On Curved Barriers.

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In two-dimensional discontinuous fluid motion one point of considerable importance has not hitherto been given sufficient attention. I raise it formally in a paper to be published soon (Fluid Motion past Circular Barriers, Scripta Universitatis atque Bibliothecae Heirosolymitanarum, 1923, Vol. I., XI., 1–14) in the following manner. Given the form of the barrier by means of, say, the radius of curvature in terms of the angle of contingence, how does the solution take into account the angular extent of the barrier? Clearly barriers which are defined by the same curve, but differ in the extent of curve used, must necessarily give rise to different solutions. Further, there must be a limiting extent of barrier, so that if it extends beyond this limit the part of the barrier in excess must lie in the "dead" fluid.

The case of the circular barrier is dealt with in the paper referred to above, where the solution is obtained for circular barriers both concave and convex to the streaming fluid. It is shown how a simple criterion establishes the limiting extent for the convex circular barrier, this limiting extent being the one defined and discussed in another paper (*Proc. Roy. Soc.*, A. 102, 542–553. 1923). The limiting extent is in fact to about 55° on either side of the "nose" for the circular barrier.

The limiting extents for convex elliptic barriers of various eccentricities or "fineness ratios" are also given in the latter paper.

It is my object here to lay down the criterion generally, without any assumption of symmetry. Ultimately the process is based on an idea mentioned by M. Brillouin (*Ann. Chim. Phys.* (VIII.), XXIII., 1911, 145–230); but he leaves it rather vague.

Brillouin proves that the free stream line must be convex on the side of the streaming fluid, concave on the side of the dead

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fluid. Now let us have some definite functional form of a barrier. Starting off with a small angular extent, practically a plane, the free stream line at either end has, to commence with, zero radius of curvature. As the angular extent increases, the stream line changes, but the radius of curvature of the stream line is still zero where the stream line leaves the barrier. If we suppose the extent to continue to increase, we reach a stage where the radius of curvature of the free stream line at the point where it leaves the barrier suddenly becomes finite. This is the critical extent of the barrier—beyond this the barrier lies in the dead fluid.

To prove this I shall show that when the radius of curvature becomes finite at the beginning of the free stream line, we have the limit beyond which the free stream line would be concave to the streaming fluid and convex to the dead fluid. Further, I shall prove that at this stage the initial radius of the curvature of the free stream line is the same as that of the barrier where the free stream line leaves it.

Using the notation of my paper (*Proc. Roy. Soc.*, A. 102, 1922, 361–72) let the complex variable $z\,(\equiv x + \iota y)$ define position in a plane perpendicular to the generators of the two-dimensional, *i.e.* cylindrical, barrier, the x axis being parallel to the direction of the stream at infinity. If u, v are the velocity components we define ϕ, ψ so that

$$u = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \quad v = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}.$$

Let $w \equiv \phi + \iota\psi$ and define $\Omega = \log\,(dz/dw) \equiv r + \iota\theta$. The solution of any problem is given by defining some relation between Ω and w.

Levi-Civita's method is somewhat as follows. Put

$$\sqrt{w} = \frac{1}{2\iota}\left(\tau - \frac{1}{\tau}\right) - \sin\sigma_0$$

where τ is a new complex variable defined as $\rho e^{\iota\sigma}$. Clearly $\rho = 1$, $\sigma = \sigma_0$, gives $w = 0$, and this defines the point in the τ plane corresponding to the nose, *i.e.*, the point in the barrier in the z plane where the fluid divides into two streams on the two sides of the barrier. We get a semicircle in the τ plane, so that $\tau = e^{\iota\sigma}$ is a point on the barrier $\left(\sigma \text{ between } -\frac{\pi}{2} \text{ and } +\frac{\pi}{2}\right)$; $\tau = \iota\rho$ is a point on either free stream line, $\rho = 1$ to 0 for one and $\rho = -1$ to 0 for the other. The general solution is

$$\Omega = \lambda \log \frac{1 + e^{\iota\sigma_0}\tau}{1 - e^{-\iota\sigma_0}\tau} + A_1\tau + \tfrac{1}{2}\iota A_2\tau^2 + \tfrac{1}{3}A_3\tau^3 + \tfrac{1}{4}\iota A_4\tau^4 + \ldots$$

where $\lambda\pi$ is the angle formed by the barrier at the "nose"—measured on the side away from the moving fluid. The values of λ, σ_0, A_1, A_2, ... define any particular barrier.

Consider the free stream line, $\tau = \iota\rho$, $\rho = 1$ to 0. For brevity put

$$\Omega = a_1\tau + \tfrac{1}{2}\,\iota a_2\tau^2 + \tfrac{1}{3}a_3\tau^3 + \tfrac{1}{4}\iota a_4\tau^4 + \ldots$$

where $a_n = A_n +$ the coefficient due to the first part of Ω. We get

$$\log r + \iota\theta = \iota\,(a_1\rho - \tfrac{1}{2}a_2\rho^2 - \tfrac{1}{3}a_3\rho^3 + \tfrac{1}{4}a_4\rho^4 - \ldots).$$

Hence $r = 1$ and

$$\theta = a_1\rho - \tfrac{1}{2}a_2\rho^2 - \tfrac{1}{3}a_3\rho^3 + \tfrac{1}{4}a_4\rho^4 + \tfrac{1}{5}a_5\rho^5 - \ldots$$

Hence
$$\frac{d\theta}{d\rho} = a_1 - a_2\rho - a_3\rho^2 + a_4\rho^3 + a_5\rho^4 - \ldots$$

