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ON A 2-KNOT GROUP WITH NONTRIVIAL CENTER

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Jonathan A. Hillman asked "Must a 2-knot whose group has non-trivial center be fibered?" We will answer this question negatively.

1. Introduction

An *n-knot* k is a locally flat submanifold of S^{n+2} which is homeomorphic to S^n. The fundamental group of $S^{n+2} - \mathring{N}(k)$ is called the *group* of k, where $N(k)$ is a tubular neighborhood of k in S^{n+2}.

In [*6*], Neuwirth showed that the center of a 1-knot group is trivial or infinite cyclic. On the other hand, Hausmann and Kervaire [*1*] proved that any finitely generated abelian group is the center of an n-knot group $(n \geq 3)$. For $n = 2$, the author [*8*] showed that there are fibered 2-knots whose groups have the centers 1, Z, $Z \oplus Z_2$ and $Z \oplus Z$ respectively. Moreover, in [*2*], Hillman investigated centers of 2-knot groups and obtained some results. In particular, he shows that if a 2-knot is fibered, then the center of its group is 1, Z, $Z \oplus Z_2$ or $Z \oplus Z$, and he asks if a 2-knot whose group has nontrivial center must be fibered. In this paper we will answer his question negatively. That is:

THEOREM. *There exists a 2-knot which is not fibered and whose group has nontrivial center.*

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2. Preliminaries

For an element g of a group H, $\langle H : g\rangle$ denotes the factor group of H by the normal closure of g in H. The subgroup of H generated by a subset S of H will be denoted by $\mathrm{gp}(S)$.

Let K_1 be a 2-knot and V_1 a tubular neighborhood of K_1. Let V be a tubular neighborhood of a trivial 2-knot and $h : V \to V_1$ a homeomorphism of V onto V_1. Let K_2 be a 2-knot contained in the interior of V. Then we obtain a 2-knot $K = h(K_2)$ (*cf.* [7]).

We will calculate the group G of K by the van Kampen theorem. Let V_2 be a tubular neighborhood of K_2 in S^4 which is contained in the interior of V and let G_i be the group of K_i $(i = 1, 2)$; that is, $G_i = \pi_1\left(S^4 - \mathring{V}_i\right)$. From the definition of K, we have the following commutative diagram of homomorphisms induced by inclusions:

$$
\begin{array}{ccccc}
 & & \pi_1(\partial V_1) & & \\
 & \swarrow^{i_{1*}} & & \searrow^{i_{2*}} & \\
(1)\quad G_1 = \pi_1\left(S^4 - \mathring{V}_1\right) & & & & \pi_1\left(V_1 - h(\mathring{V}_2)\right) \\
 & \searrow & & \swarrow & \\
 & & \pi_1\left(S^4 - h(\mathring{V}_2)\right) = G\,. & &
\end{array}
$$

Furthermore, it is easy to see that the inclusion j of $V - \mathring{V}_2$ into $S^4 - \mathring{V}_2$ induces the isomorphism j_* of $\pi_1\left(V - \mathring{V}_2\right)$ onto G_2. Therefore, we get the diagram, (2), of isomorphisms:

$$
(2)\qquad \pi_1\left(V_1 - h(\mathring{V}_2)\right) \xleftarrow[\cong]{(h|V-\mathring{V}_2)_*} \pi_1\left(V - \mathring{V}_2\right) \xrightarrow[\cong]{j_*} G_2\,.
$$

Put $y = i_{1*}(\tilde{y})$ and $c = j_*\left(h \mid V-\mathring{V}_2\right)_*^{-1} i_{2*}(\tilde{y})$, where $\tilde{y}$ is a generator of the infinite cyclic group $\pi_1(\partial V_1)$. Then, from diagrams (1)

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and (2), we obtain $G = \left\langle G_1 * G_2 : yc^{-1} \right\rangle$.

Let μ be the order of c in G_2 (if it is infinite, then put $\mu = 0$) and let $\tilde{G}_1 = \left\langle G_1 : y^{\mu} \right\rangle$. Then y has the order μ in $\tilde{G}_1$. Thus it follows that G is a free product of $\tilde{G}_1$ and G_2 with subgroups $\mathrm{gp}(y)$ and $\mathrm{gp}(c)$ amalgamated under the mapping $y \to c$.

