ON ABSOLUTE SUMMABILITY BY RIESZ AND GENERALIZED CESÀRO MEANS. I

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1. The Cesàro methods for ordinary [**9**, p. 17; **6**, p. 96] and for absolute [**9**, p. 25] summation of infinite series can be generalized by the Riesz methods [**7**, p. 21; **12**; **9**, p. 52; **6**, p. 86; **5**, p. 2] and by "the generalized Cesàro methods" introduced by Burkill [**4**] and Borwein and Russell [**3**]. (Also cf. [**2**]; for another generalization, see [**8**].) These generalizations raise the question as to their equivalence.

We shall consider series

$$\sum_{n=0}^{\infty} a_n \tag{1}$$

with complex terms a_n. Throughout, we will assume that

$$0 \leqq \lambda_0 < \lambda_1 < \ldots < \lambda_n \to \infty, \qquad \kappa \geqq 0, \tag{2}$$

and we call (1) Riesz summable to a sum s relative to the *type* $\lambda = (\lambda_n)$ and to the *order* κ, or summable (R, λ, κ) to s briefly, if the *Riesz means*

$$\sigma^{(\kappa)}(x) = \sum_{\lambda_\nu < x} \left(1 - \frac{\lambda_\nu}{x}\right)^{\kappa} a_\nu \qquad (x > \lambda_0), \quad \sigma^{(\kappa)}(\lambda_0) = 0$$

(of the partial sums of (1)) tend to s as $x \to \infty$. If, moreover,

$$\int^{\infty} |d\sigma^{(\kappa)}(x)| = \int^{\infty} \left|\frac{d}{dx}\sigma^{(\kappa)}(x)\right| dx < \infty \tag{3}$$

holds for some lower limit of integration $\geqq \lambda_0$, the series (1) is called summable $|R, \lambda, \kappa|$ to s. (1) is called summable (C, λ, κ) to s if the *generalized Cesàro means*

$$\tau_n^{(\kappa)} = \sum_{\nu=0}^{n} \left(1 - \frac{\lambda_\nu}{\lambda_{n+1}}\right) \ldots \left(1 - \frac{\lambda_\nu}{\lambda_{n+k}}\right)\left(1 - \frac{\lambda_\nu}{\lambda_{n+k+1}}\right)^{\delta} a_\nu, \qquad n = 0, 1, \ldots,$$

$$\kappa = k + \delta,\ k \text{ the integer such that } 0 \leqq \delta < 1,$$

tend to s as $n \to \infty$. (In the case that $\lambda_n = n$, these $\tau_n^{(\kappa)}$ reduce to the κth Cesàro means if κ is an integer, and at least define a method equivalent to

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the Cesàro method of order κ if κ is non-integral [**3**; **2**], i.e., it is the *method* that is generalized.) If, moreover,

$$\sum_n |\tau_n^{(\kappa)} - \tau_{n-1}^{(\kappa)}| < \infty \tag{4}$$

holds, we call (1) summable $|C, \lambda, \kappa|$ to s. This method generalizes the absolute Cesàro method of order κ, as was (for non-integral κ) proved by Borwein [**2**]. "Summable" means "summable to some s".

We may assume that $\lambda_0 > 0$. (Changing $\lambda_0 = 0$ to a new $\lambda_0 = \lambda_1/2$, e.g., we arrive at new means $\sigma^{(\kappa)}(x)$ and $\tau_n^{(\kappa)}$ which differ from the old ones by a function and a sequence, respectively, tending to zero monotonically as $x, n \to \infty$.) Therefore we may write

$$\sigma^{(\kappa)}(x) = \sum_{\lambda_\nu \leqq x} \left(\frac{1}{\lambda_\nu} - \frac{1}{x}\right)^\kappa a_\nu^{(\kappa)}, \qquad x \geqq \lambda_0,$$

$$\tau_n^{(\kappa)} = \sum_{\nu=0}^{n} \left(\frac{1}{\lambda_\nu} - \frac{1}{\lambda_{n+1}}\right) \cdots \left(\frac{1}{\lambda_\nu} - \frac{1}{\lambda_{n+k}}\right)\left(\frac{1}{\lambda_\nu} - \frac{1}{\lambda_{n+k+1}}\right)^\delta a_\nu^{(\kappa)},$$

where $a_\nu^{(\kappa)} = \lambda_\nu^\kappa a_\nu$.

