ON THE SECOND-LARGEST SYLOW SUBGROUP OF A FINITE SIMPLE GROUP OF LIE TYPE

S. P. GLASBY, ALICE C. NIEMEYER and TOMASZ POPIEL[✉]

Abstract

Let T be a finite simple group of Lie type in characteristic p, and let S be a Sylow subgroup of T with maximal order. It is well known that S is a Sylow p-subgroup except for an explicit list of exceptions and that S is always 'large' in the sense that $|T|^{1/3} < |S| \leqslant |T|^{1/2}$. One might anticipate that, moreover, the Sylow r-subgroups of T with $r \neq p$ are usually significantly smaller than S. We verify this hypothesis by proving that, for every T and every prime divisor r of $|T|$ with $r \neq p$, the order of the Sylow r-subgroup of T is at most $|T|^{2\lfloor \log_r(4(\ell+1)r) \rfloor / \ell} = |T|^{O(\log_r(\ell)/\ell)}$, where ℓ is the Lie rank of T.

1. Introduction

Given a finite simple group T of Lie type, it is natural to ask: what is the order of the largest Sylow subgroup of T? This question dates back to at least 1955 and the articles [1, 2] of Artin, who showed that if T is a classical group in characteristic p and S is a Sylow subgroup of T with maximal order, then S is a Sylow p-subgroup of T unless

- $T \cong \mathrm{PSL}(2, p)$ with p a Mersenne prime,
- $T \cong \mathrm{PSL}(2, r-1)$ with r a Fermat prime,
- $T \cong \mathrm{PSL}(2, 8)$, $T \cong \mathrm{PSU}(3, 3)$ or $T \cong \mathrm{PSU}(4, 2)$.

In these cases, S is a Sylow s-subgroup with $s = 2$, r, 3, 2 or 3, respectively. Artin's investigations were extended by Kimmerle *et al.* [8] in 1990 to the cases where T has exceptional Lie type, with the conclusion that S is always a Sylow p-subgroup. Moreover, as one immediately observes upon inspecting the order formulae for the finite simple groups of Lie type [5, Table 6], S is always 'large' (in both the

classical and the exceptional cases) in the sense that $|S| > |T|^c$ for some constant c. Indeed one can take $c = 1/3$ by [8, Theorem 3.5]. (On the other hand, $|S| \leqslant |T|^{1/2}$ by [8, Theorem 3.6].)

With the aforementioned question settled, it is natural to ask: how large are the *other* Sylow subgroups of T? Buekenhout [3] approached this question by investigating the ratios $a_i = \log(p_1^{n_1})/\log(p_i^{n_i})$ in the prime factorisation $|T| = \prod_{i=1}^{s} p_i^{n_i}$, where the labelling is such that $p_1^{n_1} > p_2^{n_2} > \cdots > p_s^{n_s}$. Specifically, he asked when $a_i \leqslant \log(3)/\log(2)$, calling the corresponding primes p_i *good contributors* to the order of T and explaining that the choice of constant $\log(3)/\log(2)$ was in some sense arbitrary but motivated by computational evidence and applications in geometry. The conclusion was that if T does not have Lie type A_1, A_2 or $^2A_2$, then $|T|$ has at most one good contributor. More precisely, only 21 such groups have a good contributor r distinct from the characteristic; in all of these cases, $r \leqslant 5$ by [3, Theorem 4.1]. The cases where T has type A_1, A_2 or $^2A_2$ produce many further examples, and these were left open. In addition to having geometric applications as alluded to in [3], Buekenhout's result has also recently been used to study certain profinite groups [4].

The purpose of this note is to prove the following result.

THEOREM 1.1. *Let q be a prime power and let $T = T(q)$ be a finite simple group of Lie type, as listed Table 1. Let r be a prime dividing $|T|$ but not q and let R be a Sylow r-subgroup of T. Then, for K and M as in Table 1, we have $|R| \leqslant |T|^{(\lfloor \log_r(M) \rfloor + 1)/K}$, except in the cases listed in the final column of the table.*

REMARK 1.2.

