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COEFFICIENT BEHAVIOR OF A CLASS OF MEROMORPHIC FUNCTIONS

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1. Statement of results. With $k \geqq 2$, denote by Λ_k the class of functions f of the form

$$(1) \quad f(z) = \frac{1}{z} + a_0 + \sum_{n=1}^{\infty} a_n z^n$$

which are analytic in $\gamma = \{z : 0 < |z| < 1\}$ and which map γ onto the complement of a domain with boundary rotation at most $k\pi$. It is known [**2**] that $f \in \Lambda_k$ if and only if there exist regular starlike functions s_1 and s_2, with

$$(k + 2)s_1''(0) = (k - 2)s_2''(0),$$

such that

$$(2) \quad f'(z) = -\frac{1}{z^2} \frac{(s(z)/z)^{(k-2)/4}}{(s_1(z)/z)^{(k+2)/4}}.$$

Using this representation, the author proved [**2**] that for any $k \geqq 2$, there exists $r(k) < 1$ such that for $r(k) < r < 1$ and for all $f \in \Lambda_k$, we have the sharp inequality

$$(3) \quad r^2 M(r, f') \leqq \left(1 + r^2 - 2r\frac{k-2}{k+2}\right)^{(k+2)/4} (1 - r)^{1-k/2},$$

where $M(r, f') = \max \{|f'(z)| : |z| \leqq r\}$. In addition, $|a_1| \leqq k/2$ and $|a_2| \leqq k/6$. Although both inequalities are sharp, the extremal functions are different.

The purpose of this note is to examine the asymptotic behavior of the maximum modulus and Laurent coefficients of functions of class Λ_k. These results are similar in spirit to previous results of the author for the well-known class V_k [**3**] and for the class $K(\beta)$ of analytic close-to-convex functions of order $\beta > 0$ [**4**].

THEOREM 1. *Suppose* $k > 2$ *and* $f \in \Lambda_k$ *is given by* (2). *Then*

$$\omega = \lim_{r \to 1} (1 - r)^{k/2-1} M(r, f')$$

exists, is finite, and equals 0 *unless* s_2 *is of the form* $z/(1 - ze^{-i\theta})^2$. *If* $\omega > 0$,

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there exists θ *such that*

$$\omega = \lim_{r \to 1} (1 - r)^{k/2-1} |f'(re^{i\theta})|.$$

The proofs of the analagous results for V_k and $K(\beta)$ depend on the linear invariance (in the sense of Pommerenke) of these two classes. Since Λ_k is not a linear invariant family, Theorem 1 requires a different method of proof.

An application of the major-minor arc technique of W. K. Hayman yields the following result.

THEOREM 2. *Suppose* $k > 2$ *and* $f \in \Lambda_k$ *is given by* (1). *Then*

$$\lim_{n \to \infty} \frac{|a_n|}{n^{(k/2)-3}} = \frac{\omega}{\Gamma((k/2) - 1)}.$$

Define $F_k \in \Lambda_k$ by

$$(4) \quad F_k'(z) = -\frac{1}{z^2} \frac{\left(1 + z^2 - 2z\dfrac{k-2}{k+1}\right)^{(k+2)/4}}{(1-z)^{k/2-1}},$$

and set

$$F_k(z) = z^{-1} + A_0 + \sum_{n=1}^{\infty} A_n z^n.$$

THEOREM 3. *Suppose* $k > 2$, $f \in \Lambda_k$ *is given by* (1), *and* F_k *is as above. Then*

$$\lim_{n \to \infty} |a_n|/|A_n|$$

exists, is at most 1, *and equals* 1 *if and only if* $f(z) = e^{i\theta}F_k(e^{i\theta}z)$ *for some* θ.

We note that for fixed $f \in \Lambda_k$, there exists $n(f)$, depending only on f, such that $|a_n| \leqq |A_n|$ for $n \geqq n(f)$. This result is clearly false for $k = 2$, since then $F_2(z) = z^{-1} + A_0 + z$ (and $A_n = 0$ for $n \geqq 2$). It is also interesting to note that although F_k is *not* the solution (for all n) of the problem of determining max $\{|a_n| : f \in \Lambda_k\}$, Theorem 3 shows that the coefficients of F_k do in fact eventually dominate the coefficients of any fixed $f \in \Lambda_k$. In other words, F_k is the unique solution to the asymptotic coefficient problem.

2. Proof of Theorem 1. If $f \in \Lambda_k$ is given by (2) with s_2 not of the form $z/(1 - ze^{i\theta})^2$, then [**6**] there exists $d < 2$ such that $s_2(re^{i\theta}) = O(1)(1 - r)^{-d}$. It follows immediately, since $k > 2$, that $\omega = 0$.

