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ON THE SPECTRUM OF THE BIHARMONIC OPERATOR IN A BOUNDED DOMAIN

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We obtain an asymptotically sharp lower bound for the sum of the first k eigenvalues of the biharmonic operator.

Let D be a bounded domain in $\mathbb{R}^n$ with piecewise smooth boundary ∂D and consider the following two eigenvalue problems:

$$\text{(I)} \quad \begin{cases} \Delta\phi + \lambda\phi = 0 & \text{in } D , \\ \phi = 0 & \text{on } \partial D ; \end{cases}$$

$$\text{(II)} \quad \begin{cases} \Delta^2\phi - \mu\phi = 0 & \text{in } D , \\ \phi = \dfrac{\partial\phi}{\partial n} = 0 & \text{on } \partial D . \end{cases}$$

We let

$$0 < \lambda_1 < \lambda_2 \le \lambda_3 \cdots ,$$

$$0 < \mu_1 \le \mu_2 \le \mu_3 \cdots$$

denote the successive eigenvalues of (I) and (II) respectively. We have the well-known asymptotic formulas of Weyl [4]:

$$\lambda_k \sim C_n(k/V)^{2/n} ,$$

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$$\mu_k \sim C_n^2(k/V)^{4/n} ,$$

where $C_n = (2\pi)^2 B_n^{-2/n}$ with B_n the volume of the unit n-ball and V the volume of D. In [3] (see also [2], p. 310) Polya proved that

$$\lambda_k \geq C_n(k/V)^{2/n} .$$

holds for $\mathbb{R}^n$-covering domains (that is, domains that tile $\mathbb{R}^n$) and conjectured this inequality for arbitrary domains.

Now it follows from the asymptotic formulas that

$$\sum_{i=1}^{k} \lambda_i \sim \frac{n}{n+2} C_n V^{-2/n} k^{(n+2)/n} ,$$

$$\sum_{i=1}^{k} \mu_i \sim \frac{n}{n+4} C_n^2 V^{-4/n} k^{(n+4)/n} ,$$

and recently Li and Yau in [2] proved that

$$\sum_{i=1}^{k} \lambda_i \geq \frac{n}{n+2} C_n V^{-2/n} k^{(n+2)/n} .$$

Since $k\lambda_k \geq \sum_{i=1}^{k} \lambda_i$ it follows that

$$\lambda_k \geq \frac{n}{n+2} C_n (k/V)^{2/n} .$$

For problem (I), we have the following well-known variational characterization ([1], p. 395),

$$\lambda_n = \max_{\{v_1,\ldots,v_{n-1}\}} \min_{\substack{u \perp v_1,\ldots,v_{n-1} \\ u=0 \text{ on } \partial D \\ \int u^2=1}} - \int_D u\Delta u$$

and on going through the proof in [1], p. 395, we can see that problem (II) can be similarly characterized by

$$\mu_n = \max_{\{v_1,\ldots,v_{n-1}\}} \min_{\substack{u\perp v_1,\ldots,v_{n-1} \\ u=\partial u/\partial n=0 \text{ on } \partial D \\ \int u^2=1}} \int_D (\Delta u)^2$$

and hence

$$\mu_n \geq \max_{\{v_1,\ldots,v_{n-1}\}} \min_{\substack{u\perp v_1,\ldots,v_{n-1} \\ u=0 \text{ on } \partial D \\ \int u^2=1}} \int (\Delta u)^2$$

and so it follows from the Cauchy-Schwarz inequality that

$$\mu_k \geq \lambda_k^2 .$$

Therefore by applying Polya's theorem, we obtain

THEOREM 1. *If D is a $\mathbb{R}^n$-covering domain, then*

$$\mu_k \geq C_n^2 (k/V)^{4/n} .$$

In view of these, it is natural to propose the following:

CONJECTURE. For any bounded domain in $\mathbb{R}^n$,

$$\mu_k \geq C_n^2 (k/V)^{4/n} .$$

In this note we shall make a slight modification of the proof in [2] and show that:

THEOREM 2. *For any bounded domain D in $\mathbb{R}^n$, we have*

$$\sum_{i=1}^{k} \mu_k \geq \frac{n}{n+4} C_n^2 V^{-4/n} k^{(n+4)/n} .$$

COROLLARY. $\mu_k \geq \big(n/(n+4)\big) C_n^2 (k/V)^{4/n}$.