(i) The conditions listed in the fifth column of Table 1 mitigate occurrences of isomorphisms between groups in different rows. In particular, observe that $G_2(2) \cong {}^2A_2(3).2$, $^2G_2(3) \cong A_1(8).3$ and that $^2B_2(2)$ is solvable. Note also that we include the Tits group $^2F_4(2)'$ in Theorem 1.1.

(ii) It follows from Table 1 that, for all T, we have $K \geqslant \ell/2$ and $M \leqslant 4(\ell + 1)$, where ℓ is the Lie rank of T as in [9, Tables 5.1.A and 5.1.B]. The upper bound on $|R|$ given in Theorem 1.1 therefore implies the bound $|R| \leqslant |T|^{2\lfloor \log_r(4(\ell+1)r) \rfloor/\ell}$ claimed in the Abstract.

(iii) Theorem 1.1 may be viewed as a refinement of [3, Theorem 4.1], in the following sense. Let S and R denote the largest and second-largest Sylow subgroups of T, so that $|S| = p_1^{n_1}$ and $|R| = p_2^{n_2}$ in the above notation. As noted earlier, $|S| > |T|^c$ for some constant c. On the other hand, as noted above, Theorem 1.1 implies that $|R| \leqslant |T|^{c' \log_{p_2}(\ell)/\ell}$ for some constant c', where ℓ is the Lie rank of T. We therefore obtain the following lower bound on a_2, which explains why Buekenhout's *good contributors* are so rare:

$$a_2 = \frac{\log|S|}{\log|R|} > \frac{\ell c \log(|T|)}{c' \log_{p_2}(\ell) \log(|T|)} = \frac{c\ell}{c' \log_{p_2}(\ell)}.$$

TABLE 1. Data for the bound $|R| \leqslant |T|^{(\lfloor \log_r(M) \rfloor + 1)/K}$ in Theorem 1.1 and the Lie rank ℓ of each group T (from [9, Tables 5.1.A and 5.1.B]).

T	K	M	ℓ	Conditions on T	Exceptions
$A_n(q)$	n	$n+1$	n	$n \geqslant 1, (n, q) \notin \{(1,2),(1,3)\}$	
$^2A_n(q)$	$n/2$	$2(n+1), \ 2 \mid n$ $2n, \qquad 2 \nmid n$	$\lfloor \frac{n+1}{2} \rfloor$	$n \geqslant 2, (n, q) \neq (2,2)$	
$B_n(q)$	n	$2n$	n	$n \geqslant 2, (n, q) \neq (2,2)$	
$C_n(q)$	n	$2n$	n	$n \geqslant 3, \ q$ odd	
$D_n(q)$	$n/2$	$2(n-1)$	n	$n \geqslant 4$	
$^2D_n(q)$	$n/2$	$2n$	$n-1$	$n \geqslant 4$	
$^2B_2(q)$	2	4	1	$q = 2^{2m+1}, m \geqslant 1$	
$^3D_4(q)$	6	12	2		$(q, r) = (3, 13)$
$E_6(q)$	12	12	6		$(q, r) = (3, 13)$
$^2E_6(q)$	12	18	4		
$E_7(q)$	18	18	7		
$E_8(q)$	29	30	8		$(q, r) = (2, 31)$
$F_4(q)$	12	12	4		$(q, r) = (3, 13)$
$^2F_4(q)'$	6	12	2	$q = 2^{2m+1}, m \geqslant 0$	
$G_2(q)$	6	6	2	$q \geqslant 3$	$(q, r) = (3, 13)$
$^2G_2(q)$	$7/2$	6	1	$q = 3^{2m+1}, m \geqslant 1$	

(iv) The questions considered here have also been investigated for the remaining nonabelian finite simple groups, namely, the alternating groups and the 26 sporadic simple groups. Precise answers can, of course, be obtained for the sporadic groups, and these are recorded in [8, Table L.5] and [3, Section 2]. Orders of Sylow subgroups of the alternating group Alt_n may be computed using the classical formula of Legendre [10] for the prime factorisation of $n!$. As one might anticipate, $p_1 = 2$ and $p_2 = 3$ almost always, indeed, unless $n \in \{5, 6, 7, 9\}$ by [3, Theorem 3.7]. Moreover, $p_1^{n_1} \leqslant (n!/2)^{0.363}$ by [8, Table L.4].

