EIGENVALUES OF THE LAPLACIAN
WITH NEUMANN BOUNDARY CONDITIONS

H. P. W. GOTTLIEB[1]

Abstract

Various geometrical properties of a domain may be elicited from the asymptotic expansion of a spectral function of the Laplacian operator for that region with appropriate boundary conditions. Explicit calculations, using analytical formulae for the eigenvalues, are performed for the cases of Neumann and mixed boundary conditions, extending earlier work involving Dirichet boundary conditions. Two- and three-dimensional cases are considered. Simply-connected regions dealt with are the rectangle, annular sector, and cuboid. Evaluations are carried out for doubly-connected regions, including the narrow annulus, annular cylinder, and thin concentric spherical cavity. The main summation tool is the Poisson summation formula. The calculations utilize asymptotic expansions of the zeros of the eigenvalue equations involving Bessel and related functions, in the cases of curved boundaries with radius ratio near unity. Conjectures concerning the form of the contributions due to corners, edges and vertices in the case of Neumann and mixed boundary conditions are presented.

1. Introduction

In a previous paper [12], the author performed an explicit analytical calculation involving the eigenfunctions of the Laplacian for a narrow annular region with Dirichlet boundary conditions, and showed that certain of the geometrical properties of this doubly-connected region could be obtained from an asymptotic expansion of the spectral function.

For eigenfunctions ψ_n and eigenvalues λ_n of the two-dimensional Helmholtz equation

$$\nabla_2^2 \psi_n + \lambda_n \psi_n = 0 \tag{1.1}$$

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[1] School of Science, Griffith University, Nathan, Queensland 4111.

with Dirichlet conditions

$$\psi = 0 \tag{1.2}$$

on the (smooth curved) boundary, the spectral function

$$E = \sum_n \exp(-\lambda_n t) \tag{1.3}$$

has the asymptotic expansion [17], [23], [30],

$$E_2^{(D)} \sim \frac{A}{4\pi t} - \frac{L}{8\sqrt{\pi t}} + \frac{1-h}{6} + O(\sqrt{t}) \text{ as } t \to 0, \tag{1.4}$$

where the subscript on E denotes the dimension and the superscript (D) stands for Dirichlet. A is the area of the region with total perimeter length L and containing h smooth holes. (Some factors in (1.4) differ from those in [12] since we have used the standard equation (1.1) without the factor of $\frac{1}{2}$ used by Kac [17].)

If the region boundary has inward facing corners with angles θ then the expansion (1.4) picks up an extra constant term [23]

$$\sum \frac{\pi^2 - \theta^2}{24\pi\theta}. \tag{1.5}$$

In our case of the narrow annulus, with inner radius a and outer radius b, the eigenvalues in (1.1) are given by

$$\lambda = x^2/a^2 \tag{1.6}$$

where x is a root of the cross-product Bessel equation (A.1) given in Appendix A with $\nu = N$. The sum in (1.3) is double, over integer N from $-\infty$ to ∞ and root ranking parameter $s = 1$ to ∞ for each N. The result obtained in [12], recast to correspond to (1.4), is

$$\begin{aligned} E_2^{(D)} \sim \ & \frac{\pi(b^2 - a^2)}{4\pi t} \left[1 + O(\gamma - 1)^2\right] \\ & - \frac{2\pi(b + a)}{8\sqrt{\pi t}} \left[1 + O(\gamma - 1)^2\right] + 0 + O(\gamma - 1) \\ & - \frac{\sqrt{\pi t}}{8a} \left[1 + O(\gamma - 1)\right] + O[(\gamma - 1)t], \qquad t \to 0, \end{aligned} \tag{1.7}$$

where $\gamma = b/a > 1$ and $\gamma - 1$ is small. Thus the area, $A = \pi(b^2 - a^2)$, total perimeter, $L = 2\pi(b + a)$, and connectivity corresponding to $h = 1$, i.e. $1 - h = 0$ in (1.4), have all been elicited explicitly.

This result was obtained by squaring McMahon's explicit analytical asymptotic formula for the roots in the approximation (A.2) (see Appendix A) and utilising Poisson summation formulae in the forms (B.2), (B.3) (Appendix B). The case corresponds physically to a vibrating annular membrane with fixed edges ([15],

page 116; [13]), equation (1.1), with ψ = displacement amplitude, being obtained from the wave equation with free wave speed c after separating off the time dependence with angular frequency ω, so the eigenvalues of the Laplacian are $-\lambda$ where

$$\lambda = \omega^2/c^2. \tag{1.8}$$

This direct explicit calculation, using the eigenvalues themselves in evaluating (1.3), contrasts with the method for proving general formulae such as (1.4) ([17], [23], [30], [18], [9]) which proceeds by regarding the Helmholtz equation (1.1) as arising from a corresponding heat equation (with units chosen so that the diffusivity constant is unity) and the parameter λ is then the decay constant in the time dependence which is separated off. Their approach is indirect, *via* Green's functions.

Our result for the annulus complements the explicit calculation of Stewartson and Waechter [30] for the full circle and contrasts with the method of that paper which proceeded indirectly *via* the Green's function.

