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SCHRÖDINGER OPERATORS WITH MAGNETIC AND ELECTRIC POTENTIALS

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In the present paper, we consider Schrödinger operators which are formally given by $P = -\sum_{j=1}^{N} (\partial_j - ia_j)^2 + V$ in $L^2(R^N)$. In Section 2 and 3 we prove that P has a regularly accretive extension which is a self-adjoint extension of P and it is the only self-adjoint realisation of P in $L^2(R^N)$ when $\vec{a}$ satisfies $\vec{a} = (a_1, a_2, \cdots, a_N) \in L^2_{\text{loc}}(R^N)^N$, a_j real-valued, $1 \leqslant j \leqslant N$, $V \in L^1_{\text{loc}}(R^N)$, real-valued and the negative part $V_- := \max(0, -V)$ satisfys $\int_{R^N} V_- |\varphi|^2\, dx \leqslant C_1 \|\nabla\varphi\|^2 + C_2 \|\varphi\|^2$ $\varphi \in H^{1,2}(R^N)$, with constants $0 \leqslant C_1 < 1$, $C_2 \geqslant 0$ independent of V. In Section 4, we prove that P is essential self-adjoint on $C_0^\infty(R^N)$ when $\vec{a}$, V satisfy $\vec{a} \in L^4_{\text{loc}}(R^N)^N$, $\operatorname{div} \vec{a} \in L^2_{\text{loc}}(R^N)$; $V = V_1 + V_2$, V real-valued, $V_i \in L^2_{\text{loc}}(R^N)$, $i = 1, 2$, $V_1(x) \geqslant -C|x|^2$, for $x \in R^N$ with $C \geqslant 0$ and $0 \geqslant V_2 \in K_N$.

1. INTRODUCTION

In the present paper, we consider Schrödinger operators which are formally given by $P = -\sum_{j=1}^{N} (\partial_j - ia_j)^2 + V$, where V is an electric potential and $\vec{a} = (a_1, a_2, \cdots, a_N)$ is a singular magnetic vector potential. In solid state physics, this corresponds to a simple one-electron model of a crystal in a magnetic field, the (short-range) potential V describing impurities of the crystal (Reed and Simon [5, Vol.IV, Section VII.16]).

Schrödinger operators with magnetic vector potentials have been studied extensively (Leinfelder and Simader [4], Simon [9], Simader [7], Hinz and Stolz [2] and the references given therein). In Section 2 and 3, we make the general assumption s

$$\vec{a} = (a_1, a_2, \cdots, a_N) \in L^2_{\text{loc}}(R^N)^N,\ a_j \text{ real-valued},\ 1 \leqslant j \leqslant N, \tag{1.1}$$

$$V \in L^1_{\text{loc}}(R^N),\ \text{real-valued}, \tag{1.2}$$

the negative part $V_- := \max(0, -V)$ satisfying

$$\int_{R^N} V_- |\varphi|^2\, dx \leqslant C_1 \|\nabla\varphi\|^2 + C_2 \|\varphi\|^2,\ \varphi \in H^{1,2}(R^N) \tag{1.3}$$

with constants $0 \leqslant C_1 < 1$, $C_2 \geqslant 0$ independent of V.

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Since $C_0^\infty(R^N)$ is dense in $H^{1,2}(R^N)$ and $V_-^{1/2}$ is a closed operator from $L^2(R^N)$ into $L^2(R^N)$ as a multiplication operator, (1.3) can be written

$$(1.3)' \qquad \int_{R^N} V_- |\varphi|^2 \, dx \leqslant C_1 \| \nabla |\varphi| \|^2 + C_2 \|\varphi\|^2, \qquad \varphi \in H^{1,2}(R^N).$$

Condition (1.3) or (1.3)′ is for example satisfied if $V_- \in K_N$ (in fact, C_1 may be zero in this case), where

$$(1.4) \qquad K_N = \left\{ V \in L^2_{\mathrm{loc}}(R^N) : \lim_{t \downarrow 0} \omega_{N,t}(V) = 0 \right\},$$

$$(1.5) \qquad \omega_{N,t}(V) = \sup_{x \in R^N} \int_{|x-y|<t} |V(y)| \, |x-y|^{2-N} \, dy, \text{ for } t > 0 \text{ and } N \geqslant 3.$$

For $N = 2$, $|x-y|^{2-N}$ has to be replaced by $\log |x-y|^{-1}$ in Equation (1.5); for $N = 1$, K_N coincides with $L^1_{\mathrm{loc}}(R)$ (compare [**1**] for these definitions).

Now we define a sesquilinear form $h_{\vec{a},V}$ in the Hilbert space $L^2(R^N)$ by

$$(1.6) \qquad h_{\vec{a},V}(u, v) = \sum_{j=1}^{N} ((\partial_j - ia_j)u, (\partial_j - ia_j)v) + \int_{R^N} V u \bar{v} dx$$

for u, v from

$$(1.7) \quad D\left(h_{\vec{a},V}\right) = \left\{ u \in L^2(R^N) : (\partial_j - ia_j)u \in L^2(R^N), \right.$$
$$\left. 1 \leqslant j \leqslant N, \, V |u|^2 \in L^1(R^N) \right\},$$

where $(\partial_j - ia_j)u$ is defined in the sense of distribution. $h_{\vec{a},V}$ is symmetric semibounded, densely defined and closed, this is shown in [4] for $V \geqslant 0$. To accommodate V_-, it is important to note that (compare [**4**, Equation (3.6)])

$$(1.8) \qquad \partial_j |u| \leqslant |(\partial_j - ia_j)u| \qquad u \in D\left(h_{\vec{a},V_+}\right)$$

where $V_+ = V + V_-$. Hence, if (1.3) holds, V_- has relative form bound $C_1 < 1$ with respect to $h_{\vec{a},V_+}$ and [**3**, Theorem VI-1.33] applies.

