ON COMPACT NORMAL SEMIGROUPS

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0. Introduction

A semigroup S is said to be *normal* if $aS = Sa$ for each a in S. Thus the class of normal semigroups includes the class of groups and the class of Abelian semigroups. Given a compact semigroup S we write $P(S)$ for the convolution semigroup of probability regular Borel measures on S. In (3), Theorem 7, Lin asserts that a compact semigroup S is normal if and only if $P(S)$ is normal. We show in this paper that Lin's result is false. In fact, if S is the union of subsemigroups each of which has an identity element, we show that $P(S)$ is normal if and only if S is Abelian. Thus any compact non-Abelian group contradicts Lin's result. What Lin's argument does establish is that if $P(S)$ is normal then S is normal, and if S is normal then $\mu P(S) = P(S)\mu$ for each *point mass* measure μ.

In Section **1** we present some simple facts about normal semigroups. Most of the results here are probably well known but we do not know any suitable reference for them. In Section **2** we prove the result stated above about compact semigroups for which $P(S)$ is normal. We also introduce a class of semigroups, called *completely normal* semigroups, for which $P(S)$ is normal and give an example of a non-Abelian completely normal finite semigroup.

Lin's aim in Section **5** of (**3**) was to generalize some results of Glicksberg (**2**) to the class of compact normal semigroups. We show in Section **3** that some results can be obtained in this direction. In particular if S is a compact normal semigroup we show that each idempotent measure μ in $P(S)$ is supported on a group. Thus each idempotent measure in $P(S)$ is simply the canonical extension of the Haar measure on a compact subgroup of S. We show also that the kernel of $P(S)$ is simply the Haar measure m on the kernel of S, i.e. m is the zero element of $P(S)$.

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1. Preliminaries on normal semigroups

Let S be any semigroup and let N be a subset of S. Then N is said to be *normal* in S if $aN = Na$ for each $a \in S$. In particular S is said to be a *normal semigroup* if it is normal in itself. An element c of S is said to be *central* if $ac = ca$ for each $a \in S$.

Proposition 1.1. *Every idempotent of a normal semigroup is central.*

Proof. Let S be a normal semigroup and let $j \in S$ with $j^2 = j$. Given

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$a \in S$ there is $b \in S$ with $aj = jb$. Then $jaj = jb = aj$. A similar argument gives $jaj = ja$, so that j is central.

An element a of a semigroup S is said to be *regular* if there is $b \in S$ such that $aba = a$. In general a regular element of S need not belong to a subgroup of S.

Proposition 1.2. *Every regular element of a normal semigroup belongs to some subgroup of the semigroup.*

Proof. Let S be a normal semigroup and let $a, b \in S$ with $aba = a$. Let $e = ab, f = ba$ so that $e^2 = e, f^2 = f$. We have $ea = aba = a, af = aba = a$. Since e, f are central we now have

$$ba = bea = eba = ef = fe = fab = afb = ab = e.$$

Thus the subsemigroup generated by a and b is a group with identity e in which b is the inverse of a.

Let S be a normal semigroup and let T be a subsemigroup of S. Then T need not be a normal semigroup. For example in the free group on two generators a, b the subsemigroup generated by a and b is not normal. The next result gives a sufficient condition for T to be a normal semigroup.

Proposition 1.3. *Let S be a normal semigroup and let T be a subsemigroup of S such that T is the union of groups. Then T is a normal semigroup.*

Proof. Given $a \in T$ we shall show that $Ta = T \cap Sa$ and similarly $aT = T \cap aS$. Since $Sa = aS$ we then conclude that $Ta = aT$ and T is normal. It is thus sufficient to show that $T \cap Sa \subset Ta$ for $a \in T$. Suppose $y \in S$ and $ya \in T$. Since T is the union of groups there are $e, b \in T$ with $ab = e$, $ea = a$. Then

$$ya = yea = yaba \in Tba \subset Ta$$

and the proof is complete.

If S is a compact semigroup the *kernel* K of S is the (unique) minimal closed two-sided ideal of S.

Proposition 1.4. *If K is the kernel of a compact normal semigroup S then*

(i) *K is a compact subgroup of S,*

(ii) *if e is the identity of K then $ej = e$ for every idempotent j of S.*

Proof. (i) Certainly K is compact. Given $a \in K$ we have that aK is a closed right ideal and

$$SaK = aSK \subset aK,$$

so that aK is two-sided. Since K is the kernel we have $aK = K$, and similarly $Ka = K$. It is well known that K is then a compact group.