In this paper we investigate the case of Neumann boundary conditions and extend the work to include mixed boundary conditions. We shall be concerned primarily with doubly connected regions, but some simpler regions, *e.g.* rectangular, are dealt with where they shed light on the forms of various terms in the expansions.

We also extend the work to three dimensions, including the cases of the annular cylinder, and conclude with the concentric spherical cavity. The methods are similar to our earlier paper [12], but there is a greater richness in the eigenvalue structure. Formulae for zeros of eigenvalue equations, and some summation formulae, are collected in three Appendices.

2. The case of the Neumann boundary conditions

The work of Pleijel ([26], page 565) and Sleeman ([29], page 138) indicates that, for a simply connected two-dimensional region with Neumann boundary conditions

$$\frac{\partial \psi}{\partial n} = 0 \tag{2.1}$$

(where n here denotes the normal to the smooth boundary), the expansion for the spectral function (1.3) for (1.1) is

$$E_2^{(N)} \sim \frac{A}{4\pi t} + \frac{L}{8\sqrt{\pi t}} + \frac{1}{6} + O(\sqrt{t}) \tag{2.2}$$

where the superscript (N) stands for Neumann.

We may see why the coefficient of L has changed sign compared with the Dirichlet case, equation (1.4), by modifying an argument of Kac [17]. As mentioned in the Introduction, the procedure involves consideration of the heat equation. The first effect of the boundary, which leads to the term involving L, is to influence nearby points as if it were a straight line. The Green's function satisfying this approximate boundary condition is obtained from the (known) free space Green's function by the method of images. For Dirichlet boundary conditions [17], one takes the difference of source and image functions and obtains the negative coefficient of L in (1.4); for Neumann boundary conditions, one must take the sum; and it is this plus sign which, when followed through, results in the positive sign of the L term in (2.2).

2.1 Rectangular region

It is worthwhile confirming this sign change first in the simple case of a rectangular region $0 \leqslant x \leqslant a$, $0 \leqslant y \leqslant b$. With *Dirichlet* boundary conditions, the eigenfunctions and eigenvalues are

$$\psi_{M,N}^{(D)}(x, y) = \sin(M\pi x/a)\sin(N\pi y/b), \tag{2.3}$$

$$\lambda_{M,N}^{(D)} = \pi^2\left[(M^2/a^2) + (N^2/b^2)\right]; \qquad M, N = 1, 2, \dots. \tag{2.4}$$

Utilising the Poisson summation formula (B.3) twice after factorizing the form (1.3), we easily get, with $A = ab$, $L = 2(a + b)$,

$$E_2^{(D)} \sim \frac{A}{4\pi t} - \frac{L}{8\sqrt{\pi t}} + \frac{1}{4} ; \tag{2.5}$$

neglecting exponentially small terms [23], the series terminates (*cf.* [30]). The first two terms in (2.5) accord with (1.4). Since the boundary has no curvature, the third term in (1.4) is inoperative, and the constant term is given by (1.5), as may be verified for the four internal angles each of $\pi/2$.

For the case of *Neumann* boundary conditions, the eigenfunctions are

$$\psi_{M,N}^{(N)}(x, y) = \cos(M\pi x/a)\cos(N\pi y/b), \tag{2.6}$$

with eigenvalues as in (2.4) but with $M, N = 0, 1, 2, \dots$. Thus there are some eigenvalues which are smaller than those in the Dirichlet case, and furthermore, there is a zero eigenvalue corresponding to a constant eigenfunction. These features occur in the analysis of later sections and one must be careful to include them in the calculations.

The Poisson summation formula (B.4) must there be utilized, and yields

$$E_2^{(N)} \sim \frac{A}{4\pi t} + \frac{L}{8\sqrt{\pi t}} + \frac{1}{4}. \tag{2.7}$$

The change in sign for the term involving L is thereby verified explicitly in this case.

2.2 Narrow annular region

For a narrow annular region with Neumann boundary conditions, there is first of all a zero eigenvalue $\lambda = 0$ corresponding to a constant eigenfunction, as in the case of the rectangular region of the previous subsection. Most of the eigenvalues in (1.1) are given by (1.6) where x is now a root of the cross-product derivative Bessel equation (A.3) with $\nu = N$, an integer (see [15], Appendix III), and given by (A.4). Remarkably, for this case (A.3) only [20] and for $N \neq 0$ [10], the lowest root for each N is not given by a McMahon-type formula [24], but instead [32] remains finite as $\gamma \to 1$, approaching the value N. The remaining roots of (A.3), *i.e.* the remaining eigenvalues in (1.1), are given by (A.5) which was obtained recently by the author [14]. For our purposes, the first order approximation to these exceptional roots

$$x_N^1 = N\left\{1 - \tfrac{1}{2}(\gamma - 1) + O\left[(\gamma - 1)^2\right]\right\} \tag{2.8}$$

obtainable from a preliminary observation of Buchholz ([7], page 361), suffices. A more extended discussion of the exceptional roots may be found in [14]. (We stress that zero is not a solution of (A.3) but is a genuine eigenvalue of (1.1) for this boundary condition (2.1) (*cf.* [25], page 604).)