We prove some preliminary lemmas in Section 2 before giving the proof of Theorem 1.1 in Section 3.

2. Supporting lemmas

As in [1, 2, 8], we consider the cyclotomic factorisations for the finite simple groups of Lie type (compare with [8, Definition 4.4]). Writing Φ_i for the ith cyclotomic polynomial and d for the number of diagonal outer automorphisms of $T = T(q)$, this factorisation has the form

$$d|T| = q^{e_0} \prod_{i=1}^{M} \Phi_i(q)^{e_i},$$

TABLE 2. Numbers d and cyclotomic factorisations $d|T| = q^{e_0} \prod_{i=1}^{M} \Phi_i(q)^{e_i}$ for finite simple classical groups T. (Note: the q in $\Phi_i(q)$ is suppressed for brevity.)

T	$d = \gcd(\cdot, \cdot)$	cyclotomic factorisation of $d|T|$	M
A_n	$(n+1, q-1)$	$q^{\binom{n+1}{2}} \Phi_1^n \prod_{i=2}^{n+1} \Phi_i^{\lfloor (n+1)/i \rfloor}$	$n+1$
^2A_n	$(n+1, q+1)$	$q^{\binom{n+1}{2}} \Phi_2^n \prod_{2 \neq i \equiv 2(4)} \Phi_i^{\lfloor (2(n+1))/i \rfloor} \prod_{i \not\equiv 2(4)} \Phi_i^{\lfloor (n+1)/\mathrm{lcm}(2,i) \rfloor}$	$2(n+1)$, $2 \mid n$ $2n$, $2 \nmid n$
B_n, C_n	$(2, q-1)$	$q^{n^2} \prod_{i=1}^{2n} \Phi_i^{\lfloor 2n/\mathrm{lcm}(2,i) \rfloor}$	$2n$
D_n	$(4, q^n - 1)$	$q^{n(n-1)} \prod_{i \nmid n \text{ and } i \mid 2n} \Phi_i^{2n/i-1} \prod_{i \mid n \text{ or } i \nmid 2n} \Phi_i^{\lfloor 2n/\mathrm{lcm}(2,i) \rfloor}$	$2(n-1)$
^2D_n	$(4, q^n - 1)$	$q^{n(n-1)} \prod_{i \nmid n} \Phi_i^{\lfloor 2n/\mathrm{lcm}(2,i) \rfloor} \prod_{i \mid n} \Phi_i^{\lfloor 2n/\mathrm{lcm}(2,i) \rfloor - 1}$	$2n$

where $e_i \geqslant 0$ for $i \leqslant M$ and $e_M > 0$. We set $e_i = 0$ for $i > M$. The values of M and $e_0, e_1, \ldots, e_M$ can be deduced from the usual formulae for $|T|$ by noting that

$$q^i - 1 = \prod_{k \mid i} \Phi_k(q) \quad \text{and} \quad q^i + 1 = \prod_{k \mid 2i, k \nmid i} \Phi_k(q).$$

They may be also obtained from [8, Definition 4.4 and Tables C.1 and C.2]. The values of M are listed in Table 1. Table 2 lists the cyclotomic factorisations for the classical groups and duplicates the values of M for ease of reference.

We need the following lemma about cyclotomic polynomials.

LEMMA 2.1. *Let $i < j$ be integers, and suppose that r is a prime dividing both $\Phi_i(q)$ and $\Phi_j(q)$ for some prime power q. Then $j/i = r^k$ for some positive integer k. In particular, r divides j.*

PROOF. This follows immediately from [6, Theorem 2] or [7, Lemma 2]. □

The next lemma imposes an upper bound on the number of distinct cyclotomic polynomial factors of $|T|$ that are divisible by a given prime distinct from the characteristic.