(ii) Let $j \in S$ with $j^2 = j$. Since e is central ej is an idempotent and it is in K. Thus $ej = e$.

We remark that it is easy to construct examples of compact normal semigroups that are neither groups nor Abelian. In fact let G be any non-Abelian

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compact group and let T be any compact Abelian semigroup that is not a group. Then the direct product semigroup $G \times T$ is such a compact normal semigroup.

2. Compact semigroups S with $P(S)$ normal

Let S be a compact semigroup and let $P(S)$ be the convolution semigroup of probability measures on S. For convenience of notation we shall identify the elements of S with the point mass measures. Given $\mu \in P(S)$ we write supp μ for the support of μ, i.e. the unique minimal closed subset H of S with $\mu(H) = 1$. It is well known that, if $\mu, \nu \in P(S)$ and $0 < t < 1$, then

$$\text{supp } \mu\nu = \text{supp } \mu \text{ supp } \nu$$

$$\text{supp } (t\mu + (1-t)\nu) = \text{supp } \mu \cup \text{supp } \nu.$$

The theorem below shows that for a large class of compact semigroups S, $P(S)$ is normal if and only if S is Abelian.

Theorem 2.1. *Let S be a compact semigroup with $P(S)$ normal. Let $y \in S$ with $ey = y$ for some idempotent e of S. Then y is central in S.*

Proof. It follows from Lin's argument that S is a normal semigroup and so e is central in S. Suppose there is $x \in S$ with $xy \neq yx$. Let $\mu = te + (1-t)x$ where $0 < t < 1$, $t \neq \frac{1}{2}$. Since $P(S)$ is normal there is $\rho \in P(S)$ with $\mu y = \rho\mu$, i.e.

$$ty + (1-t)xy = t\rho e + (1-t)\rho x.$$

If $H = \text{supp } \rho$, then $\{y, xy\} = He \cup Hx$. If $y = xy$ then for any $w \in H$ we have $we = y = xy = wx$, and so $yx = wex = wxe = xye = xy$. This contradiction shows that $y \neq xy$. We now consider various possibilities for the sets He and Hx.

(i) $He = \{y\}$ or $Hx = \{xy\}$. It follows that there is $w \in H$ with $we = y$, $wx = xy$. This gives

$$yx = wex = wxe = xye = xy.$$

(ii) $He = \{xy\}$, $Hx = \{y, xy\}$. For each $w \in H$ we have $we = xy$ and so $wxe = wex = xyx$. But $wx = y$ or $wx = xy$ and so $wxe = y$ or $wxe = xy$. Since both possibilities occur we obtain the contradiction $y = xy$.

(iii) $He = \{xy\}$, $Hx = \{y\}$. Then $\rho e = xy$, $\rho x = y$,

$$ty + (1-t)xy = txy + (1-t)y.$$

Since $t \neq \frac{1}{2}$ this gives the contradiction $y = xy$. We are now reduced to case (iv) below.

(iv) $He = \{y, xy\}$. Then $Hxe = Hex = \{yx, xyx\}$. If $Hx = \{y\}$ then $Hx = Hxe$ and so $y = yx = xyx$. This gives $xy = x(yx) = y = yx$. This contradiction shows that $Hx \neq \{y\}$. We cannot have $Hx = \{xy\}$ by part (i), and therefore $Hx = \{y, xy\} = Hxe$. Since $yx \neq xy$, we now deduce that $yx = y$.

Using the normality of $P(S)$ again we get $\nu \in P(S)$ with $y\mu = \mu\nu$ and so

$$y = ty+(1-t)yx = te\nu+(1-t)x\nu.$$

Let $z \in \operatorname{supp} \nu$ and then $ez = y = xz$. This gives $xy = xz = y = yx$. This final contradiction shows that y must be central in S.

Corollary. *If $P(S)$ is normal and S is the union of subsemigroups each of which has an identity element, then S is Abelian. In particular if $P(S)$ is normal and S is a group then S is Abelian.*

Remark 1. Suppose $x, y \in S$ with $xy = y$. Then $x^n y = y$ for each positive integer n. It is well known that there is an idempotent e that is a closure point of $\{x^n\}$ and then $ey = y$.