The spectral function (1.3) now has a term 1, corresponding to $\lambda = 0$, plus a double sum over integral N from $-\infty$ to $+\infty$ and over parameter $s = 1$ to ∞ corresponding, *via* (1.6), to the right hand side of (A.4), plus a single sum over nonzero integer N corresponding, *via* (1.6), to the exceptional roots (2.8). For the double sum,

$$\lambda = \frac{1}{a^2}\left[\frac{\pi^2 s^2}{(\gamma - 1)^2} + \frac{N^2}{\gamma} + \frac{3}{4\gamma} + O(\gamma - 1)^2\right]. \tag{2.9}$$

Upon use of the Poisson summation formulae (B.2), (B.3), we find by explicit calculation that the t^{-1} and constant terms in the Dirichlet boundary condition result (1.7) remain unchanged and the $t^{-1/2}$ term indeed changes sign precisely as required by (2.2).

As far as we know, this is the first explicit verification of the doubly-connected region version of (2.2) (*cf.* (1.4)). It is amusing to note that the existence of a family of exceptional roots $x \approx N$ could have been deduced from the spectral expansion result (1.7) by seeking extra values of λ which would, *via* the Poisson summation formula (B.3), just cancel out the contribution 1 from $\lambda = 0$ and change the sign of the L term. More intriguingly, to maintain the order of approximation of the corresponding $t^{-1/2}$ term in (1.7), one may infer the first order correction term given in (2.8).

Regarding the $t^{1/2}$ term of the spectral function expansion (1.3), for the Neumann case we calculate a factor of (-3) times the corresponding (fourth) term in (1.7), the Dirichlet case. (This is a direct result of the relative quantities added to $4\nu^2$ in the second terms of (A.2) and (A.4): see (2.9).) This contrasts with a general result of Pleijel [26], expressed in terms of the integral of squared curvature along the boundary, which indicates a relative factor of (-7). The discrepancy may arise from the fact that Pleijel considered a simply connected domain, but may more likely be related to the procedure which led to the apparent error by Pleijel in the calculation for the Dirichlet boundary condition. In the latter case, Stewartson and Waechter [30] have detected an error of sign in the coefficient of the $t^{1/2}$ term as given by Pleijel [26], a correction also noted by Sen [28].

Our example corresponds physically to the purely transverse vibrations of a gas enclosed in the annular space between two coaxial circular cylinders (*cf.* [21], page 263; [15], page 263; [24], page 29) with ψ in (1.1) being the pressure or the velocity potential. There is also some correspondence to the so-called tidal waves on the surface of a liquid in a tank between concentric circular vertical walls. In fact, the appropriate wave equation for the elevation of the surface above the undisturbed plane is an approximation valid only for wavelengths which are large compared with the depth of the tank (*cf.* [8], pages 87–88), so the physical analogy deteriorates for the higher eigenfrequencies.

The eigenvalue equation (A.3) for Neumann boundary conditions also arises in the analysis of coaxial (annular cross-section) acoustic ([25], pages 603–604), and electromagnetic ([22], page 31) waveguides. The latter case corresponds to TE (transverse electric) waves in transmission lines. The case of narrow annular cross-section is in fact important in electrical engineering applications ([19], page 251). In these cases, the eigenvalue λ in the two-dimensional Helmholtz equation (1.1) arises after separating off the dependence on the third spatial variable along the line. Some care must be exercised in interpreting λ in terms of frequencies as it will also involve a parameter corresponding to propagating and evanescent modes in the third direction. Thus, loosely speaking, this section might be subtitled, in emulation of the titles of references [17] and [12], "Sensing the cross-section of a coaxial pipe."

3. Mixed boundary conditions

By "mixed" boundary conditions, we mean that some sides may have Dirichlet boundary condition (D.b.c.) (1.2) and some sides may have Neumann boundary condition (N.b.c.) (2.1).

3.1 Rectangular regions

There are four distinct inequivalent mixed boundary configurations for a rectangular region. In each case, proceeding as in Section 2.1, we find that we may expand the spectral function (1.3), for a rectangle of side lengths a and b, in the form

$$E_2 \sim \frac{A}{4\pi t} + \frac{L_N - L_D}{8\sqrt{\pi t}} + \left(\sum_{DD,NN} - \sum_{DN} \right)\left(\frac{1}{16} \right) \tag{3.1}$$

where A is the area of the rectangle, L_N is the length of the part of the boundary with N.b.c., and L_D is the length of the part with D.b.c.

In the summation labels in the constant term, $DD(NN)$ denotes angles at corners both of whose adjacent sides have D.b.c. (N.b.c.), whilst DN denotes angles with one adjacent side D.b.c. and the other N.b.c. Note that the number $(1/16)$ is just the value of the expression in (1.5) for $\theta = \pi/2$.