Let $H_{\vec{a},V}$ denote the self-adjoint and semibounded operator associated with $h_{\vec{a},V}$ by [**3**, Theorem VI-2.1]. Instead of $H_{0,V}$, we write $-\Delta + V$. Then $H_{\vec{a},V}$ is a self-adjoint realisation of P in $L^2(R^N)$ in the sense of form and $D\left(H_{\vec{a},V}\right) = \{u \in L^2(R^N) :$

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$(\partial_j - ia_j)u \in L^2(R^N)$, $|V|^{1/2}u \in L^2(R^N)$, $Pu \in L^2(R^N)\}$, where P acts on u in the distribution sense.

In Section 2, we consider the regularly accretive extension of P, which is also a self-adjoint extension of P. We point out that when $\vec{a}$, V satisfy (1.1)–(1.3), P has a regularly accretive extension $F_{\vec{a},V}$ and $F_{\vec{a},V} = H_{\vec{a},V}$ (Theorem 2.2 and Theorem 2.3). In Section 3 we prove that one can define a maximal self-adjoint realisation $\widetilde{H}_{\vec{a},V}$ of P in $L^2(R^N)$ as follows:

$$D\left(\widetilde{H}_{\vec{a},V}\right) = \Big\{u \in L^2(R^N) : (\partial_j - ia_j)u \in L^2_{\mathrm{loc}}(R^N),$$
$$|V|^{1/2}u \in L^2_{\mathrm{loc}}(R^N),\ Pu \in L^2(R^N)\Big\},$$
$$\widetilde{H}_{\vec{a},V}u = Pu, \quad u \in D\left(\widetilde{H}_{\vec{a},V}\right).$$

It is clear that $\widetilde{H}_{\vec{a},V}$ is an extension of $H_{\vec{a},V}$. In fact, we have $\widetilde{H}_{\vec{a},V} = H_{\vec{a},V}$ (Theorem 3.1). In [7], Simader considered a Schrödinger operator $Tu = -\Delta u + Vu$ on $D(T) = C_0^\infty(R^N)$ when the potential V satisfies

$$\text{(H)} \quad \begin{cases} V = V_1 + V_2,\ V \text{ real-valued},\ V_i \in L^2_{\mathrm{loc}}(R^N), \quad i = 1, 2, \\ V_1(x) \geqslant -C\,|x|^2 \text{ for } x \in R^N \text{ with suitable constant } C \geqslant 0 \text{ and } 0 \geqslant V_2 \in K_N. \end{cases}$$

He proved that T is essentially self-adjoint when V satisfies (H) and $V_1 \geqslant 0$, see [7, Theorem 2]. In Section 4, we consider the self-adjoint realisation of P in $L^2_{\mathrm{loc}}(R^N)$ in the sense of operator when V satisfies (H) and $\vec{a} \in L^4_{\mathrm{loc}}(R^N)^N$, $\operatorname{div}\vec{a} \in L^2_{\mathrm{loc}}(R^N)$. We prove that P is essential self-adjoint on $C_0^\infty(R^N)$. Here, we must point out that Simader's proof of the above theorem is completely dependent on the local boundedness result in [8], but this method fails to be used in our case since $-\left(\nabla - i\vec{a}\right)^2 + V$ is not a real differential operator on $C_0^\infty(R^N)$. We avoid the estimation of local boundedness by means of the self-adjoint realisation $H_{\vec{a},V}$ of P in the sense of form. Recently, Hinz and Stolz proved that when $\vec{a} \in L^4_{\mathrm{loc}}(R^N)^N$, $\operatorname{div}\vec{a} \in L^2_{\mathrm{loc}}(R^N)$, $V \in L^2_{\mathrm{loc}}(R^N)$ and $V_- \in K_N + O\left(|x|^2\right)$, P is essential self-adjoint on $C_0^\infty(R^N)$. Their methods are the same as Simader's.

2. The regularly accretive extension of P

Let H denote a complex Hilbert space with inner product $(u, v)_H$ and norm $\|u\|_H = (u, u)_H^{1/2}$. We suppose that there is a dense subspace W of H which is a Hilbert space with inner product $(u, v)_W$ and norm $\|u\|_W = (u, u)_W^{1/2}$ with $u \in W$.

Suppose the identity map $W \to H$ is a bounded operator, that is, there is a constant K_0 such that for all $u \in W$,

$$\|u\|_H \leqslant K_0 \|u\|_W . \tag{2.1}$$

Suppose further there is a bilinear form $b(u, v)$ defined on $W \times W$ with values in $\mathbb{C}$ and a constant K_1 such that for all u, v in W,

$$|b(u, v)| \leqslant K_1 \|u\|_W \|v\|_W . \tag{2.2}$$

We may define the linear operator associated with b to be that operator A with domin $D(A) \subseteq W$ such that $u \in D(A)$ and $Au = v$ if and only if $b(u, w) = (v, w)$ for all $w \in W$.