Proof. This follows since $k\mu_k \geq \sum_{i=1}^{k} \mu_i$.

Proof of Theorem 2. We shall need the following extension of a lemma of Hormander ([2], p. 311).

LEMMA 1. *Let* f *be a real-valued function defined on* $\mathbb{R}^n$ *such that* $0 \le f \le M_1$ *and let* m *be a positive integer. Suppose*

$$\int_{\mathbb{R}^n} |z|^m f(z)dz \le M_2 ,$$

then

$$\int_{\mathbb{R}^n} f(z)dz \le \left(\frac{n+m}{n}\right)^{n/(n+m)} \left(M_1 B_n\right)^{m/(n+m)} M_2^{n/(n+m)} .$$

Proof. Just replace 2 by m in the proof in [2], p. 312, and everything goes through just the same. □

Now let $\phi_1, \ldots, \phi_k$ be a set of orthonormal eigenfunctions corresponding to the eigenvalues $\mu_1, \ldots, \mu_k$ respectively and extend ϕ_i to be zero outside D. Following [2] we introduce the function

$$\Phi(x, y) = \sum_{i=1}^{k} \phi_i(x)\phi_i(y)$$

and consider its Fourier transform in x which is given by

$$\hat{\Phi}(z, y) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \Phi(x, y)e^{ix\cdot z}dx .$$

We now let

$$f(z) = \int_D |\hat{\Phi}(z, y)|^2 dy .$$

Then from equations (6) and (4) of [2] we have

$$0 \le f(z) \le (2\pi)^{-n} V$$

and

(*)
$$\int_{\mathbb{R}^n} f(z)dz = k .$$

As in [2] the crucial step now is to get an estimate of the left hand side of (*).

To do this we consider

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$$\int_{\mathbb{R}^n} |z|^4 f(z)dz$$

$$= (2\pi)^{-n} \int_{\mathbb{R}^n} \sum_{i=1}^{k} \left|\int_D |z|^2 \phi_i(x) e^{ix\cdot z} dx\right|^2 dz$$

$$= \sum_{i=1}^{k} (2\pi)^{-n} \int_{\mathbb{R}^n} \left|\int_D \sum_{j=1}^{n} z_j^2 \phi_i(x) e^{ix\cdot z} dx\right|^2 dz$$

$$= \sum_{i=1}^{k} (2\pi)^{-n} \int_{\mathbb{R}^n} \left|\int_D \phi_i(x) \Delta_x e^{ix\cdot z} dx\right|^2 dz$$

$$= \sum_{i=1}^{k} \int_{\mathbb{R}^n} \left|(2\pi)^{-n/2} \int_D (\Delta\phi_i(x)) e^{ix\cdot z} dx\right|^2 dz \quad \text{(by Green's identity)}$$

$$= \sum_{i=1}^{k} \int_{\mathbb{R}^n} |\Delta\hat{\phi}_i(z)|^2 dz$$

$$= \sum_{i=1}^{k} \int_{\mathbb{R}^n} (\Delta\phi_i(z))^2 dz \quad \text{(by Parseval formula)}$$

$$= \sum_{i=1}^{k} \int_D \phi_i(z) \Delta^2 \phi_i(z) dz$$

$$= \sum_{i=1}^{k} \mu_i .$$

Finally, by applying Lemma 1 with $m = 4$, $M_1 = (2\pi)^{-n} V$, $M_2 = \sum_{i=1}^{k} \mu_i$, we obtain

$$k \le \left(\frac{n+4}{n}\right)^{n/(n+4)} \left((2\pi)^{-n} V B_n\right)^{4/(n+4)} \left(\sum_{i=1}^{k} \mu_i\right)^{n/(n+4)}$$

which simplifies to

$$\sum_{i=1}^{k} \mu_i \ge \frac{n}{n+4} C_n^2 V^{-4/n} k^{(n+4)/n} . \qquad \square$$

ADDED REMARK. After we had completed this paper we were informed by Professor S.T. Yau that Professor M.H. Protter has obtained similar results independently.

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