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INNER FUNCTIONS AND THE MAXIMAL IDEAL SPACE OF $L^\infty(T^n)$

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Abstract

Let U^n be the unit polydisc in $\mathbf{C}^n$ and let T^n be its distinguished boundary. It is shown that a function $f \in H^\infty(U^n)$ is inner if and only if $|\hat{f}(\Phi)| = 1$ for all Φ in the maximal ideal space of $L^\infty(T^n)$. This generalizes a result of Csordas.

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1. Introduction

By identifying each $f \in H^\infty(U^n)$ with its radial boundary function, $H^\infty(U^n)$ may be regarded as a closed subalgebra of $L^\infty(T^n)$. Let X, M be the maximal ideal space of $L^\infty(T^n)$ and $H^\infty(U^n)$ respectively and let $\tau: X \to M$ be defined by mapping each complex homomorphism in X to its restriction to $H^\infty(U^n)$.

For $n = 1$ it is well known that $\tau(X)$ is the Shilov boundary of $H^\infty(U)$, see Hoffman (1962, p. 174), while Range (1972) has shown that $\tau(X)$ is strictly larger than the Shilov boundary for $n \geq 2$. Csordas (1973) has shown that $f \in H^\infty(U)$ is inner if and only if $|\hat{f}(\Phi)| = 1$ for all Φ in the Shilov boundary. Using the same approach, we show in Section 2 that $f \in H^\infty(U^n)$ is inner if and only if $|\hat{f}(\Phi)| = 1$ for all Φ in $\tau(X)$.

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2. A characterization of inner functions

Let the fibre of X over $\alpha \in T^n$ be denoted by X_α, that is,

$$X_\alpha = \{\Phi \in X: \hat{Z}_i(\Phi) = \alpha_i,\ i = 1, \ldots, n\}.$$

In the same manner as in Hoffman (1962, p. 171), we have

LEMMA 1. *Let $f \in L^\infty(T^n)$, then $\omega \notin \hat{f}(X_\alpha)$ if and only if there is a $\varepsilon > 0$ and a neighbourhood $N(\alpha)$ such that the set*

$$\{z \in T^n : |f(z) - \omega| < \varepsilon\} \cap N(\alpha)$$

has Lebesgue measure zero.

PROOF. The number $\omega \notin \hat{f}(X_\alpha)$ if and only if there are functions

$$h, g_1, \ldots, g_n \in L^\infty(T^n)$$

such that

$$(z_1 - \alpha_1) g_1 + \ldots + (z_n - \alpha_n) g_n + (f - \omega) h = 1$$

or that

$$\left\{1 - \sum_{k=1}^{n} (z_k - \alpha_k) g_k\right\} \Big/ f - \omega \in L^\infty(T^n).$$

Now suppose that $f - \omega$ is essentially bounded away from zero in a neighbourhood $N(\alpha)$, say $N(\alpha) = \{z \in T^n : |z_i - \alpha_i| < \delta\}$. Define $N_k(\alpha) = \{z \in T^n : |z_i - \alpha_i| < \delta$ for $i = 1, \ldots, k-1$ and $|z_k - \alpha_k| \geqslant \delta\}$ and

$$g_k(z) = \begin{cases} \dfrac{1 - (f - \omega)}{z_k - \alpha_k}, & z \in N_k(\alpha), \\ 0, & \text{elsewhere.} \end{cases}$$

Then $g_k \in L^\infty(T^n)$ and since T^n is the disjoint union of $N(\alpha)$ and the $N_k(\alpha)$'s, it is easy to see that

$$\left\{1 - \sum_{k=1}^{n} (z_k - \alpha_k) g_k\right\} \Big/ f - \omega = \begin{cases} 1/f - \omega, & z \in N(\alpha), \\ 1, & \text{elsewhere} \end{cases}$$

and hence is a member of $L^\infty(T^n)$. On the other hand, if $f - \omega$ is not essentially bounded away from zero in every neighbourhood of α, then clearly for every choice of $g_k \in L^\infty(T^n)$, the function $\{1 - \sum_{k=1}^{n} (z_k - \alpha_k) g_k\}/f - \omega \notin L^\infty(T^n)$ since in a small enough neighbourhood of α, $|1 - \sum_{k=1}^{n} (z_k - \alpha_k) g_k| \geqslant \frac{1}{2}$.