For definitions of the terms "inclusion" and "equivalence" applying to summation methods, see [**6**, p. 66]; we shall write $\subseteq$ and $=$, respectively.

2. In this paper we shall be concerned with integral orders $\kappa = k$ only; results on non-integral orders will appear subsequently (cf. [**10**]). After proofs had been provided for $(R, \lambda, k) = (C, \lambda, k)$ under more assumptions on λ than (2) [**4**; **13**; **1**], Russell [**13**, Theorem 4] and Meir [**11**] proved the two inclusions with no restriction other than (2) (see also [**3**] for the history of the problem). It is our aim in the present paper to establish that $|R, \lambda, k| = |C, \lambda, k|$ holds without additional assumptions. Moreover, our proof furnishes an alternative argument for the result of Russell and Meir.

3. To some extent, we shall employ the technique of proof initiated by Burkill in [**4**] and also used in subsequent work. It is worthwhile defining, for numbers $b_1, \ldots, b_r$,

$$\begin{cases} B_r^p = \sum_{1 \leqq \rho_1 < \ldots < \rho_p \leqq r} b_{\rho_1} \cdots b_{\rho_p}, & p = 1, \ldots, r, \\ B_r^0 = B_0^0 = 1, \qquad B_r^{-q} = 0 & (q = 1, 2, \ldots). \end{cases} \tag{5}$$

Part (ii) in our proof of the Theorem requires the following result.

LEMMA. *Given an integer* $k > 1$ *and numbers* $b_1, \ldots, b_{k-1} \neq 0$; *the matrix*

$$\mathfrak{M}_J = (B_{k-j}^{i-j+1} : i, j = 1, \ldots, J), \qquad J = 1, \ldots, k-1,$$

with entries (5) *has the determinant*

$$|\mathfrak{M}_J| = \sum_{\rho_1=1}^{k-J} b_{\rho_1} \sum_{\rho_2=1}^{\rho_1} b_{\rho_2} \cdots \sum_{\rho_J=1}^{\rho_{J-1}} b_{\rho_J}, \qquad J = 1, \ldots, k-1.$$

Proof. For an index $s = 1, \ldots, r \leqq k - 1$ and for $p = 1, \ldots, r$, let C_{rs}^p denote the sum of all those summands of B_r^p that are products $b_{\rho_1} \ldots b_{\rho_p}$ with $\rho_1 = s$. Hence, the decomposition

$$B_r^p = \sum_{s=1}^{r-p+1} C_{rs}^p$$

holds. Applying it to the last row of $\mathfrak{M}_J$, we obtain:

$$(6) \qquad |\mathfrak{M}_J| = \sum_{\rho_1=1}^{k-J} \begin{vmatrix} & & & 0 \\ & & & \cdot \\ & \mathfrak{M}_{J-1} & & \cdot \\ & & & \cdot \\ & & & 0 \\ & & & 1 \\ C_{k-1,\rho_1}^J & \cdots & C_{k-J+1,\rho_1}^2 & b_{\rho_1} \end{vmatrix}.$$

The last row of $\mathfrak{M}_{J-1}$ reads

$$B_{k-t}^{J-t} = \sum_{\rho_2=1}^{k-J+1} C_{k-t,\rho_2}^{J-t}, \qquad t = 1, \ldots, J-1;$$

in the matrices on the right-hand side of (6), these elements are neighbouring

$$C_{k-t,\rho_1}^{J-t+1} = b_{\rho_1} \sum_{\rho_2=\rho_1+1}^{k-J+1} C_{k-t,\rho_2}^{J-t}, \qquad t = 1, \ldots, J-1,$$

respectively. Thus, some basic operations yield

$$|\mathfrak{M}_J| = \sum_{\rho_1=1}^{k-J} b_{\rho_1} \sum_{\rho_2=1}^{\rho_1} \begin{vmatrix} & & & 0 \\ & & & \cdot \\ & \mathfrak{M}_{J-2} & & \cdot \\ & & & \cdot \\ & & & 0 \\ & & & 1 \\ C_{k-1,\rho_2}^{J-1} & \cdots & C_{k-J+2,\rho_2}^2 & b_{\rho_2} \end{vmatrix}.$$

This again is the situation of (6), and iteration completes the proof.