LEMMA 2.2. *Let $T = T(q)$ be a finite simple group of Lie type defined over a field of order q, and let $d|T| = q^{e_0} \prod_{i=1}^{M} \Phi_i(q)^{e_i}$ be the cyclotomic factorisation of T, where d is the number of diagonal outer automorphisms of T. If r is a prime dividing $|T|$ but not q, then r divides at most $\lfloor \log_r(M/m) \rfloor + 1$ of the factors $\Phi_1(q)^{e_1}, \ldots, \Phi_M(q)^{e_M}$, where m is the order of q modulo r.*

PROOF. Since r divides $|T|$ but not q, it divides some factor $\Phi_i(q)^{e_i}$ of $d|T|$. Hence $e_i > 0$. Moreover, the minimal such i is the order m of q modulo r. By Lemma 2.1, r might also divide some or all of $\Phi_{mr}(q)^{e_{mr}}, \ldots, \Phi_{mr^k}(q)^{e_{mr^k}}$, where k is maximal such that $\Phi_{mr^k}(q)$ divides $|T|$, but r cannot divide any of the other $\Phi_j(q)^{e_j}$. Since $mr^k \leqslant M$, r divides at most $1 + k = 1 + \lfloor \log_r(M/m) \rfloor$ of the factors $\Phi_j(q)^{e_j}$ of $d|T|$. □

The final lemma bounds the contribution of each cyclotomic polynomial factor to the order of a finite simple classical group.

LEMMA 2.3. *Let $T = T(q)$ be a finite simple classical group defined over a field of order q, as listed in Table 1 or Table 2. Let $Q(T)$ denote the largest factor of the form $\Phi_i(q)^{e_i}$ dividing $d|T|$. Then*

$$Q(T) \leqslant |T|^{a/n}, \quad \text{where } a = \begin{cases} 1 & \text{if } T \text{ has type } A_n, B_n \text{ or } C_n, \\ 2 & \text{if } T \text{ has type } {}^2A_n, D_n \text{ or } {}^2D_n. \end{cases}$$

PROOF. First, suppose that $n = 1$. Then $T = A_1(q)$ (with $q \geqslant 4$) and $d|T| = q(q^2 - 1)$, so $Q(T) = \Phi_2(q)$. The desired bound is $Q(T) \leqslant |T|$, and this holds because

$$|T| \geqslant \frac{q(q^2 - 1)}{2} = \frac{1}{2}q(q - 1)(q + 1) \geqslant q + 1 = \Phi_2(q).$$

Suppose from now on that $n \geqslant 2$. We need the following inequality, which follows from [11, Lemma 3.5]:

$$1 - q^{-1} - q^{-2} < \prod_{i=1}^{\infty}(1 - q^{-i}) \leqslant 1 - q^{-1} - q^{-2} + q^{-3} \quad \text{for all } q \geqslant 2. \tag{2.1}$$

We now divide the proof into four cases.

Case 1: $T \cong A_n(q)$ with $n \geqslant 2$. Here $d = \gcd(n + 1, q - 1) < q$ (see Table 2) so (2.1) yields

$$|T| = \frac{q^{n(n+1)/2}}{d}\prod_{i=2}^{n+1}(q^i - 1) = \frac{q^{n(n+2)}}{d}\prod_{i=2}^{n+1}(1 - q^{-i}) > \frac{1 - q^{-1} - q^{-2}}{q} \cdot q^{n^2+2n}. \tag{2.2}$$

Suppose that a cyclotomic polynomial $\Phi_i(q)$ divides $|T|$. According to Table 2, we have $1 \leqslant i \leqslant n + 1$, $e_1 = n$ and $e_i = \lfloor (n + 1)/i \rfloor \leqslant n + 1$ for $i \geqslant 2$. We now show that $\Phi_i(q)^{e_i} \leqslant q^{n+1}/(q - 1)$. This is true when $i = 1$ because $(q - 1)^{n+1} < q^{n+1}$; if $2 \leqslant i \leqslant n + 1$ then $\Phi_i(q)$ divides $(q^i - 1)/(q - 1)$, and so

$$\Phi_i(q)^{e_i} \leqslant \left(\frac{q^i - 1}{q - 1}\right)^{(n+1)/i} < \frac{(q^i)^{(n+1)/i}}{(q - 1)^{(n+1)/i}} \leqslant \frac{q^{n+1}}{q - 1}.$$

Therefore $Q(T) \leqslant q^{n+1}/(q - 1)$, and it follows from (2.2) that

$$Q(T)^n \leqslant \frac{q^{n(n+1)}}{(q - 1)^n} = \frac{q^{n(n+2)}}{q^n(q - 1)^n} < \frac{q^{n(n+2)}}{4(q - 1)} \leqslant (1 - q^{-1} - q^{-2})\frac{q^{n(n+2)}}{d} = |T|.$$

Hence $Q(T) \leqslant |T|^{1/n}$, as claimed.