For example, with D.b.c. on $x = 0, a, y = 0$ and N.b.c. on $y = b$, the eigenfunctions are

$$\psi = \sin[M\pi x/a]\sin\left[\left(N + \tfrac{1}{2}\right)\pi y/b\right] \tag{3.2}$$

with $M - 1, N = 0, 1, 2, \ldots$, so the Poisson summation formula in the form (B.7) is also used, and

$$E_2 \sim \frac{ab}{4\pi t} - \frac{2b}{8\sqrt{\pi t}}. \tag{3.3}$$

Based on this analysis and the corroborating results (2.5) and (2.7), we conjecture that for a simply-connected convex plane domain, with piecewise differentiable boundary consisting of a finite number of smooth segments with pure or mixed boundary conditions, the spectral function (1.3) for (1.1) has the expansion

$$E_2 \sim \frac{A}{4\pi t} + \frac{L_N - L_D}{8\sqrt{\pi t}} + \text{constant} + O(\sqrt{t}), \tag{3.4}$$

where the contribution to the constant term from the corners with inward-facing angles is $(cf. (1.5))$

$$\left(\sum_{DD,NN} - \sum_{DN} \right)\left(\frac{\pi^2 - \theta^2}{24\pi\theta} \right). \tag{3.5}$$

There appears to be a "parity factor" of (-1) associated with each D.b.c.

3.2 Narrow annular region

For a narrow annular region with Neumann boundary condition on the inner radius and Dirichlet boundary condition on the outer, the eigenvalues in (1.1) are given by (1.6) where x is a root of the mixed cross-product Bessel equation (A.6)

with $\nu = N$. (The opposite boundary configuration can be obtained by appropriate redefinitions of x and γ [5], [20].) As indicated in the expression (A.7), the roots tend to infinity as $\gamma \to 1$ (*cf.* [5]). That this must be so can be deduced immediately from the fact that the left side of (A.6) in the case $\gamma = 1$ is just the Wronskian ([2], page 500)

$$W[J_\nu, Y_\nu] \equiv J_\nu(x) Y_\nu'(x) - Y_\nu(x) J_\nu'(x) = 2/(\pi x) \qquad (3.6)$$

so if the L.H.S. of (3.6) tend to zero, $x \to \infty$. Thus there can be no finite roots, unlike the previous section.

A careful analysis of page 374 of [1] leads to (A.7) and to

$$a^2 \lambda = \pi^2 \frac{(s - \tfrac{1}{2})^2}{(\gamma - 1)^2} + \frac{1}{\gamma - 1} - \frac{1}{4} + N^2 - \frac{1}{4\pi^2} \frac{1}{(s - \tfrac{1}{2})^2} + O(\gamma - 1). \quad (3.7)$$

The fifth explicit term in (3.7) does not cancel away; it must be included to this order so that the fourth term facilitates the summation over N. But then, even the Poisson summation formula in the form (B.7) will not facilitate the summation over s, so we do not proceed further analytically in this manner.

In any case, we note that the exponential factor in the spectral function (1.3) corresponding to the single term $(\gamma - 1)^{-1}$ on the right side of (3.7) results in the vanishing of the spectral function as $\gamma \to 1$, in contrast with the previous case of annuli with pure Dirichlet (Section 1) or pure Neumann (Section 2.2) boundary conditions where the term representing the total length of the perimeter remained finite (see (1.7)). This is consistent with our conjecture (3.4) for this doubly-connected region, since for this mixed boundary condition case, $L_N - L_D$ is proportional to $(\gamma - 1)$ and hence indeed vanishes as $\gamma \to 1$.

3.3. Annular sector

The annular sector bounded by plane polar coordinates $r = a, b$, $\phi = 0, \alpha$ $(0 < \alpha \leqslant 2\pi)$ is simply connected but not convex. In this case, the Bessel function order ν in the radial part of the eigenfunction (*cf.* [13]) is in general not an integer. For instance, for Dirichlet boundary conditions,

$$\nu = N\pi/\alpha; \qquad N = 1, 2, 3, \ldots, \qquad (3.8)$$

and this is to be used in equation (A.2).

There are six distinct inequivalent configurations with D.b.c., or with N.b.c., on both curved sides and either b.c. on the straight side segments. In all cases, proceeding as in previous sections for small $(\gamma - 1)$, we find by calculations using Appendices A, B that the spectral function expansion (1.3) has the form of equations (3.4), (3.5) with $A = (b^2 - a^2)\, \alpha/2$, curved side lengths $b\alpha$ and $a\alpha$, and two radial segments of length $(b - a)$. Although the latter are of order $(\gamma - 1)$

compared with the curved side lengths, they are indeed elicited explicitly where appropriate by these calculations, with neglected error being of order $(\gamma - 1)^2$. The corner contributions come out correctly according to (3.5).

The case $\alpha = 2\pi$ is of physical interest, being a septate annulus. For purely D.b.c., it corresponds to a vibrating annular membrane fixed along a radius. For purely N.b.c., it corresponds to the transverse vibrations of gas in a coaxial cylinder with a single rigid radial wall. The lowest frequency is lower than for an uninterrupted annulus. (Cf. [27], page 300; and [22], page 31, for the electromagnetic transmission line case.)