LEMMA. *Suppose that* G_2 *is not infinite cyclic. If* $c\ (\neq 1) \in [G_2, G_2]$ *and* $\tilde{G}_1 \ncong Z_\mu$ *, then the commutator subgroup* $[G, G]$ *is not finitely generated.*

Proof. To complete the proof, we use the subgroup theorem for a free product with an amalgamated subgroup [3, Theorem 5]. Let generating systems of $\tilde{G}_1$ and G_2 be α- and β-generating systems in [3], respectively. Let x be an element of G_2 mapped on a generator of $G_2/[G_2, G_2]$ by abelianization. We choose $\{x^s : s = 0, \pm 1, \ldots\}$ as α- and β-representative systems for a compatible regular extended Schreier system for G mod $[G, G]$ (see [3]). Then the associated α- and β-double coset representative systems $\{D_\alpha\}$, $\{D_\beta\}$ for G mod $([G, G], \tilde{G}_1)$ and G mod $([G, G], G_2)$ are $\{x^s : s = 0, \pm 1, \ldots\}$ and $\{1\}$ respectively, and the v-double coset representatives $\{D_\beta E_v\}$ for G mod $([G, G], \mathrm{gp}(c))$ are $\{x^s : s = 0, \pm 1, \ldots\}$. Therefore, in Theorem 5 of [3], there is no t-symbol. Moreover, since $y = c \in [G_2, G_2] \subset [G, G]$, it follows that $x^s\tilde{G}_1x^{-s} \subset [G, G]$ for each s . Hence, from Theorem 5, $[G, G]$ is a tree product of an infinite number of factors $\left\{[G_2, G_2],\ x^s\tilde{G}_1x^{-s},\ s = 0, \pm 1, \ldots\right\}$ with the subgroups $x^s\mathrm{gp}(c)x^{-s}$ and $x^s\mathrm{gp}(y)x^{-s}$ amalgamated under the mapping $x^scx^{-s} \to x^syx^{-s}$ $(s = 0,\ 1, \ldots)$. Since $\tilde{G}_1 \ncong Z_\mu \cong \mathrm{gp}(y)$, we have $x^s\tilde{G}_1x^{-s} \neq x^s\mathrm{gp}(y)x^{-s}$ for each s . Hence, by [4, p. 53], $[G, G]$ is not finitely generated.

3. Proof of theorem

We will give two examples. One has center Z and the other has center Z_2. We note that the latter can not be realized as a center of any fibered 2-knot group.

EXAMPLE 1. Let K_1 and K_2 be the 2- and 6-twist-spun 2-knots of the trefoil respectively [9]. Then we have

$$G_1 = \langle y, d : ydy^{-1} = d^{-1}, d^3 \rangle$$

and

$$G_2 = \langle x, a, b : xax^{-1} = b, xbx^{-1} = a^{-1}b, [[a, b], a], [[a, b], b] \rangle$$

[8].

Let V_i be a tubular neighborhood of K_i $(i = 1, 2)$ in S^4. Let C be a simple closed curve in $S^4 - V_2$ which represents an element $c = [a, b]$ of G_2 and N a tubular neighborhood of C in S^4 such that $N \cap V_2 = \emptyset$. Then, since N is homeomorphic to $S^1 \times B^3$, the manifold $S^4 - \mathring{N} \approx S^2 \times B^2$ is considered as a tubular neighborhood of a trivial 2-knot in S^4. Therefore, in the previous section, we can take $V = S^4 - \mathring{N}$. Let $h : V \to V_1$ be a homeomorphism of V onto V_1 such that $j_*(h \mid V - \mathring{V}_2)_*^{-1} i_{2*}(\tilde{y}) = c$ for a generator $\tilde{y}$ of $\pi_1(\partial V_1)$ with $i_{1*}(\tilde{y}) = y$. Then, from Section 2, we obtain a 2-knot $K = h(K_2)$ with the group $G = \left\langle \tilde{G}_1 * G_2 : yc^{-1} \right\rangle$.

The element c has infinite order in G_2. Therefore, we have $\tilde{G}_1 = G_1$. Thus G is a free product of G_1 and G_2 with amalgamated subgroups $\mathrm{gp}(y)$ and $\mathrm{gp}(c)$. Hence, by [5, p. 211], the center of G is $\mathrm{gp}(y) \cap C(G_1) \cap C(G_2)$, where $C(G_i)$ is the center of G_i $(i = 1, 2)$. Consequently, G has the non-trivial center $\mathrm{gp}(y^2) \cong Z$ because

$C(G_1) = \text{gp}(y^2)$ and $C(G_2) = \text{gp}(x^6, c)$ [8].

Furthermore, by virtue of the lemma, it follows that K is not fibered.

EXAMPLE 2. Let K_1 and K_2 be the 2- and 5-twist-spun 2-knots of the trefoil, respectively. Then the group G_2 of K_2 is

$$\langle x, a, b : xax^{-1} = b, xbx^{-1} = a^{-1}b, a^5 = (ab)^3 = (aba)^2 \rangle ,$$

and the center $C(G_2)$ is $\text{gp}(x[a, b^{-1}], (aba)^2) \cong Z \oplus Z_2$ [8], [9]. We choose the element aba of G_2 as c. Then, in the same way as above, we can construct a 2-knot whose group has center Z_2 and which is not fibered.

Note. Recently, T. Kanenobu communicated to the author that he has obtained another example of such a 2-knot by Fox's hyperplane cross section method.

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