The following measure theoretic result will also be needed.

LEMMA 2. *Let E be a measurable subset of a regular measure space (X, μ), with $\mu(E) > 0$. Then there exists $E^1 \subseteq E$, with $\mu(E^1) > 0$ and for each $\alpha \in E^1$ and $N(\alpha)$, the set $E^1 \cap N(\alpha)$ has positive measure.*

PROOF. Since μ is regular, we may assume E to be compact. Suppose that for every $\alpha \in E$, there is a $N(\alpha)$ such that $E \cap N(\alpha)$ has zero measure. Since E is compact there exists $N(\alpha_k)$ such that $E \subseteq \bigcup_k N(\alpha_k)$, but then

$$\mu(E) = \mu\left(E \cap \left(\bigcup_k N(\alpha_k)\right)\right) \leqslant \sum_k \mu(E \cap N(\alpha_k)) = 0,$$

contradicting $\mu(E) > 0$.

Let $E_0 = \{\alpha \in E: \text{for some } N(\alpha), \mu(E \cap N(\alpha)) = 0\}$ and suppose that $\mu(E_0) > 0$, then as shown there exists a $\alpha \in E_0$ such that for all $N(\alpha)$,

$$\mu(E \cap N(\alpha)) \geqslant \mu(E_0 \cap N(\alpha)) > 0$$

i.e. $\alpha \notin E_0$! Thus $\mu(E_0) = 0$.

Let $E^1 = E \setminus E_0$, then $\mu(E^1) = \mu(E) > 0$ and for each $\alpha \in E^1$, $N(\alpha)$,

$$\mu(E^1 \cap N(\alpha)) = \mu(E \cap N(\alpha)) > 0.$$

For $\alpha \in T^n$ and $f \in H^\infty(U^n)$, the radial cluster set of f at α, denoted by $C_\rho(f, \alpha)$, is the set of all ω such that there exists a sequence of real $r_n \to 1$ with $0 \leqslant r_n < 1$ and $f(r_n \alpha) \to \omega$. An extension of a result of Csordas (1973) can now be given.

THEOREM 3. *Let $f \in H^\infty(U^n)$, and $\alpha \in T^n$. Then the set*

$$E = \{\alpha \in T^n: C_\rho(f, \alpha) \cap \hat{f}(\tau(X_\alpha)) = \emptyset\}$$

has Lebesgue measure zero.

PROOF. Let f^* be the radial boundary function of f. Suppose E has positive Lebesgue measure, then we can suppose that f^* is defined on E. By a well-known result of Lusin, f^* is continuous on a subset $E_0 \subseteq E$, of positive measure. Choose E_0^1 as in Lemma 2, then f^* is also continuous on $E_0^1 \subseteq E$. Let $\alpha \in E_0^1$, then $f^*(\alpha) \notin \hat{f}(\tau(X_\alpha))$ and by Lemma 1 there exists $\varepsilon > 0, N(\alpha)$ such that

$$V = \{z \in T^n: |f^*(z) - f^*(\alpha)| < \varepsilon\} \cap N(\alpha)$$

has Lebesgue measure zero.

Since f^* is continuous at $\alpha \in E_0^1$, there is a neighbourhood $N^1(\alpha)$ such that $|f^*(z) - f^*(\alpha)| < \varepsilon$ for all $z \in N^1(\alpha) \cap E_0^1$. By our choice of E_0^1, $N^1(\alpha) \cap N(\alpha) \cap E_0^1 (\subseteq V)$ has positive measure but V has measure zero!

COROLLARY 4. *A function $f \in H^\infty(U^n)$ is inner if and only if $|\hat{f}(\Phi)| = 1$ for all $\Phi \in \tau(X)$.*

PROOF. If $|\hat{f}(\Phi)| = 1$ for all Φ in $\tau(X)$, then f is inner by Theorem 3. Conversely, if f is inner, then f is invertible in $L^\infty(T^n)$ with $\|f\| = \|f^{-1}\| = 1$ and hence $|\hat{f}(\Phi)| = 1$ for all $\Phi \in \tau(X)$.

REMARK. The example given in Range (1972) to show that the Shilov boundary of $H^\infty(U^n)$ for $n \geqslant 2$ is a proper subset of $\tau(X)$ also shows that $|\hat{f}(\Phi)| = 1$ for all Φ in the Shilov boundary does not imply that f is inner.

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