4. The three-dimensional case

For a three-dimensional simply-connected domain bounded by a convex surface, Waechter [33] has determined, indirectly via the diffusion (heat) equation and Green's functions and by other considerations, the expansion for the spectral function

$$E_3 \equiv \sum e^{-\lambda t} \tag{4.1}$$

involving eigenvalues of the Laplacian (three-dimensional Helmholz equation)

$$\nabla^2 \psi + \lambda \psi = 0 \tag{4.2}$$

for the case of *Dirichlet* boundary conditions to be

$$E_3^{(D)} \sim \frac{V}{8(\pi t)^{3/2}} - \frac{S}{16\pi t} + \frac{C}{2(\pi t)^{1/2}} + \text{constant} + O(t^{1/2}) \quad \text{as } t \to 0, \tag{4.3}$$

where V is the volume of the region and S is its surface area. C is a constant which, if there are edges present, receives contributions in the form

$$\int \left[\frac{\pi^2 - \theta^2}{24\pi\theta} \right] dL(\theta). \tag{4.4}$$

In this integral, $L(\theta)$ is the length of the edge formed by two faces inclined at angle θ, and the bracketed term is the corner correction (1.5) of the *two-dimensional* plane case. (There is a misprint in equation (4.9) in [33]: see [30] and [23].)

We may fairly readily extend our previous work using direct calculations involving explicit eigenvalues and Poisson summation formulae to three-dimensional examples with Dirichlet (1.2) and/or Neumann (2.1) boundary conditions. Purely Neumann boundary conditions correspond physically to oscillations of a gas confined within the volume, and ψ in (4.2) may represent the pressure or the velocity potential ([25], page 243).

4.1 Rectangular parallelepiped

Before dealing with doubly-connected regions, which are the main concern of this paper, it is worthwhile considering the simpler case of the cuboid to obtain an understanding of the various terms which arise and of the effects of edges and vertices.

For the case of a cuboidal region (rectangular parallelepiped) with either D.b.c. or N.b.c on any face, there are ten distinct inequivalent configurations. Using calculations similar to those in Sections (2.1) and (3.1), with eigenfunctions the product of three appropriate trigonometric functions, and utilizing the Poisson summation formulae (B.3), (B.4), (B.7), in three factored summations, we find that in each case the spectral function expansion for a cuboid of side lengths a, b and c, may be written in the form

$$E_3 \sim \frac{abc}{8(\pi t)^{3/2}} + \frac{S_N - S_D}{16\pi t} + \frac{L_{DD} + L_{NN} - L_{DN}}{16(2\sqrt{\pi t})} + \left(\sum_{\substack{NNN, \\ DDN}} - \sum_{\substack{DDD, \\ DNN}} \right) \frac{1}{64}$$

$$(4.5)$$

where S_N (S_D) is the total area of those surfaces with N.b.c. (D.b.c.), and L_{DD} denotes the total length of edges formed by faces which both have D.b.c., *etc.* In the last, constant term in (4.5), the summation label NNN, for instance, denotes (three-dimensional) vertices subtended by three faces with N.b.c., *etc.* (For purely D.b.c., (4.5) corresponds to a result quoted by Waechter [33], the last term taking the value $-\frac{1}{8}$.)

For example, for faces $x = 0, a, y = 0, b$ D.b.c., and $z = 0, c$ N.b.c., we find

$$E_3 \sim \frac{abc}{8(\pi t)^{3/2}} + \frac{2ab - 2bc - 2ac}{16\pi t} + \frac{4c - (4a + 4b)}{32(\pi t)^{1/2}} + \frac{1}{8}, \qquad (4.6)$$

whilst for $x = 0, y = 0, z = 0$ D.b.c. and $x = a, y = b, z = c$ N.b.c. we obtain the first term in (4.6) only. We note that the number (1/16) in the third term in (4.5) is, as mentioned in Section (3.1), the value of the two-dimensional corner expression (1.5) for the case $\theta = \pi/2$.

Based on (4.5) and (4.4), we conjecture that, for a simply connected convex volume, the spectral function (4.1) for the three-dimensional case (4.2) with pure or mixed boundary conditions has the expansion

$$E_3 \sim \frac{V}{8(\pi t)^{3/2}} + \frac{S_N - S_D}{16\pi t} + \frac{C}{2(\pi t)^{1/2}} + \text{constant} + O(t^{1/2}) \qquad (4.7)$$

where the contribution from the edges to the constant C in the third term is

$$\int \left[\frac{\pi^2 - \theta^2}{24\pi\theta} \right] [dL_{DD}(\theta) + dL_{NN}(\theta) - dL_{DN}(\theta)]. \qquad (4.8)$$

The contribution to the constant (fourth term) in (4.7) due to the vertices is conjectured, from (4.5), to have the form

$$\left(\sum_{\substack{NNN, \\ DDN}} - \sum_{\substack{DDD, \\ DNN}} \right) x \tag{4.9}$$

where the contribution for a purely Dirichlet vertex is $-x$, and x assumes the value $1/64$ for a *solid* angle of $4\pi/8 = \pi/2$. As in the two-dimensional case, a "parity factor" (-1) appears to be associated with each D.b.c.

4.2 Annular cylinder

For the case of a finite annular cylinder, the eigenfunctions are the product of the two-dimensional annular eigenfunctions (see [13]) and a single trigonometric function of the appropriate form for the boundary conditions in the z direction, as in Section 2.1. The eigenvalue λ in (4.1) is the sum of the squares of certain roots in the annular problem, as in (1.6), and a single Cartesian square of the type appearing in (2.4). The spectral function (4.1) therefore just has as factors, the spectral function for the annular case and a single Cartesian term, and so the special function expansion may be obtained from the product of the result already obtained in Sections 1 and 2.2 and (B.3), (B.4) or (B.7).