We now make the fundamental

DEFINITION: A linear operator A is said to be regularly accretive if it is associated with a bilinear form b which in addition to satisfying (2.2), also satisfies

$$\|u\|_W^2 \leqslant K_2 \left(\operatorname{Re} b(u, u) + K_3 \|u\|_H^2 \right) \tag{2.3}$$

for all u in W and fixed constants K_2 and K_3.

It can be shown that a regularly accretive operator is densely defined and closed. In addition its spectrum is contained in some half-space $\operatorname{Re} \lambda > K$ of the complex-plane. If b is symmetric, then A is a semibounded self-adjoint operator.

Now suppose A_0 is a linear operator in H whose domain $D(A_0)$ is not necessarily dense in H. The following lemma will be useful for us.

LEMMA 2.1. *Let U be a dense subspace of W which contains $D(A_0)$. Suppose $b(\cdot, \cdot)$ is a bilinear form on $U \times U$ which satisfies inequalities (2.1), (2.2) and (2.3) for all u and v in U. If*

$$b(u, v) = (A_0 u, v)_H \tag{2.4}$$

for all u in $D(A_0)$ and v in U, it follows that A_0 has a regularly accretive extension A.

The proof follows directly from the observation that the inequalities (2.1), (2.2) and (2.3) as well as the form b itself extend to all of W by continuity. That the regularly accretive operator A associated with b and W is an extension of A_0 follows from (2.4).

A fuller account of the ideas here can be found in Schechter [6] and Kato [3].

In the sequel, we consider the regularly accretive extension of P. Here, suppose $\vec{a}$, V satisfy (1.1)–(.13). Define the operator F_0 on $C_0^\infty(R^N)$ as follows:

$$D(F_0) = \left\{u \in C_0^\infty(R^N) : |V|^{1/2}\, u \in L^2(R^N),\, Pu \in L^2(R^N)\right\}$$
$$F_0 u = Pu, \quad u \in D(F_0),$$

where P acts on u in the distribution sense.

Obviously, F_0 is a linear operator in $L^2(R^N)$.

THEOREM 2.2. *Let $\vec{a}$, V and F_0 as above, then F_0 has a regularly accretive extension $F_{\vec{a},V}$.*

PROOF: Let U denote the space $C_0^\infty(R^N)$ and let W be the closure of U with respect to the norm

$$\|u\|_W = \left(\int_{R^N}\left(\sum_{j=1}^N |(\partial_j - ia_j)u|^2 + V_+\,|u|^2\right)dx + \|u\|^2\right)^{1/2} \tag{2.5}$$

where $V_+ = \max(0, V)$.

By (1.3) and (1.8), we can easily deduce

$$\int_{R^N} V_-\,|u|^2\,dx \leqslant C_1 \sum_{j=1}^N \|(\partial_j - ia_j)u\|^2 + C_2\,\|u\|^2$$

for all u in W.

Further we define a bilinear form b on $W \times W$ by the equation

$$b(u, v) = \int_{R^N} V u \bar{v}\,dx + \sum_{j=1}^N ((\partial_j - ia_j)u,\, (\partial_j - ia_j)v). \tag{2.6}$$

Then for all u in $D(F_0)$ and $v \in U$, $b(u, v) = (F_0 u, v)$ and (2.1) is clear from (2.5). We see by Lemma 2.1 that we need only verify inequalities (2.2) and (2.3), that is, we need to find three positive constants K_1, K_2 and K_3 such that for all u, v in W,

$$|b(u, v)| \leqslant K_1\,\|u\|_W \cdot \|v\|_W \tag{2.7}$$

$$\|u\|_W^2 \leqslant K_2\left(b(u, u) + K_3\,\|u\|^2\right). \tag{2.8}$$

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In fact,

$$
\begin{aligned}
|b(u, u)| &\leqslant \int_{R^N} \left(\sum_{j=1}^{N} |(\partial_j - ia_j)u|^2 + |V|\,|u|^2 \right) dx \\
&\leqslant \int_{R^N} \left(\sum_{j=1}^{N} |(\partial_j - ia_j)u|^2 + V_+\,|u|^2 \right) dx \\
&\qquad + C_1 \int_{R^N} \sum_{j=1}^{N} |(\partial_j - ia_j)u|^2\, dx + C_2 \|u\|^2 \\
&\leqslant (1 + C_1) \int_{R^N} \left(\sum_{j=1}^{N} |(\partial_j - ia_j)u|^2 + V_+\,|u|^2 \right) dx + C_2 \|u\|^2 .
\end{aligned}
$$

Thus, there is a constant $K_1 > 0$ such that (2.7) holds. Also,

$$
b(u, u) = \|u\|_W^2 - \int_{R^N} V_-\,|u|^2\, dx - \|u\|^2 ,
$$

so we have

$$
\begin{aligned}
\|u\|_W^2 &= b(u, u) + \int_{R^N} V_-\,|u|^2\, dx + \|u\|^2 \\
&\leqslant b(u, u) + C_1 \|u\|_W^2 + (1 + C_2) \|u\|^2 .
\end{aligned}
$$

Since $0 \leqslant C_1 < 1$, there exist $K_2, K_3 > 0$ such that (2.8) holds. By Lemma 2.1, F_0 has a regularly accretive extension $F_{\vec{a},V}$ in $L^2(R^N)$. ∎

Obviously, $F_{\vec{a},V}$ is also a self-adjoint extension in the sense of form. What is the connection between $F_{\vec{a},V}$ and $H_{\vec{a},V}$? The following result answers this question.