THEOREM. $|R, \lambda, k| = |C, \lambda, k|$, $k = 0, 1, \ldots$.

Proof. Since $(R, \lambda, k) = (C, \lambda, k)$, the problem is as follows: A series (1) is summable $|R, \lambda, k|$ if and only if it is summable $|C, \lambda, k|$. The cases $k = 0$ and $k = 1$ are trivial; the latter one since, on every interval $[\lambda_n, \lambda_{n+1}]$, the function $x\sigma^{(1)}(x)$ is linear and so $\sigma^{(1)}(x)$ is monotone. We therefore assume that $k \geqq 2$.

(i) Suppose (1) to be summable $|R, \lambda, k|$; we have to prove (4), $\kappa = k$, that is,

$$(7) \qquad \sum_n \left(\frac{1}{\lambda_n} - \frac{1}{\lambda_{n+k}}\right) \left| \sum_{\nu=0}^n \left(\frac{1}{\lambda_\nu} - \frac{1}{\lambda_{n+1}}\right) \cdots \left(\frac{1}{\lambda_\nu} - \frac{1}{\lambda_{n+k-1}}\right) a_\nu^{(k)} \right| < \infty.$$

We assume an integral-valued sequence $m = m(n)$, $n \leqq m \leqq n + k - 1$, chosen in a way to satisfy

$$\frac{1}{\lambda_m} - \frac{1}{\lambda_{m+1}} = \max_{i=1,\ldots,k} \left(\frac{1}{\lambda_{n+i-1}} - \frac{1}{\lambda_{n+i}}\right),$$

and we intend to prove (7) by means of

$$(8) \quad \sum_n \left(\frac{1}{\lambda_m} - \frac{1}{\lambda_{m+1}}\right) \left| \sum_{\nu=0}^{m} \left(\frac{1}{\lambda_\nu} - \frac{1}{\lambda_{n+1}}\right) \cdots \left(\frac{1}{\lambda_\nu} - \frac{1}{\lambda_{n+k-1}}\right) a_\nu^{(k)} \right| < \infty.$$

When $[1/\lambda_{m+1}, 1/\lambda_m]$ is subdivided by

$$\frac{1}{\mu_{ni}} = \frac{1}{\lambda_m} - \frac{i-1}{2k-1}\left(\frac{1}{\lambda_m} - \frac{1}{\lambda_{m+1}}\right), \qquad i = 1, \ldots, 2k$$

(we write $\mu_i = \mu_{ni}$), there exist certain mean values $\theta_{nj} \in (\mu_{2j-1}, \mu_{2j})$ $j = 1, \ldots, k$, such that

$$\sigma^{(k)}(\mu_{2j-1}) - \sigma^{(k)}(\mu_{2j}) = k\left(\frac{1}{\mu_{2j-1}} - \frac{1}{\mu_{2j}}\right) \sum_{\nu=0}^{m} \left(\frac{1}{\lambda_\nu} - \frac{1}{\theta_{nj}}\right)^{k-1} a_\nu^{(k)}$$

holds. (3) then implies

$$(9) \qquad \sum_n \left(\frac{1}{\lambda_m} - \frac{1}{\lambda_{m+1}}\right) \left| \sum_{\nu=0}^{m} \left(\frac{1}{\lambda_\nu} - \frac{1}{\theta_{nj}}\right)^{k-1} a_\nu^{(k)} \right| < \infty, \qquad j = 1, \ldots, k,$$

since m takes on the same value for at most k different n.

In order to infer (8) from (9), we propose the following: There are sequences $q_{nj} = O(1)$, $j = 1, \ldots, k$, independent of ν, such that

$$(10) \quad \left(\frac{1}{\lambda_\nu} - \frac{1}{\lambda_{n+1}}\right) \cdots \left(\frac{1}{\lambda_\nu} - \frac{1}{\lambda_{n+k-1}}\right) = \sum_{j=1}^{k} q_{nj} \left(\frac{1}{\lambda_\nu} - \frac{1}{\theta_{nj}}\right)^{k-1}, \quad \nu = 0, \ldots, m,$$

is satisfied. After division by l_m^{k-1}, $l_m = 1/\lambda_m - 1/\lambda_{m+1}$, (10) becomes

$$(11) \qquad \sum_{j=1}^{k} q_{nj}(x_{n\nu} + \vartheta_{nj})^{k-1} = (x_{n\nu} + b_{n1}) \ldots (x_{n\nu} + b_{n,k-1}),$$

where

$$x_{n\nu} = \left(\frac{1}{\lambda_\nu} - \frac{1}{\lambda_m}\right) \Big/ l_m, \quad \vartheta_{nj} = \left(\frac{1}{\lambda_m} - \frac{1}{\theta_{nj}}\right) \Big/ l_m, \quad b_{n\rho} = \left(\frac{1}{\lambda_m} - \frac{1}{\lambda_{n+\rho}}\right) \Big/ l_m$$