There are twelve distinct inequivalent configurations corresponding to various combinations of Dirichlet and/or Neumann boundary conditions on the several surfaces, but because of the results of Section (3.2) we are only able to discuss the six configurations with D.b.c. (or N.b.c.) on both curved surfaces.

We find that in each case the spectral function expansion, for small $(\gamma - 1)$, may be written in the form

$$E_3 \sim \frac{V}{8(\pi t)^{3/2}} + \frac{S_N - S_D}{16\pi t} + \frac{L_{DD} + L_{NN} - L_{DN}}{16\left[2(\pi t)^{1/2}\right]} + \text{constant} + O(t^{1/2})$$

$$\tag{4.10}$$

which agrees with our conjectures (4.7), (4.8). For instance, for the case of D.b.c. on the annular ends and N.b.c. on the curved surfaces, we obtain, for cylinder length c,

$$E_3 \sim \frac{\pi(b^2 - a^2)c}{8(\pi t)^{3/2}} + \frac{2\pi(b + a)c - 2\pi(b^2 - a^2)}{16\pi t}$$

$$+ \frac{-2(2\pi)(b + a)}{16(2\sqrt{\pi t})} + \frac{3c}{16a} + O(\sqrt{t}). \tag{4.11}$$

(In the cases of D.b.c. on the curved surfaces, the fourth, constant, term turns out to be $-c/(16a)$.)

Our result (4.10) for the mixed annular cylinder complements and extends a calculation of Waechter [33] *via* Green's functions for the full cylinder with purely D.b.c. which agrees with the Dirichlet terms in the first three terms in (4.10) and also includes curvature terms, the first of which is absent in (4.10) because of the double-connectedness (*cf.* (1.4)).

5. Concentric spherical cavity

The eigenfunctions for the doubly-connected region bounded by two concentric spherical surfaces, radii $r = a, b$, involve spherical Bessel functions and associated Legendre functions (see [8], pages 11–12). Because of the degeneracy associated with the azimuthal angle, there is a factor $(2N + 1)$ for each order N of spherical Bessel functions j_N, y_N, where $N = 0, 1, 2, \ldots$.

For D.b.c., the eigenvalue equation is given by (A.1) with $\nu = N + \frac{1}{2}$, with roots given by (A.2). For nearby concentric surfaces, *i.e.* small $(\gamma - 1)$, the spectral function (4.1) leading to the first 2 terms in the asymptotic expansion has the form

$$E_3 \sim \left[\frac{a(\gamma - 1)}{\sqrt{\pi t}} - 1 \right] \sum_{N=0}^{\infty} (N + \tfrac{1}{2}) \exp\left[-(N + \tfrac{1}{2})^2 t/(a^2 \gamma) \right]. \qquad (5.1)$$

The first term in square parentheses arises *via* (B.3) from the sum over s just as for the annular case of Section 1 (the factor $\frac{1}{2}$ being carried through to the next summation). The sum in (5.1) is not obtainable by manipulation of the Poisson summation formulae in Appendix B.

Up till now we have carried through explicit calculations to verify formulae such as (1.4) and (4.3). Now we are going to turn the problem on its head, and use the first two terms of (4.3) for the *Dirichlet* case to evaluate the leading asymptotic form of the sum remaining in (5.1). Then we shall use that result to follow our original procedure for the Neumann case.

The ratio of the first to second terms in (4.3) for the case of a concentric spherical cavity with $V = (4/3)\pi(b^3 - a^3)$, $S = 4\pi(a^2 + b^2)$, is just

$$\frac{a(\gamma - 1)}{\sqrt{\pi t}} + O(\gamma - 1)^2. \qquad (5.2)$$

This is exactly the ratio of first to second terms in the first square parantheses in (5.1). We therefore deduce by comparing the second term in (4.3) with the second term in (5.1) for small $(\gamma - 1)$ that

$$\sum_{N=0}^{\infty} (N + \tfrac{1}{2}) \exp\left(-z(N + \tfrac{1}{2})^2 \right) \sim \frac{1}{2z} \quad \text{as } z \to 0, \qquad (5.3)$$

where $z = t/(a^2\gamma)$ in this instance. The correction is of order constant. That this is not a Poisson-summation-type formula can be seen by the fact that there is no factor $\pi^{1/2}$ on the right side, and the correction is not of order $z^{-1/2}$ (*cf.* equations (B.4), (B.7)).

Now we return to our usual procedure of previous sections in carrying out the explicit calculation of the spectral function using analytical expressions for the eigenvalues, for the case of *Neumann* boundary conditions. We shall demonstrate the change in sign in the surface area term of (4.7) for N.b.c., utilizing (5.3).

For Neumann boundary conditions on a concentric spherical cavity, there is first of all a zero eigenvalue corresponding to constant eigenfunction which produces a constant term 1 in the spectral function (4.1) and therefore is not required if we work to the first two leading orders in (4.7). The remaining eigenvalues satisfy the eigenvalue equation (A.8) with $\nu = N = 0, 1, 2, \ldots$, involving spherical Bessel functions ([1], page 437; [2], page 52). Most of the eigenvalues in (4.2) are given by (1.6) where x is given, for small $(\gamma - 1)$, by (A.9) (see [24], and [15], Appendix III) with ν a nonnegative integer. This leads to the same contributions to the first two terms in the spectral function (4.1) expansion as in the Dirichlet case (5.1) with (5.3).