Theorem 2.3. $F_{\vec{a},V} = H_{\vec{a},V}$.

Proof: From the proof of Theorem 2.2, we have

$$
\left(H_{\vec{a},V}u, v\right) = \int_{R^N} \left(\sum_{j=1}^{N} (\partial_j - ia_j)u \cdot \overline{(\partial_j - ia_j)v} + Vu\bar{v} \right) dx
$$

for $u \in D\left(H_{\vec{a},V}\right)$, $v \in W$.

Since $F_{\vec{a},V}$ is a regularly accretive extension of F_0, we have $u \in D\left(F_{\vec{a},V}\right)$ and $F_{\vec{a},V}u = H_{\vec{a},V}u$. Therefore, $F_{\vec{a},V}$ is an extension of $H_{\vec{a},V}$ and $H_{\vec{a},V} = F_{\vec{a},V}$ for $F_{\vec{a},V}$ and $H_{\vec{a},V}$ are both self-adjoint. ∎

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3. The maximal self-adjoint realisation of P

Given the differential operator P, we can define a maximal realisation $\widetilde{H}_{\vec{a},V}$ of P in $L^2(R^N)$ as follows:

$$D\left(\widetilde{H}_{\vec{a},V}\right) = \Big\{u \in L^2(R^N) : (\partial_j - ia_j)u \in L^2_{\mathrm{loc}}(R^N),$$
$$|V|^{1/2} u \in L^2_{\mathrm{loc}}(R^N),\ Pu \in L^2(R^N)\Big\},$$
$$\widetilde{H}_{\vec{a},V} u = Pu, \qquad u \in D\left(\widetilde{H}_{\vec{a},V}\right),$$

where P acts on u in the distribution sense. It is clear that $\widetilde{H}_{\vec{a},V}$ is the extension of $H_{\vec{a},V}$ obtained in Section 1. In fact, we have

Theorem 3.1. $\widetilde{H}_{\vec{a},V} = H_{\vec{a},V}$.

Corollary 3.2. *$H_{\vec{a},V}$ is the only self-adjoint realisation of P in $L^2(R^N)$.*

From (1.3) and $\left(H_{\vec{a},V}u, v\right) = \sum_{j=1}^{N}((\partial_j - ia_j)u, (\partial_j - ia_j)v) + \int_{R^N} Vu\bar{v}dx$ for $u, v \in C_0^\infty(R^N)$, we can easily find $k > 0$ such that

$$\begin{aligned} k\|\varphi\|^2 + \sum_{j=1}^{N}\int_{R^N} |(\partial_j - ia_j)\varphi|^2\,dx + \int_{R^N}|V|\cdot|\varphi|^2\,dx &\geqslant \left(\left(H_{\vec{a},V} + k\right)\varphi, \varphi\right) \\ \geqslant \|\varphi\|^2 + \frac{1-C_1}{2}\left(\int_{R^N}\sum_{j=1}^{N}|(\partial_j - ia_j)\varphi|^2\,dx + \int_{R^N}|V|\cdot|\varphi|^2\,dx\right) \end{aligned} \tag{3.1}$$

for all $\varphi \in C_0^\infty(R^N)$. Thus, we may define a norm on $C_0^\infty(R^N)$ as follow:

$$\|\varphi\|_1 = \left(\left(H_{\vec{a},V} + k\right)\varphi, \varphi\right)^{1/2}.$$

By completing $C_0^\infty(R^N)$ in the norm $\|\cdot\|_1$, we obtain a Hilbert space which we denote by M. From [**9**, Theorem 2.1], we have

$$M = \{u \in L^2(R^N) : (\partial_j - ia_j)u \in L^2(R^N),\ |V|^{1/2}u \in L^2(R^N)\}. \tag{3.2}$$

For the proof of the Theorem 3.1, we need

Lemma 3.3. *If there is a $k' > 0$ such that for all $\varphi \in C_0^\infty(R^N)$,*

$$\sum_{j=1}^{N}\|(\partial_j - ia_j)\varphi\|^2 + \int_{R^N} V|\varphi|^2\,dx + k\|\varphi\|^2 \geqslant k'\|\varphi\|^2,$$

where k is as in (3.1). Then the map $u \to (P+k)u$ is an injective map from $D\left(\widetilde{H}_{\vec{a},V}+k\right)$ into $L^2(R^N)$.