Remark 2. Let S be any semigroup with the discrete topology and replace $P(S)$ by co (S). It is then clear that Theorem 2.1 holds if *compact* is replaced by *discrete*.

Since co (S) is weak* dense in $P(S)$ the result below may be established by a routine argument; we omit the proof.

Proposition 2.2. *Let S be a compact semigroup. Then $P(S)$ is normal if anp only if for each $x \in S$, $\mu \in$ co (S) there are $\rho, \nu \in P(S)$ such that $\mu x = \rho\mu$, $x\mu = \mu\nu$.*

The above result leads to a sufficient condition on S that $P(S)$ be normal. Suppose that $\mu \in$ co (S) so that

$$\mu = \sum_1^n t_r y_r,\ t_r \geqq 0,\ \sum_1^n t_r = 1.$$

Given $x \in S$ suppose there is $z \in S$ such that

$$y_r x = zy_r \quad (r = 1, \ldots, n). \tag{1}$$

The first condition of Proposition 2.2. will now be satisfied with $\rho = z$. If S is normal then $E_y = \{z \in S\colon yx = zy\}$ is non-empty for each y and is clearly closed. If condition (1) above holds for any finite set $\{y_1, \ldots, y_n\}$ then the closed sets $\{E_y\colon y \in S\}$ satisfy the finite intersection property. Since S is compact $\cap\{E_y\colon y \in S\}$ must be non-empty. We are thus led to the following definition.

A semigroup S is said to be *completely normal* if for each $x \in S$ there are $\phi(x), \psi(x) \in S$ such that

$$yx = \phi(x)y \quad (y \in S)$$

$$xy = y\psi(x) \quad (y \in S).$$

The result below is now clear.

Proposition 2.3. *Let S be a completely normal compact semigroup. Then $P(S)$ is normal.*

We give next an example of a completely normal compact semigroup which is not Abelian.

Example 2.4. Let $S = \{a, b, c, d, e\}$ with the following multiplication table.

	a	b	c	d	e
a	d	d	e	d	d
b	e	d	d	d	d
c	d	e	d	d	d
d	d	d	d	d	d
e	d	d	d	d	d

The multiplication is associative since the product of any three elements is d. Since $ab \neq ba$, S is a finite non-Abelian semigroup. Define $\phi: S \to S$ by

$$\phi(a) = c, \quad \phi(b) = a, \quad \phi(c) = b, \quad \phi(d) = d, \quad \phi(e) = e$$

and define $\psi = \phi^{-1}$. It is readily verified that

$$yx = \phi(x)y, \, xy = y\psi(x) \quad (x, y \in S)$$

so that S is completely normal.

We remark without proof that the above example is the smallest possible example of a non-Abelian completely normal semigroup. Also if S is any completely normal semigroup it is easy to see that S^2 is a subset of the centre of S.

3. The semigroup $P(S)$ with S compact normal

Throughout this section S will be a compact normal semigroup. We write K for the kernel of S so that K is a compact group by Proposition 1.4. We write m for the Haar measure on K.

The results below generalize known results for compact groups and compact Abelian semigroups.

Theorem 3.1. *The support of each idempotent measure in $P(S)$ is a group. Thus the idempotent measures in $P(S)$ are the Haar measures on compact subgroups of S.*

Proof. Let $\mu \in P(S)$ with $\mu^2 = \mu$, and let $T = \operatorname{supp} \mu$. By Collins (**1**), T is a simple semigroup and hence is a union of subgroups. By Proposition 1.3 we have that T is a normal semigroup. Since T is simple we deduce that $Tx = xT = T \, (x \in T)$. It is now well known that T must be a group. Finally it is well known that the only idempotent measure supported on a compact group is the Haar measure of the group.

We write $K(P(S))$ for the kernel of $P(S)$.

Theorem 3.2. $K(P(S)) = \{m\}$.

Proof. It is sufficient to show that m is the zero element of $P(S)$. Given $x \in S$ we have $\operatorname{supp}(xm) \subset K$. Since m is the zero element in $P(K)$ we have $xm = xm^2 = m$. It follows by the standard density argument that $\mu m = m$ for each μ in $P(S)$ and similarly $m\mu = m$.

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