Case 2: $T \cong {}^2A_n(q)$. Here $d = \gcd(n+1, q+1) \leqslant q+1$, so

$$|T| = \frac{q^{n(n+2)}}{d} \prod_{i=2}^{n+1} (1 - (-q)^{-i}) > \frac{q^{n(n+2)}}{q+1} (1 + q^{-3}) \prod_{j=1}^{\infty} (1 - q^{-2j}). \qquad (2.3)$$

That is, for the estimate we have omitted the factors in the original product with odd $i > 3$. Using (2.1), the identity $(1 + q^{-3})/(1 + q^{-1}) = 1 - q^{-1} + q^{-2}$ and the inequality $1 - q^{-1} + q^{-2} \geqslant 11/16$ gives

$$\frac{1 + q^{-3}}{q+1} \prod_{j=1}^{\infty} (1 - q^{-2j}) > \frac{(1 + q^{-3})(1 - q^{-2} - q^{-4})}{q(1 + q^{-1})} \geqslant \frac{11}{16}\left(\frac{1 - q^{-1} + q^{-2}}{q}\right) > \frac{1}{2q}.$$

Together with (2.3), this yields

$$|T| > \frac{q^{n(n+2)-1}}{2}. \qquad (2.4)$$

We now show that $Q(T) \leqslant q^{2(n+1)}$. If a cyclotomic polynomial $\Phi_i(q)$ divides $d|T|$, then $1 \leqslant i \leqslant 2(n+1)$ by Table 2. If $i = 1$, then $e_1 = \lfloor (n+1)/2 \rfloor \leqslant (n+1)/2$ (by Table 2), so $\Phi_1(q)^{e_1} \leqslant (q-1)^{(n+1)/2} < q^{2(n+1)}$. For $i = 2$, similarly, $e_2 = n$ and $\Phi_2(q)^{e_2} < (q^2-1)^n \leqslant q^{2(n+1)}$. Finally, if $i \geqslant 3$, then $e_i \leqslant 2(n+1)/i$, so

$$\Phi_i(q)^{e_i} < (q^i - 1)^{2(n+1)/i} < (q^i)^{2(n+1)/i} = q^{2(n+1)}.$$

Therefore $Q(T) \leqslant q^{2(n+1)}$, which together with (2.4) yields

$$Q(T)^n < q^{2n^2+2n} \leqslant \frac{q^{2n^2+4n-2}}{4} = \left(\frac{q^{2(n+1)-1}}{2}\right)^2 < |T|^2.$$

Thus $Q(T) < |T|^{n/2}$, as claimed.

Case 3: $T \cong B_n(q)$ or $C_n(q)$. Here $d = \gcd(2, q-1) \leqslant 2$, which together with (2.1) gives

$$|T| = \frac{q^{2n^2+n}}{d} \prod_{i=1}^{n} (1 - q^{-2i}) > \frac{1 - q^{-2} - q^{-4}}{2} \cdot q^{2n^2+n} \geqslant \frac{11}{32} q^{2n^2+n} > \frac{q^{2n^2+n}}{3}.$$

Suppose that $\Phi_i(q)$ divides $d|T|$. Then $1 \leqslant i \leqslant 2n$ and $e_i = \lfloor 2n/\mathrm{lcm}(2, i) \rfloor \leqslant 2n/i$ by Table 2. Since $\Phi_i(q) \leqslant q^i - 1 < q^i$,

$$\Phi_i(q)^{e_i} < (q^i)^{2n/i} = q^{2n}.$$

Hence $Q(T) < q^{2n}$, and because $n \geqslant 2$,

$$Q(T)^n < q^{2n^2} < \frac{q^{2n^2+n}}{3} < |T|.$$

TABLE 3. Values of $Q(T)$ for the exceptional Lie type groups T, and constants K, d_0 and q_0 such that $Q(T)^K \leqslant (d/d_0)|T|$ for all $q \geqslant q_0$.