PROOF: Suppose $u \in D\left(\widetilde{H}_{\vec{a},V}+k\right)$ such that $\left(\widetilde{H}_{\vec{a},V}+k\right)u=0$. For any $\varepsilon>0$, define $u_\varepsilon = u/(1+\varepsilon|u|)$, then we have

(i) $u_\varepsilon \in L^2_{\text{loc}}(R^N)$, $(\partial_j - ia_j)u_\varepsilon \in L^2_{\text{loc}}(R^N)$,

(ii) $u_\varepsilon \to u$, $(\partial_j - ia_j)u_\varepsilon \to (\partial_j - ia_j)u$ in $L^2_{\text{loc}}(R^N)$ as $\varepsilon \to 0$.

In fact, $u_\varepsilon \in L^2_{\text{loc}}(R^N)$ is obvious, and by $|u_\varepsilon - u| = \left(\varepsilon|u|^2\right)/(1+\varepsilon|u|) \leqslant |u|$ and the dominated convergence theorem, we obtain $u_\varepsilon \to u$ as $\varepsilon \to 0$ in $L^2_{\text{loc}}(R^N)$. Also since $u \in L^2(R^N) \subset L^1_{\text{loc}}(R^N)$, $a_j u \in L^1_{\text{loc}}(R^N)$, $1 \leqslant j \leqslant N$, we can deduce $\partial_j u \in L^1_{\text{loc}}(R^N)$ and $D\left(\widetilde{H}_{\vec{a},V}\right) \subset H^{1,1}_{\text{loc}}(R^N)$. By (1.8) we have $\partial_j|u| \in L^2_{\text{loc}}(R^N)$. For any φ in $C_0^\infty(R^N)$,

$$\partial_j|u_\varepsilon\varphi| = u_\varepsilon(\partial_j\varphi) + \varphi\cdot\frac{\partial_j u - \varepsilon u_\varepsilon \partial_j|u|}{1+\varepsilon|u|}. \tag{3.3}$$

This implies

$$\begin{aligned}
&|(\partial_j - ia_j)(u_\varepsilon\varphi) - (\partial_j - ia_j)(u\varphi)| \\
&\quad = \left|(u_\varepsilon - u)(\partial_j\varphi) + \varphi\cdot\frac{-\varepsilon u_\varepsilon\partial_j|u| - \varepsilon|u|(\partial_j - ia_j)u}{1+\varepsilon|u|}\right| \\
&\quad \leqslant |(\partial_j\varphi)u| + |(\partial_j - ia_j)u|\cdot|\varphi| \in L^2(R^N),
\end{aligned}$$

therefore $(\partial_j - ia_j)(u_\varepsilon\varphi) \in L^2(R^N)$. Using (3.3) and the dominated convergence theorem, we obtain $(\partial_j - ia_j)u_\varepsilon \to (\partial_j - ia_j)u$ as $\varepsilon \to 0$ in $L^2_{\text{loc}}(R^N)$. So we have proved (i) and (ii).

For any real function $\varphi \in C_0^\infty(R^N)$, $u_\varepsilon\varphi^2 \in M \cap L^\infty(R^N)$. By (3.2), we have

$$\sum_{j=1}^N \left((\partial_j - ia_j)u, (\partial_j - ia_j)(u_\varepsilon\varphi^2)\right) + \int_{R^N}(V+k)u\bar{u}_\varepsilon\varphi^2\,dx = 0.$$

Then

$$\begin{aligned}
&\int_{R^N}\left((\partial_j - ia_j)u\cdot\overline{(\partial_j - ia_j)(u_\varepsilon\varphi^2)} + V|u_\varepsilon|^2\varphi^2\right)dx + \int_{R^N}k|u_\varepsilon|^2\varphi^2\,dx \\
&= \int_{R^N}k(u_\varepsilon - u)\bar{u}_\varepsilon\varphi^2\,dx \\
&\quad + \int_{R^N}\left((\partial_j - ia_j)(u_\varepsilon - u)\cdot\overline{(\partial_j - ia_j)(u_\varepsilon\varphi^2)} - V(u - u_\varepsilon)\bar{u}\varphi^2\right)dx \\
&=: I_\varepsilon.
\end{aligned}$$

Since $(u-u_\varepsilon)\overline{u}_\varepsilon$ $\left(=\left(\varepsilon|u|^3\right)/\left((1+\varepsilon|u|)^2\right)\geqslant 0\right)$ is real, $\lim_{\varepsilon\to 0}\int_{R^N}(u_\varepsilon-u)\overline{u}_\varepsilon\varphi^2dx=0$ and $\lim_{\varepsilon\to 0}\int_{R^N}(\partial_j-ia_j)(u_\varepsilon-u)\cdot\overline{(\partial_j-ia_j)(u_\varepsilon\varphi^2)}dx=0$, we have $\lim_{\varepsilon\to 0}\operatorname{Im}I_\varepsilon=0$.