($j = 1, \ldots, k$; $\rho = 1, \ldots, k-1$). Dropping the index n in (11), we may write, by notation (5),

$$\sum_{i=0}^{k-1} x_\nu^{k-1-i} \binom{k-1}{i} \sum_{j=1}^{k} \vartheta_j^i q_j = \sum_{i=0}^{k-1} x_\nu^{k-1-i} B_{k-1}^i.$$

Thus, the q_{nj} are determined by a linear system the (Vandermonde) determinant of which is

$$|\vartheta_{nj}^{i-1}: i, j = 1, \ldots, k| = \prod_{1 \leqq s < t \leqq k} (\vartheta_{nt} - \vartheta_{ns}).$$

When applying Cramer's rule, we observe that

$$(\vartheta_{nt} - \vartheta_{ns})l_m = \frac{1}{\theta_{ns}} - \frac{1}{\theta_{nt}} \geqq \frac{1}{\mu_{n,2s}} - \frac{1}{\mu_{n,2t-1}}$$

$$= \frac{2(t-s)-1}{2k-1} l_m \geqq \frac{1}{2k-1} l_m, \qquad 1 \leqq s < t \leqq k,$$

and $0 < \vartheta_{nj} < 1$, $j = 1, \ldots, k$,

$$|b_{n\rho}|l_m = \left|\frac{1}{\lambda_m} - \frac{1}{\lambda_{n+\rho}}\right| \leqq (k-1)l_m, \qquad \rho = 1, \ldots, k-1.$$

Consequently, there exist bounded q_{nj} satisfying (10).

(ii) Suppose (1) to be summable $|C, \lambda, k|$; we have to prove (3), that is,

$$(12) \quad \sum_n \int_{1/\lambda_{n+1}}^{1/\lambda_n} \left|\frac{d}{dt} \sum_{\nu=0}^{n} \left(\frac{1}{\lambda_\nu} - t\right)^k a_\nu^{(k)}\right| dt$$

$$= k \sum_n \left(\frac{1}{\lambda_n} - \frac{1}{\lambda_{n+1}}\right) \left| \sum_{\nu=0}^{n} \left(\frac{1}{\lambda_\nu} - \frac{1}{\theta_n}\right)^{k-1} a_\nu^{(k)} \right| < \infty$$

with certain mean values $\theta_n \in (\lambda_n, \lambda_{n+1})$.

First, we propose the following:

$$(13) \quad \begin{aligned} &\sum_n |d_n^{(j)}| < \infty, \qquad j = 1, \ldots, k, \\ &d_n^{(j)} = \left(\frac{1}{\lambda_n} - \frac{1}{\lambda_{n+1+k-j}}\right)^j \sum_{\nu=0}^{n} \left(\frac{1}{\lambda_\nu} - \frac{1}{\lambda_{n+1}}\right) \cdots \left(\frac{1}{\lambda_\nu} - \frac{1}{\lambda_{n+k-j}}\right) a_\nu^{(k)}, \end{aligned}$$

where the empty product (for $j = k$) is taken to be one. The hypothesis, that is (7), yields (13), $j = 1$. With the abbreviations

$$l_{nj} = \frac{1}{\lambda_n} - \frac{1}{\lambda_{n+1+k-j}}, \qquad d_n^{(j)} = l_{nj}^j S_n^{(j)} \qquad (j = 1, \ldots, k),$$

the recurrence

$$S_n^{(j)} - S_{n-1}^{(j)} = l_{n,j+1} S_n^{(j+1)}, \qquad j = 1, \ldots, k-1,$$

and therefore

$$(14) \quad d_n^{(j+1)} = l_{n,j+1}^j \left(\frac{d_n^{(j)}}{l_{nj}^j} - \frac{d_{n-1}^{(j)}}{l_{n-1,j}^j}\right), \qquad j = 1, \ldots, k-1,$$

holds. Hence,

$$|d_n^{(j+1)}| \leqq |d_n^{(j)}| + |d_{n-1}^{(j)}|, \qquad j = 1, \ldots, k-1,$$

is true, and (13) follows by induction.