We now assume $s_2(z) = z/(1 - ze^{i\theta})^2$ for some θ. For notational ease we assume $\theta = 0$. Set

$$\omega_1 = \limsup_{r \to 1} (1 - r)^{k/2-1} M(r, f')$$

$$\omega_2 = \liminf_{r \to 1} (1 - r)^{k/2-1} M(r, f'),$$

and note that $0 \leqq \omega_2 \leqq \omega_1 < \infty$. The remainder of the proof will be divided into a sequence of lemmas. We first state a definition: a sequence $\{z_n\}_1^\infty$, with $|z_n| < |$, $\lim_{n\to\infty} z_n = 1$, is said to approach 1 *strictly tangentially* if, given any Stolz angle S with vertex 1, there exists $N(S)$ such that $z_n \notin S$ for $n \geqq N(S)$.

LEMMA 1. *If* $\{z_n\}_1^\infty$ *approaches* 1 *strictly tangentially, then*

$$\lim_{n\to\infty} (1 - |z_n|)^{k/2-1}|f'(z_n)| = 0.$$

Proof. Since f is given by (2) with $s_2(z) = z/(1-z)^2$,

$$|z_n|^2(1 - |z_n|)^{k/2-1}|f'(z_n)| = \left|\frac{1-|z_n|}{1-z_n}\right|^{k/2-1}\left|\frac{z_n}{s_1(z_n)}\right|^{(k+2)/4}.$$

The lemma now follows upon noting that $|z/s_1(z)| \leqq 4$ [**1**, p. 353] and that $(1 - |z_n|)/|1 - z_n| \to 0$ as $n \to \infty$ (since $\{z_n\}_1^\infty$ approaches 1 strictly tangentially).

LEMMA 2. *If* $\omega_1 > 0$, *then* $\omega_1 = \omega_2$.

Proof. Choose $\{z_n\}_1^\infty$ such that $z_n \to 1$,

$$|f'(z_n)| = M(|z_n|, f'),$$

and

$$\omega_1 = \lim_{n\to\infty} (1 - |z_n|)^{k/2-1}|f'(z_n)|.$$

(Since $s_2(z) = z/(1 - z)^2$, such a sequence exists.) Lemma 1 and the hypothesis $\omega_1 > 0$ together imply the existence of a Stolz angle S and a subsequence $\{z_{n_j}\}$ such that $z_{n_j} \in S$ for all j. Therefore, we may choose a second subsequence (denoted by $\{z_{n_j}\}$ for notational ease) such that

$$\lim_{j\to\infty} (1 - |z_{n_j}|)/|1 - z_{n_j}|$$

exists, is finite, and is non-zero. Since

$$\omega_1 = \lim_{j\to\infty}\left|\frac{1-|z_{n_j}|}{1-z_{n_j}}\right|^{k/2-1}\left|\frac{z_{n_j}}{s_1(z_{n_j})}\right|^{(k+2)/4} > 0,$$

we conclude that

$$(5) \quad 0 < \lim_{j\to\infty}\left|\frac{s_1(z_{n_j})}{z_{n_j}}\right| < \infty.$$

In view of the fact that

$$v(\theta) = \lim_{r\to 1} \arg s_1(re^{i\theta})$$

is continuous at $\theta = 0$ (see [**6**, Lemma 1]) and recalling that $s_2(z) = z/(1 - z)^2$),

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we see from (5) and [**7**, Theorem 7] that $\lim_{r\to 1} s_1(r)$ exists and is finite. We can now use a theorem of Lindelöf [**1**, p. 260] and the fact that $z_{n_j} \in S$ for all j to conclude that

$$\lim_{j\to\infty} |s_1(z_{n_j})| = \lim_{r\to 1} |s_1(r)|,$$

and so

$$0 < \omega_1 \leqq \lim_{j\to\infty} |s_1(z_{n_j})|^{-(k+2)/4}$$
$$= \lim_{r\to 1} |s_1(r)|^{-(k+2)/4}$$
$$= \lim_{r\to 1} (1-r)^{k/2-1}|f'(r)| \leqq \omega_1.$$

Therefore

$$\omega_1 = \lim_{r\to 1} (1-r)^{k/2-1}|f'(r)|.$$

It is now clear that $\omega_2 = \omega_1$, as required.

Theorem 1 now follows immediately. If $\omega_1 = 0$, then clearly $\omega_1 = \omega_2 = \omega = 0$. If $\omega_1 > 0$, then Lemma 2 states that ω exists and is finite. In the course of the proof of Lemma 2, we showed that if $\omega > 0$, then

$$\omega = \lim_{r\to 1} (1-r)^{k/2-1}|f'(r)|.$$

If in place of $s_2(z) = z/(1-z)^2$ we have $s_2(z) = z/(1-ze^{-i\theta})^2$, then

$$\omega = \lim_{r\to 1} (1-r)^{k/2-1}|f'(re^{i\theta})|.$$

3. Proof of Theorem 2. Suppose first that $\omega = 0$. Since

$$n^2|a_n| = O(1)M(r_n, f')$$

where $r_n = 1 - 1/n$ [**8**], we have $a_n = o(1)n^{k/2-3}$, as required.

