$, Corollary 4.3 implies that $X \cup V''$ is a general position set of $\widetilde{G}$. Conversely, assume that X is not independent and let $x, y \in X$ be adjacent vertices. Then the path $xyy'y''$ is a geodesic, which in turn implies that $X \cup V''$ is not a general position set of $\widetilde{G}$.

We next claim that $\alpha(G) \geq k$ if and only if $\mathrm{gp}(\widetilde{G}) \geq k + |V|$. It suffices to show that if S is a general position set of $\widetilde{G}$, then there exists a general position set $\widetilde{S}$ of $\widetilde{G}$ such that $\widetilde{S} = X \cup V''$, where X is an independent set of G and $|\widetilde{S}| \geq |S|$. For any two vertices

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x and y of V and the corresponding vertices x' and y' of V' and x'' and y'' of V'', $x''x'y'y''$ is a geodesic in $\widetilde{G}$. For some $u \in V$, if both u' and u'' are in S, then no other vertices $v'' \neq u''$ will be in S. This will contradict the maximality of the gp-set of $\widetilde{G}$ when $|V| \geq 3$. If $u' \in S$ and $u'' \notin S$, then consider set $\widetilde{S} = S \cup \{u''\} \setminus \{u'\}$. We conclude that given a general position set S of $\widetilde{G}$, there exists a general position set $\widetilde{S}$ of $\widetilde{G}$ such that $\widetilde{S} \cap V' = \emptyset$. From here the claim follows and this completes the argument. □

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