(13) will infer (12) if there exist $q_{nj} = O(1)$, $j = 1, \ldots, k$, to satisfy

$$\left(\frac{1}{\lambda_n} - \frac{1}{\lambda_{n+1}}\right) \sum_{\nu=0}^{n} \left(\frac{1}{\lambda_\nu} - \frac{1}{\theta_n}\right)^{k-1} a_\nu^{(k)} = \sum_{j=1}^{k} q_{nj} d_n^{(j)},$$

or, equivalently,

$$(15)\quad \left(\frac{1}{\lambda_n}-\frac{1}{\lambda_{n+1}}\right)\left(\frac{1}{\lambda_\nu}-\frac{1}{\theta_n}\right)^{k-1}=\sum_{j=1}^{k} q_{nj}l_{nj}^{j}\left(\frac{1}{\lambda_\nu}-\frac{1}{\lambda_{n+1}}\right)\cdots\left(\frac{1}{\lambda_\nu}-\frac{1}{\lambda_{n+k-j}}\right),$$

$\nu = 0, \ldots, n$, with the empty product equal to one. (While following the analogy between part (i) and part (ii) of the proof, the denotations in both need not coincide!) We set

$$x_{n\nu}=\frac{1}{\lambda_\nu}-\frac{1}{\theta_n}, \qquad b_{n\rho}=\frac{1}{\theta_n}-\frac{1}{\lambda_{n+\rho}} \quad (\rho = 1, \ldots, k-1);$$

for brevity, we may drop the index n in $x_{n\nu}$, $b_{n\rho}$ and, for the moment, in q_{nj}, l_{nj}. By this and by notation (5), (15) is given the form:

$$\begin{aligned} l_k x_\nu^{k-1} &= \sum_{j=1}^{k} q_j l_j^j \sum_{p=0}^{k-j} B_{k-j}^{p} x_\nu^{k-j-p} \\ &= \sum_{i=1}^{k}\left(\sum_{j=1}^{i} l_j^j B_{k-j}^{i-j} q_j\right) x_\nu^{k-i}, \end{aligned}$$

so that we have a linear system for the q_j with the matrix

$$(l_j^j B_{k-j}^{i-j}:\ i, j = 1, \ldots, k),$$

the determinant of which is $l_1^1 \ldots l_k^k \neq 0$. This yields $q_1 = l_k l_1^{-1}$, $q_j = l_k l_j^{-j} |\mathfrak{B}_j|$ $(j = 2, \ldots, k)$, where $\mathfrak{B}_j$ results from $(B_{k-j}^{i-j}:\ i, j = 1, \ldots, k)$ by deleting the first row and the jth column. Hence, $|\mathfrak{B}_j| = |\mathfrak{M}_{j-1}|$, $j = 2, \ldots, k$, with the matrix $\mathfrak{M}_J$ defined as in the Lemma. This determinant is, by virtue of the Lemma and since $0 < b_{\rho_1} < b_{\rho_2}$ for any $\rho_1 < \rho_2$, a positive number less than a constant multiple of

$$b_{k-j+1}^{j-1} = \left(\frac{1}{\theta_n}-\frac{1}{\lambda_{n+k+1-j}}\right)^{j-1} < l_j^{j-1}, \qquad j = 2, \ldots, k.$$

Thus we arrive at

$$q_{nj} = O\left(\frac{l_{nk}}{l_{nj}}\right) = O(1), \qquad j = 1, \ldots, k.$$

4. We add the following observation. From parts of the proof above, one may easily obtain, by new interpretation, that $\sigma^{(k-1)}(x) = o(1)$ holds if and only if $\tau_n^{(k-1)} = o(1)$, $k = 2, 3, \ldots$. (To infer the first from the latter, take the former mean values θ_n in (15) to mean an arbitrary sequence of numbers $\theta_n \in [\lambda_n, \lambda_{n+1})$ and replace (13) by an appropriate statement.) Thus again, $(R, \lambda, k) = (C, \lambda, k)$, $k = 1, 2, \ldots$.

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