As discussed in Section (2.2) and Appendix A, for N.b.c. equations of the derivative cross-product type (A.3), (A.8) there is an exceptional set of lowest roots which remain finite as $\gamma \to 1$, *i.e.* which are not given by the McMahon-type formulae such as (A.9) and which, for equation (A.8), do not appear to have received attention. They are given by (A.10) (see [14]); for our purposes, the first-order approximation suffices:

$$x_N^1 = \sqrt{N(N + 1)} \left\{ 1 - \tfrac{1}{2}(\gamma - 1) + O\left[(\gamma - 1)^2\right] \right\}. \tag{5.4}$$

(The zeroth order approximation in (5.4) may be found in Rayleigh [27], page 272.) Use of (5.4) in (1.6) gives an extra contribution to (4.1) which, for small $(\gamma - 1)$ and leading order in $(1/t)$, is just twice the sum in the right side of (5.1). Thus adding this to the contribution (5.1) from McMahon-type roots indeed changes the sign of the second, area, term. Use of (5.3) then yields the first two terms in (4.7) for the Neumann case, with volume V and total surface area S_N elicited explictly (the corrections being of order $(\gamma - 1)^2$).

The asymptotic result (5.3) may be derived by direct mathematical methods (Appendix C), but in this instance we prefer to proceed in the above manner to show how the Dirichlet boundary condition results may contribute towards obtaining the Neumann boundary condition results.

With this rather spectacular demonstration of the importance of the exceptional eigenvalues associated with (A.8) we close our series of calculated cases of the spectral function expansion involving Neumann boundary conditions. The

conjectures (3.4) with (3.5) (two dimensions) and (4.7) with (4.8) and (4.9) (three dimensions) are recorded for comparison with any prospective extensions of general procedures.

Acknowledgements

Appendix A. Formulae for zeros

Asymptotic expansions of zeros of various cross-products of Bessel and related functions obtained by McMahon and others (see [24]; [15], pages 260–263; [16], pages 617–618; [1], page 374) and used in this paper are recorded here. In each case a is the inner radius, b is the outer radius, and $\gamma = b/a$ is near 1, *i.e.* $\gamma - 1$ is small and positive. $x_\nu^{(s)}$ is the sth root in order of magnitude. J_ν and Y_ν are Bessel functions of order ν of the first and second kind respectively [1]. A prime on a function denotes differentiation with respect to argument. j_ν and y_ν are spherical Bessel functions of order ν [1], [2].

We note that ν is unrestricted [24]. Furthermore, as indicated in [7] and [25] page 604, the formulae are valid not only for large s but also for low s values if $\gamma - 1$ is sufficiently small.

Finally, in the case of equation (A.3), the lowest root for $\nu \neq 0$ is not given by a McMahon-type formula, and is overlooked by mathematical formula handbooks (although numerical values are available, *e.g.* [11], [6], [3]). Equation (A.5) for this exceptional root has been derived by the author [14]. The $\nu = 0$ case of (A.3) is just the $\nu = 1$ case of (A.1), since $J_0' = -J_1$, $Y_0' = -Y_1$ (*cf.* the second terms in (A.4) and (A.2)). Equation (A.8) has an exceptional lowest root if $\nu \neq 0$ (−1), given by (A.10), which has been derived by the author. Higher order terms for equations (A.5) and (A.10) may be found in [14].

$$J_\nu(x)Y_\nu(\gamma x) - J_\nu(\gamma x)Y_\nu(x) = 0, \tag{A.1}$$

$$x_\nu^{(s)} = \frac{\pi s}{\gamma - 1} + \frac{4\nu^2 - 1}{8\pi\gamma}\left(\frac{\gamma - 1}{s}\right) + O\left[\left(\frac{\gamma - 1}{s}\right)^3\right], \tag{A.2}$$

$$J_\nu'(x)Y_\nu'(\gamma x) - J_\nu'(\gamma x)Y_\nu'(x) = 0, \tag{A.3}$$

$$x_\nu^{(s')} = \frac{\pi s}{\gamma - 1} + \frac{4\nu^2 + 3}{8\pi\gamma}\left(\frac{\gamma - 1}{s}\right) + O\left[\left(\frac{\gamma - 1}{s}\right)^3\right], \tag{A.4}$$

where $s' = s$, $\nu = 0$, and $s' = s + 1$, $\nu \neq 0$ [14],

$$x_\nu^1 = \nu\left\{1 - \tfrac{1}{2}(\gamma - 1) + \frac{7}{24}(\gamma - 1)^2 + O\left[(\gamma - 1)^3\right]\right\}, \tag{A.5}$$