By $(u-u_\varepsilon)\overline{u}_\varepsilon\geqslant 0$, we have

$$\begin{aligned}\operatorname{Re}I_\varepsilon&=\operatorname{Re}\left(\int_{R^N}(\partial_j-ia_j)(u_\varepsilon-u)\cdot\overline{(\partial_j-ia_j)(u_\varepsilon\varphi^2)}dx+k\int_{R^N}(u_\varepsilon-u)\overline{u}_\varepsilon\varphi^2dx\right)\\&\quad-\int_{R^N}V(u-u_\varepsilon)\overline{u}_\varepsilon\varphi^2dx\\&\leqslant\operatorname{Re}(\cdots)+\int_{R^N}V_-(u-u_\varepsilon)\overline{u}_\varepsilon\varphi^2dx\\&\leqslant\operatorname{Re}(\cdots)+\left(\int_{R^N}V_-|u-u_\varepsilon|^2\varphi^2dx\right)^{1/2}\left(\int_{R^N}V_-\varphi^2|u_\varepsilon|^2dx\right)^{1/2}.\end{aligned}$$

So from (i) and (ii), $\overline{\lim_{\varepsilon\to 0}}(\operatorname{Re}I_\varepsilon+\operatorname{Im}I_\varepsilon)\leqslant 0$. Also, since

$$\begin{aligned}&(\partial_j-ia_j)u_\varepsilon\cdot\overline{(\partial_j-ia_j)(u_\varepsilon\varphi^2)}\\&\quad=\sum_{j=1}^N|(\partial_j-ia_j)(u_\varepsilon\varphi)|^2-|u_\varepsilon|^2\cdot|\nabla\varphi|^2+2i\operatorname{Im}\left(\sum_{j=1}^N\overline{u}_\varepsilon\varphi\partial_j\varphi\cdot(\partial_j-ia_j)u_\varepsilon\right),\end{aligned}$$

we have

$$\begin{aligned}&\int_{R^N}\left(\sum_{j=1}^N|(\partial_j-ia_j)(u_\varepsilon\varphi)|^2-|u_\varepsilon|^2|\nabla\varphi|^2\right.\\&\quad\left.+2i\operatorname{Im}\sum_{j=1}^N(\overline{u}_\varepsilon\varphi\cdot\partial_j\varphi\cdot(\partial_j-ia_j)u_\varepsilon)+V|u_\varepsilon|^2\varphi^2+k|u_\varepsilon|^2\varphi^2\right)dx\equiv I_\varepsilon.\end{aligned}$$

By $\lim_{\varepsilon\to 0}\operatorname{Im}I_\varepsilon=0$,

$$\lim_{\varepsilon\to 0}\left|\operatorname{Im}\sum_{j=1}^N\int_{R^N}\overline{u}_\varepsilon\varphi\partial_j\varphi\cdot(\partial_j-ia_j)u_\varepsilon dx\right|=0.$$

Then from (3.2) and the condition, we have

$$\sum_{j=1}^N\|(\partial_j-ia_j)(u_\varepsilon\varphi)\|^2+\int_{R^N}V|u_\varepsilon\varphi|^2dx+\int_{R^N}k|u_\varepsilon\varphi|^2dx\geqslant k'\|u_\varepsilon\varphi\|^2$$

and

$$
\begin{aligned}
0 &\geqslant \overline{\lim_{\varepsilon\to 0}}(\operatorname{Re} I_\varepsilon + \operatorname{Im} I_\varepsilon) \\
&= \overline{\lim_{\varepsilon\to 0}} \operatorname{Re} \int_{R^N} \left(\sum_{j=1}^{N} |(\partial_j - ia_j)(u_\varepsilon \varphi)|^2 - |u_\varepsilon|^2 |\nabla\varphi|^2 + k|u_\varepsilon|^2 \varphi^2 + V|u_\varepsilon\varphi|^2 \right) dx \\
&\geqslant \overline{\lim_{\varepsilon\to 0}} \left(k' \|u_\varepsilon \varphi\|^2 - \int_{R^N} |u_\varepsilon|^2 |\nabla\varphi|^2 \, dx \right) \\
&= k' \|u\varphi\|^2 - \int_{R^N} |u|^2 |\nabla\varphi|^2 \, dx,
\end{aligned}
$$

that is, $k' \|u\varphi\|^2 \leqslant \int_{R^N} |u|^2 |\nabla\varphi|^2 \, dx$. Taking $\varphi_\varepsilon(x) = \Psi(x/\varepsilon)$, where $\Psi \in C_0^\infty(R^N)$, $\Psi(x) = 1$ when $|x| \leqslant 1$; $\Psi(x) = 0$ when $|x| \geqslant 2$ and $0 \leqslant \Psi \leqslant 1$, $|\partial_j \Psi(y/\varepsilon)| = o(1/\varepsilon)$ $(\varepsilon \to \infty)$, we have $k' \|u\| \leqslant 0$ and $u \equiv 0$. ∎

PROOF OF THEOREM 3.1: $\left(H_{\vec a,V} + k\right)^{-1}$ is a bounded linear operator in $L^2(R^N)$ for suitable $k > 0$. Suppose $u \in D\left(\widetilde{H}_{\vec a,V} + k\right)$, $v = u - \left(H_{\vec a,V} + k\right)^{-1} \left(\left(\widetilde{H}_{\vec a,V} + k\right)u\right)$. Since $D\left(H_{\vec a,V}\right) \subset D\left(\widetilde{H}_{\vec a,V}\right)$, we have $v \in D\left(\widetilde{H}_{\vec a,V} + k\right)$. Also since $(P+k)v = 0$ and from (3.1), we have $v \equiv 0$ by Lemma 3.3. So $u \in D\left(H_{\vec a,V} + k\right)$ and $\widetilde{H}_{\vec a,V} = H_{\vec a,V}$. ∎

4. THE ESSENTIAL SELF-ADJOINTNESS OF P ON $C_0^\infty(R^N)$

In this section, we consider the essential self-adjoint extension of the Schrödinger operator $P = -\sum_{j=1}^{N} (\partial_j - ia_j)^2 + V$, where $\vec a \in L^4_{\text{loc}}(R^N)^N$, $\operatorname{div} \vec a \in L^2_{\text{loc}}(R^N)$, $V = V_1 + V_2$, $V_i \in L^2_{\text{loc}}(R^N)$, $i = 1, 2$, $V_1(x) \geqslant -C|x|^2$ $(C \geqslant 0)$, $0 \geqslant V_2 \in K_N$.