θ represents the direction at any point of the free stream line. If ds is an element of length on it we have

$$ds = rd\phi = d\phi$$

while
$$\phi = \left\{\tfrac{1}{2}\left(\rho + \frac{1}{\rho}\right) - \sin\sigma_0\right\}^2.$$

Hence
$$\frac{ds}{d\rho} = \frac{ds}{d\phi}\frac{d\phi}{d\rho} = -\frac{1-\rho^2}{\rho^2}\left\{\tfrac{1}{2}\left(\rho + \frac{1}{\rho}\right) - \sin\sigma_0\right\}^2.$$

To get the radius of curvature we need $ds/d\theta$, *i.e.*, $\dfrac{ds}{d\rho}\Big/\dfrac{d\theta}{d\rho}$. We get

$$\frac{ds}{d\theta} = -\frac{\dfrac{1-\rho^2}{\rho^2}\left\{\tfrac{1}{2}\left(\rho + \dfrac{1}{\rho}\right) - \sin\sigma_0\right\}^2}{a_1 - a_2\rho - a_3\rho^2 + a_4\rho^3 + \ldots}$$

It is clear that in general we get zero radius of curvature for $\rho = 1$, namely, at the point where the free stream line leaves the barrier.

For finite radius of curvature at $\rho = 1$ we must evidently have $d\theta/d\rho$ zero at $\rho = 1$. We therefore have the condition

$$a_1 - a_2 - a_3 + a_4 + a_5 - a_6 - a_7 + a_8 + \ldots = 0.$$

We have already mentioned that the free stream line has zero radius of curvature for small angular extent, so that it is convex to the moving fluid at $\rho = 1$, and $d\theta/d\rho$ is positive. In the limiting case just mentioned we have $d\theta/d\rho$ zero, so that we are at the passage from convexity to concavity.

Putting $\rho = 1 - \epsilon$ and proceeding to the limit where $\epsilon \to 0$, with the assumption that the condition for finite radius of curvature at $\rho = 1$ is satisfied, we get

$$\frac{ds}{d\theta} = -\frac{2(1-\sin\sigma_0)}{a_2 + 2a_3 - 3a_4 - 4a_5 + 5a_6 + \ldots} = \frac{2(1-\sin\sigma_0)}{a_1 - 2a_2 - 3a_3 + 4a_4 + 5a_5 - 6a_6 - \ldots}$$

Now consider the barrier itself; we do this by putting $\tau = e^{\iota\sigma}$. We get

$$\log r + \iota\theta = a_1 e^{\iota\sigma} + \tfrac{1}{2}\iota a_2 e^{2\iota\sigma} + \ldots$$

so that $\quad \log r = a_1 \cos\sigma - \frac{1}{2}a_2 \sin 2\sigma + \frac{1}{3}a_3 \cos 3\sigma - \frac{1}{4}a_4 \sin 4\sigma + \ldots,$

$$\theta = a_1 \sin\sigma + \tfrac{1}{2}a_2 \cos 2\sigma + \tfrac{1}{3}a_3 \sin 3\sigma + \tfrac{1}{4}a_4 \cos 4\sigma + \ldots$$

Hence $\quad \dfrac{d\theta}{d\sigma} = a_1 \cos\sigma - a_2 \sin 2\sigma + a_3 \cos 3\sigma - a_4 \sin 4\sigma + \ldots$

To find $ds/d\theta$ we need to know the value of $ds/d\sigma$, *i.e.*, of $\dfrac{ds}{d\phi}\cdot\dfrac{d\phi}{d\sigma}$, *i.e.*, of $r d\phi/d\sigma$. Also we have now

$$\phi = (\sin\sigma - \sin\sigma_0)^2.$$

Hence

$$\frac{ds}{d\sigma} = 2\cos\sigma(\sin\sigma - \sin\sigma_0)\, e^{a_1 \cos\sigma - \frac{1}{2}a_2 \sin 2\sigma + \frac{1}{3}a_3 \cos 3\sigma - \frac{1}{4}a_4 \sin 4\sigma + \ldots}$$

The radius of curvature is therefore

$$\frac{ds}{d\theta} = \frac{2\cos\sigma(\sin\sigma - \sin\sigma_0)\, e^{a_1 \cos\sigma - \frac{1}{2}a_2 \sin 2\sigma + \ldots}}{a_1 \cos\sigma - a_2 \sin 2\sigma + a_3 \cos 3\sigma - a_4 \sin 4\sigma + \ldots}.$$

When σ is taken equal to $\pi/2$ we get

$$\frac{ds}{d\theta} = \frac{2(1-\sin\sigma_0)}{a_1 - 2a_2 - 3a_3 + 4a_4 + 5a_5 - 6a_6 - \ldots}.$$

Thus the radius of curvature of the free stream line at $\rho = 1$ is the same as that of the barrier at $\sigma = \pi/2$, which is what we set out to prove.

Exactly similar reasoning applies to the other stream line, ρ from -1 to 0. At $\rho = -1$ we have in general zero radius of curvature. But in the limit, when the radius of curvature is finite, the free stream line is just at the limit of convexity to the moving fluid, and the radius of curvature is the same as that of the barrier at $\sigma = -\pi/2$. It is not necessary for both sides of the barrier to have the same property; one can extend to the limit or beyond, while the other has an extent less than the limit. Of course, for the symmetrical case we must have the same conditions on both sides.