$$J_\nu'(x)Y_\nu(\gamma x) - J_\nu(\gamma x)Y_\nu'(x) = 0, \tag{A.6}$$

$$x_\nu^{(s)} = \frac{\pi\left(s - \tfrac{1}{2}\right)}{(\gamma - 1)} + \frac{1}{2\pi\gamma\left(s - \tfrac{1}{2}\right)} + \frac{(4\nu^2 + 3)}{8\pi}\frac{(\gamma - 1)}{\left(s - \tfrac{1}{2}\right)}$$

$$- \frac{1}{4\pi^3}\frac{(\gamma - 1)}{\left(s - \tfrac{1}{2}\right)^3} + O\left[\frac{(\gamma - 1)^2}{\left(s - \tfrac{1}{2}\right)^3}\right], \tag{A.7}$$

$$j_\nu'(x)y_\nu'(\gamma x) - j_\nu'(\gamma x)y_\nu'(x) = 0, \tag{A.8}$$

$$x_\nu^{(s')} = \frac{\pi s}{\gamma - 1} + \frac{4\left(\nu + \tfrac{1}{2}\right)^2 + 7}{8\pi\gamma}\left(\frac{\gamma - 1}{s}\right) + O\left[\left(\frac{\gamma - 1}{s}\right)^3\right], \tag{A.9}$$

where $s' = s$, $\nu = 0\,(-1)$, and $s' = s + 1$, $\nu \neq 0\,(-1)$ [14],

$$x_\nu^1 = \sqrt{\nu(\nu + 1)}\left\{1 - \tfrac{1}{2}(\gamma - 1) + \frac{5}{24}(\gamma - 1)^2 + O\left[(\gamma - 1)^3\right]\right\}. \tag{A.10}$$

Appendix B. Poisson summation formulae

The extended Poisson summation formula for Gaussians may conveniently be written in the form (see [34], page 124 and page 476; [31], page 347)

$$\sum_{N=-\infty}^{\infty} \exp\left[-z(N + a)^2\right] = \left(\sqrt{\pi}/\sqrt{z}\right)\sum_{N=-\infty}^{\infty}\exp\left(-\pi^2 N^2/z\right)\cos(2\pi a N). \tag{B.1}$$

If $a = 0$, we have

$$\sum_{N=-\infty}^{\infty}\exp(-zN^2) \sim \sqrt{\pi}/\sqrt{z} \quad \text{as } z \to 0 \tag{B.2}$$

and

$$\sum_{N=1}^{\infty}\exp(-zN^2) \sim \sqrt{\pi}/(2\sqrt{z}) - \tfrac{1}{2} \quad \text{as } z \to 0, \tag{B.3}$$

$$\sum_{N=0}^{\infty}\exp(-zN^2) \sim \sqrt{\pi}/(2\sqrt{z}) + \tfrac{1}{2} \quad \text{as } z \to 0. \tag{B.4}$$

If $a = \tfrac{1}{2}$, we get

$$\sum_{N=-\infty}^{\infty}\exp\left[-z\left(N + \tfrac{1}{2}\right)^2\right] = \left(\sqrt{\pi}/\sqrt{z}\right)\sum_{N=-\infty}^{\infty}(-1)^N\exp\left(-\pi^2 N^2/z\right) \tag{B.5}$$

$$\sim \sqrt{\pi}/\sqrt{z} \quad \text{as } z \to 0 \tag{B.6}$$

and

$$\sum_{N=0}^{\infty} \exp\left[-z\left(N + \tfrac{1}{2}\right)^2\right] \sim \sqrt{\pi}/(2\sqrt{z}) \quad \text{as } z \to 0, \tag{B.7}$$

where in each asymptotic case the exponentially small terms are neglected (*cf.* [23]).

Appendix C. Riemann integral summation method

We may evaluate the asymptotic expansion result (5.3) in a rather direct way by approximating the sum by a Riemann integral, utilising a method described by Bender and Orszag ([4], page 303). Writing $z = \zeta^2$, we have

$$\sum_{N=0}^{\infty} \left(N + \tfrac{1}{2}\right) e^{-z(N+\frac{1}{2})^2} = \zeta^{-2} \sum_{N=0}^{\infty} \left[\left(N + \tfrac{1}{2}\right)\zeta\right] e^{-\{(N+\frac{1}{2})\zeta\}^2}(\zeta)$$

$$\sim \zeta^{-2} \int_0^{\infty} u e^{-u^2}\, du \quad \text{as } \zeta \to 0$$

$$= \zeta^{-2}/2, \tag{C.1}$$

since as $\zeta \to 0$ the second series above becomes a Riemann sum for the given integral. Thus (5.3) is obtained.

A more general result obtained similarly is

$$\sum_{N=0}^{\infty} (N + \beta)^p e^{-z(N+\beta)^q} \sim \frac{1}{q}\Gamma[(p+1)/q] \frac{1}{z^{[(p+1)/q]}} \tag{C.2}$$

as $z \to 0$, for $p > -1$, $q > 0$, and arbitrary finite β, where $\Gamma(\ )$ is the Gamma function. Note that this method only gives the leading asymptotic term, which is independent of β. For $p = 0$, $q = 2$ and $\beta = 0$ or $\beta = \tfrac{1}{2}$, we recover the leading term of (B.4) or (B.7). Equation (5.3) corresponds to $p = 1$, $q = 2$ and $\beta = \tfrac{1}{2}$.

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