First, we prove the following result.

LEMMA 4.1. *Let $\vec a$, V be as above. Then there exist constants $C_3 > 0$, $C_4 > 0$ such that for all $u \in C_0^\infty(R^N)$,*

$$
\sum_{j=1}^{N} \int_{B_m} |(\partial_j - ia_j)u|^2 \, dx \leqslant C_3 \int_{B_m} |Pu|^2 \, dx + C_4 m^2 \int_{B_m} |u|^2 \, dx,
$$

where $B_m = \{x \in R^N : m/2 \leqslant |x| \leqslant 3m\}$, $m > 0$.

PROOF: Take $\xi \in C_0^\infty(R^N)$, $0 \leqslant \xi \leqslant 1$, $\xi(x) = 1$ when $1 \leqslant |x| \leqslant 2$; $\xi(x) = 0$ when $|x| \geqslant 3$ or $|x| \leqslant 1/2$. For any positive integer m, $\xi_m = \xi(x/m)$. By $V_2 \in K_N$, for any $\varepsilon > 0$, there exists $M(\varepsilon, V_2) > 0$ such that

$$|(V_2\xi_m u, \xi_m u)| \leqslant \varepsilon \int_{R^N} |\nabla(\xi_m u)|^2 \, dx + M(\varepsilon, V_2) \int_{R^N} |\xi_m u|^2 \, dx$$

for $u \in C_0^\infty(R^N)$. Set $K = \max_{y \in R^N} |\nabla \xi(y)|$, then $|\nabla \xi_m(x)| \leqslant K/m$ and

$$|\partial_j(\xi_m u)|^2 \leqslant 2(\partial_j \xi_m)^2 |u|^2 + 2\xi_m^2 |\partial_j u|^2 . \tag{4.1}$$

Therefore, there exists a constant $C_\varepsilon > 0$ such that for any $u \in C_0^\infty(R^N)$.

$$\left|(V_2 u, \xi_m^2 u)\right| \leqslant 2\varepsilon \int_{R^N} \xi_m^2 |\nabla|u||^2 \, dx + C_\varepsilon \int_{R^N} |u|^2 \, dx.$$

Taking $\varepsilon = 1/16$, we have

$$\begin{aligned} \left|(Pu, \xi_m^2 u)\right| &= \left|\left(\left(\nabla - i\vec{a}\right)u, \left(\nabla - i\vec{a}\right)(\xi_m^2 u)\right) + (Vu, \xi_m^2 u)\right| \\ &\geqslant \left|\sum_{j=1}^N ((\partial_j - ia_j)u, (\partial_j - ia_j)(\xi_m^2 u))\right| - C(3m)^2 \int_{B_m} |u|^2 \, dx \\ &\quad - \frac{1}{8} \int_{B_m} |\nabla|u||^2 \, dx - C_{1/16} \int_{B_m} |u|^2 \, dx. \end{aligned} \tag{4.2}$$

Also since

$$\begin{aligned} &\sum_{j=1}^N ((\partial_j - ia_j)u, (\partial_j - ia_j)(\xi_m^2 u)) \\ &\quad = \sum_{j=1}^N ((\partial_j - ia_j)u, \xi_m^2(\partial_j - ia_j)u) + \sum_{j=1}^N ((\partial_j - ia_j)u, 2\xi_m(\partial_j \xi_m)u), \end{aligned}$$

we have

$$\begin{aligned} \left|(Pu, \xi_m^2 u)\right| \geqslant &\sum_{j=1}^N \int_{B_m} |(\partial_j - ia_j)u|^2 \, dx - \frac{1}{4} \sum_{j=1}^N \int_{B_m} |(\partial_j - ia_j)u|^2 \, dx \\ &- 4 \int_{B_m} |u|^2 |\nabla \xi_m|^2 \, dx - C(3m)^2 \int_{B_m} |u|^2 \, dx \\ &- \frac{1}{8} \int_{B_m} |\nabla|u||^2 \, dx - C_{1/16} \int_{B_m} |u|^2 \, dx. \end{aligned}$$

This implies that

$$\sum_{j=1}^N \int_{B_m} |(\partial_j - ia_j)u|^2 \, dx \leqslant C_3' \int_{B_m} |Pu|^2 \, dx + C_4' m^2 \int_{B_m} |u|^2 \, dx + \frac{1}{8} \int_{B_m} |\nabla|u||^2 \, dx$$

for suitable constants $C_3', C_4' > 0$. Also by (1.8), we have

$$\sum_{j=1}^{N} \int_{B_m} |(\partial_j - ia_j)u|^2 \, dx \leqslant C_3 \int_{B_m} |Pu|^2 \, dx + C_4 m^2 \int_{B_m} |u|^2 \, dx$$

for suitable constants $C_3, C_4 > 0$. ∎

THEOREM 4.2. *Let $\vec{a}$, V be as above, then $P = -\sum_{j=1}^{N} (\partial_j - ia_j)^2 + V$ is essential self-adjoint on $C_0^\infty(R^N)$.*