T	${}^2B_2$	${}^3D_4$	E_6	${}^2E_6$	E_7	E_8	F_4	${}^2F_4'$	G_2	${}^2G_2$
$Q(T)$	Φ_4	Φ_3^3	Φ_3^3	Φ_2^6	Φ_2^7	Φ_2^8	Φ_2^4	Φ_4^2	Φ_2^2	Φ_6
K	2	6	12	12	18	29	12	6	6	7/2
d_0	1	1	3	3	2	1	1	1	1	1
q_0	8	4	5	7	9	7	7	8	4	27

Therefore $Q(T) \leqslant |T|^{1/n}$, as claimed.

Case 4: $T \cong D_n(q)$ or ${}^2D_n(q)$. Here $d = \gcd(4, q \pm 1) \leqslant 4$ and

$$|T| = \frac{q^{n^2-n}(q^n \pm 1)}{d} \prod_{i=1}^{n-1}(q^{2i} - 1) > \frac{q^{2n^2-n}}{4} \prod_{i=1}^{\infty}(1 - q^{-2i}) > \frac{(1 - q^{-2} - q^{-4})q^{2n^2-n}}{4}.$$

Since $\prod_{i=1}^{\infty}(1 - q^{-2i}) > (1 - q^{-2} - q^{-4}) \geqslant 11/16$ by (2.1), we have $|T| > q^{2n^2-n-3}$. If $\Phi_i(q)$ divides $d|T|$, then $i \leqslant 2n$ and $e_i \leqslant 2n/i$ by Table 2, so $\Phi_i(q)^{e_i} \leqslant (q^i - 1)^{2n/i} < q^{2n}$. Hence $Q(T) \leqslant q^{2n}$, so $Q(T)^n \leqslant q^{2n^2} < (q^{2n^2-n-3})^2 < |T|^2$ and $Q(T) \leqslant |T|^{2/n}$, as claimed. This completes the proof. □

3. Proof of the main theorem

PROOF OF THEOREM 1.1. As in Section 2, write $d|T| = q^{e_0} \prod_{i=1}^{M} \Phi_i(q)^{e_i}$ and let $Q(T)$ denote the maximum of $\Phi_1(q)^{e_1}, \ldots, \Phi_M(q)^{e_M}$. Let r be a prime dividing $|T|$ but not q, and let R be a Sylow r-subgroup of T. It follows from Lemma 2.2 that

$$|R| \leqslant Q(T)^{1+\lfloor \log_r(M) \rfloor}. \tag{3.1}$$

We now seek 'large' constants K satisfying $Q(T)^K \leqslant |T|$ to deduce the claimed bounds of the form $|R| \leqslant |T|^{(\lfloor \log_r(M) \rfloor + 1)/K}$.

First, suppose that T is a classical group. Then the values of M may be deduced from the cyclotomic factorisations listed in Table 2, and they are as listed in both Table 1 and Table 2. The values of K appearing in Table 1 are obtained from Lemma 2.3.

Now suppose that T is an exceptional Lie-type group. Then $d|T| = q^{e_0} \prod_{i=1}^{30} \Phi_i(q)^{e_i}$, where the values of $e_1, \ldots, e_{30}$ are listed in [8, Table C.2]. In particular, we obtain the values of M appearing in Table 1. The values of e_0 appear in the last row of [8, Table C.3] and in [5, Table 6]. By inspecting these factorisations, one finds the value of i such that $Q(T) = \Phi_i(q)^{e_i}$ (and such that this value is independent of q). Table 3 lists constants K, d_0 and q_0 such that

$$\Phi_i(q)^{e_iK} \leqslant \frac{d}{d_0}|T| \quad \text{for all } q \geqslant q_0.$$

Note that $d = d_0 = 1$ in all cases, except when $T = E_6(q)$, ${}^2E_6(q)$ or $E_7(q)$, where $(d, d_0) = (\gcd(3, q - 1), 3)$, $(\gcd(3, q + 1), 3)$ or $(\gcd(2, q - 1), 2)$, respectively.

In particular, $d \leqslant d_0$ in all cases, and so

$$Q(T)^K \leqslant |T| \quad \text{for all } q \geqslant q_0.$$

The constants K agree with those in Table 1, and so by combining the above bound with (3.1) we obtain the claimed bounds $|R| \leqslant |T|^{(\lfloor \log_r (M) \rfloor + 1)/K}$ for $q \geqslant q_0$. It remains to consider the cases where $q < q_0$. In these cases, we check manually, for each prime r dividing $|T|$ but not q, whether the bound $|R| \leqslant |T|^{(\lfloor \log_r (M) \rfloor + 1)/K}$ (with K and M as in Table 1) holds. The exceptions, which were checked both manually and using the MAGMA code available at the first author's website (http://school.maths.uwa.edu.au/~glasby/ExceptionsCheck.mag), are recorded in Table 1. □

REMARK 3.1. One can slightly improve the values of K listed in Table 1 in some cases. For example, if $T \cong {}^2B_2(q)$, then K can be increased to $\log(|{}^2B_2(8)|)/\log(\Phi_2(8)) \approx 2.46$. As such improvements seem tedious to achieve and do not change the form of our generic bound $|R| \leqslant |T|^{O(\log_r(\ell)/\ell)}$, where ℓ is the Lie rank of T, we chose not to pursue them here.

References

[1] E. Artin, 'The orders of the linear groups', *Comm. Pure Appl. Math.* **8** (1955), 355–365.
[2] E. Artin, 'The orders of the classical simple groups', *Comm. Pure Appl. Math.* **8** (1955), 455–472.
[3] F. Buekenhout, 'Good contributors to the order of the finite simple groups', *Arch. Math.* **44** (1985), 289–296.
[4] L. Cimetta and A. Lucchini, 'Normally ζ-reversible profinite groups', *Algebra Discrete Math.* **1** (2016), 24–50.
[5] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson, *ATLAS of Finite Groups* (Oxford University Press, Oxford, 1985).
[6] G. Dresden, 'Resultants of cyclotomic polynomials', *Rocky Mountain J. Math.* **42** (2012), 1461–1469.
[7] M. Filaseta, 'Coverings of the integers associated with an irreducibility theorem of A. Schinzel', in: *Number Theory for the Millennium II* (ed. M. A. Bennett) (A. K. Peters, Natick, MA, 2002), 1–24.
[8] W. Kimmerle, R. Lyons, R. Sandling and D. N. Teague, 'Composition factors from the group ring and Artin's theorem on orders of simple groups', *Proc. Lond. Math. Soc. (3)* **60** (1990), 89–122.
[9] P. B. Kleidman and M. W. Liebeck, *The Subgroup Structure of the Finite Classical Groups* (Cambridge University Press, Cambridge, 1990).
[10] A. M. Legendre, *Essai sur la théorie des nombres*, 2nd edn (Duprat, Paris, 1808).
[11] P. M. Neumann and C. E. Praeger, 'Cyclic matrices in classical groups over finite fields', *J. Algebra* **234** (2000), 367–418.

S. P. GLASBY, Centre for the Mathematics of Symmetry and Computation,
The University of Western Australia, Crawley WA 6009, Australia
e-mail: Stephen.Glasby@uwa.edu.au

ALICE C. NIEMEYER, Lehrstuhl B für Mathematik,
Lehr- und Forschungsgebiet Algebra,
RWTH Aachen University, Pontdriesch 10-16, 52062 Aachen, Germany
e-mail: alice.niemeyer@mathb.rwth-aachen.de

TOMASZ POPIEL, School of Mathematical Sciences,
Queen Mary University of London, Mile End Road,
London E1 4NS, UK
and
Centre for the Mathematics of Symmetry and Computation,
The University of Western Australia, Crawley WA 6009, Australia
e-mail: tomasz.popiel@uwa.edu.au