PROOF: Since P is symmetric, if we want to prove that P is essential self-adjoint on $C_0^\infty(R^N)$, we only need to prove that for any $\varphi \in C_0^\infty(R^N)$, if $f \in L^2(R^N)$, $(f, P\varphi) = 0$, then $f \equiv 0$. Thus, in the sequel, we suppose $f \in L^2(R^N)$, $(f, P\varphi) = 0$ for any $\varphi \in C_0^\infty(R^N)$.

If V satisfies (H), put

$$V_1^{(k)}(x) = \begin{cases} V_1(x) & |x| \leqslant k, \\ -Ck^2 & |x| > k. \end{cases}$$

Then $\left(V_1^{(k)}\right)_-$ is a bounded function. By the discussion in Section 1, we have $P_k := -\sum_{j=1}^{N} (\partial_j - ia_j)^2 + V_2 + V_1^{(k)}$ is essential self-adjoint on $C_0^\infty(R^N)$ in the sense of form and we denote the self-adjoint realisation of P_k by $\overline{P}_k$. Moreover, by (4.2) we have

$$\left|(\overline{P}_k u, \xi_m^2 u)\right| = \left|\left(\left(\nabla - i\vec{a}\right)u, \left(\nabla - i\vec{a}\right)(\xi_m^2 u)\right) + \left(\left(V_2 + V_1^{(k)}\right)u, \xi_m^2 u\right)\right|$$

for $u \in D(\overline{P}_k)$. Using the same methods in the proof of Lemma 4.1, we have there exist constants $C_5, C_6 > 0$ such that

$$\sum_{j=1}^{N} \int_{B_m} |(\partial_j - ia_j)u|^2 \, dx \leqslant C_5 \int_{B_m} \left|\overline{P}_k u\right|^2 dx + C_6 m^2 \int_{B_m} |u|^2 \, dx. \tag{4.3}$$

Take $\eta \in C_0^\infty(R^N)$, $\eta(x) = 1$ for $|x| \leqslant 1$; $\eta(x) = 0$ for $|x| \geqslant 2$ and set $\eta_m(x) = \eta(x/m)$. For any $u \in C_0^\infty(R^N)$,

$$(P + i)(u\eta_m) = \eta_m(P + i) - 2\nabla\eta_m \cdot \vec{D}u - (\Delta\eta_m)u$$

where $\vec{D} = (\partial_1 - ia_1, \partial_2 - ia_2, \cdots, \partial_N - ia_N)$. From this, we have

$$(f, \eta_m(T + i)u) = (f, (\Delta\eta_m)u) + 2\left(f, \nabla\eta_m \cdot \vec{D}u\right)$$

and for any $u \in C_0^\infty(R^N)$, $k \geqslant 3m$,

$$\eta_m(T+i)u = \eta_m(P_k+i)u.$$

Taking $k = 3m$, we have

$$(f\eta_m, (P_{3m}+i)u) = (f, (\Delta\eta_m)u) + 2\left(f, \nabla\eta_m \cdot \vec{D}u\right). \tag{4.4}$$

Since $\overline{P}_{3m}$ is essential self-adjoint on $C_0^\infty(R^N)$ in the sense of form, (4.4) also holds for $u \in D(\overline{P}_k)$. Therefore there exists $u_m \in D(\overline{P}_{3m})$ such that $(\overline{P}_{3m}+i)u_m = f\eta_m$. So, we have

$$\begin{aligned} \left\| |f|^2 \eta_m^2 \right\|^2 &= \left\| (\overline{P}_{3m}+i)u_m \right\|^2 = (f, (\Delta\eta_m)u_m) + 2\left(f, \nabla\eta_m \cdot \vec{D}u_m\right) \\ &\leqslant \|f\|_{L^2(B_m)} \left(Mm^{-2}\|u_m\| + 2Mm^{-1}\left\| \left|\vec{D}u_m\right| \right\|_{L^2(B_m)} \right) \end{aligned} \tag{4.5}$$

for suitable constant $M > 0$. Since $\overline{P}_{3m}$ is a self-adjoint operator and $(\overline{P}_{3m}+i)u_m = f\eta_m$, we have $\|u_m\| \leqslant \|f\eta_m\| \leqslant \|f\|$. Also by (4.3),

$$\left\| \left|\vec{D}u_m\right| \right\|^2_{L^2(B_m)} \leqslant C_5 \int_{B_m} \left|(\overline{P}_{3m}+i)u_m\right|^2 dx + C_6 m^2 \int_{B_m} |u_m|^2 dx. \tag{4.6}$$

So from (4.5) and (4.6), there exists $C_7 > 0$ such that

$$\left| |f|^2 \eta_m^2 \right\|^2 \leqslant C_7 \|f\|_{L^2(B_m)}.$$

Let $m \to \infty$, then we have $\left\| |f|^2 \right\| = 0$; thus $f \equiv 0$. □

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