If $\omega > 0$, we apply the major-minor arc technique to the function $zf'(z)$. Since this technique is well-known, we shall merely sketch the proof. Set $g(z) = -zf'(z)$, and note that

$$g'(z) = -z^{-2} - \sum_{n=1}^{\infty} n^2 a_n z^{n-1}$$

$$= \frac{k-2}{4} z^{-2} \frac{1+z}{(1-z)^{k/2}} \left(\frac{z}{s_1(z)}\right)^{(k+2)/4} - \frac{k+2}{4} z^{-2} \frac{s_1'(z)}{(1-z)^{k/2-1}} \left(\frac{z}{s_1(z)}\right)^{(k+6)/4}. \tag{6}$$

(Again we assume $s_2(z) = z/(1-z)^2$).

LEMMA 3. *Suppose* $k > 2$ *and*

$$\omega = \lim_{r\to 1} (1-r)^{k/2-1}|f'(r)| > 0.$$

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Then given $\delta > 0$, there exists $C(\delta) > 0$ and $r(\delta) < 1$ such that

$$\int_E |g'(re^{i\theta})|d\theta < \frac{\delta}{(1-r)^{k/2-1}}$$

for $r(\delta) < r < 1$, where $E = \{\theta : C(\delta)(1-r) \leqq |\theta| \leqq \pi\}$.

Proof. It follows from (6) that there exist constants A_1 and A_2 such that, with $z = re^{i\theta}$,

$$\int_E |g'(z)|d\theta \leqq A_1 \int_E |1-z|^{-k/2}d\theta + A_2 \int_E |s_1'(z)/s_1(z)|\, |1-z|^{1-k/2}d\theta.$$

As in [**3**, Lemma 3.1] we see that

$$\int_E |1-z|^{-k/2}d\theta < \delta/(1-r)^{k/2-1},$$

provided $C(\delta)$ is chosen sufficiently large.

We next choose conjugate indices p and q, both greater than 1, such that $p(k/2-1) > 1$. Then

$$\int_E |s_1'(z)/s_1(z)|\, |1-z|^{1-k/2}d\theta$$
$$\leqq \left\{\int_E |1-z|^{-p(k/2-1)}d\theta\right\}^{1/p} \left\{\int_{-\pi}^{\pi} \left|\frac{s_1'(z)}{s_1(z)}\right|^q d\theta\right\}^{1/q}.$$

As above, we choose $C(\delta)$ so that

$$\int_E |1-z|^{-p(k/2-1)}d\theta < \delta/(1-r)^{1-p(k/2-1)}.$$

In addition, since $zs_1'(z)/s_1(z)$ is subordinate to $(1+z)/(1-z)$, we have

$$\int_{-\pi}^{\pi} |s_1'(z)/s_1(z)|^q d\theta = O(1)(1-r)^{-q+1}.$$

The lemma now follows upon combining the above estimates.

LEMMA 4. *Suppose $k > 2$, $f \in \Lambda_k$, $\omega > 0$. For $n \geqq 2$, set $r_n = 1 - 1/n$, $\omega_n = (k/2-1)s_1(r_n)^{-(k+2)/4}$, $g_n'(z) = \omega_n(1-z)^{-k/2}$. Put*

$$I_n = \{\theta : 0 \leqq |\theta| \leqq c(1-r_n)\}.$$

Then with $z = r_n e^{i\theta}$, $g'(z)/g_n'(z) \to 1$ uniformly for $\theta \in I_n$, as $n \to \infty$.

Proof. This lemma follows fairly easily from (6). An application of Lindelöf's theorem (as in Lemma 2) shows that the quotient of the first summand in (6) and $g_n'(z)$ approaches 1 uniformly for $\theta \in I_n$, as $n \to \infty$. The quotient of the second summand in (6) and $g_n'(z)$ approaches 0, as may be seen by combining Lindelöf's theorem with the fact that the starlike function s_1 has a Stieltjes integral representation in which the integrator is continuous at $\theta = 0$ (and hence $(1 - z)s_1'(z) = o(1)$ as $|z| \to 1, \theta \in I_n$).

If we now apply Lemmas 3 and 4 in the standard fashion (see, for example, [**3**, p. 401]), we arrive at the conclusion of Theorem 2. The details are left to the interested reader.

4. Proof of Theorem 3. Set

$$\omega^* = \lim_{r\to 1}\left(1 + r^2 - 2r\frac{k-2}{k+2}\right)^{(k+2)/4} = \left(\frac{8}{k+2}\right)^{(k+2)/4}.$$

Given $f \in \Lambda_k$, we see from (3) that $\omega \leqq \omega^*$, with equality for F_k as defined in (4). Hence, with

$$F_k(z) = z^{-1} + A_0 + \sum_{n=1}^{\infty} A_n z^n,$$

it follows from Theorem 2 that $\lim_{n\to\infty} |a_n/A_n|$ exists and is at most 1. It remains to show that $\omega = \omega^*$ only when f is a rotation of F_k. The proof of this fact is somewhat technical, and will be divided into a sequence of lemmas.

LEMMA 5. *Suppose $f \in \Lambda_k (k > 2)$ is given by* (2). *Assume that*

$$\omega = \lim_{r\to 1} (1 - r)^{k/2-1}|f'(r)| > 0,$$

and suppose that s_1 in (2) *is given by*

$$s_1(z) = z \exp\left\{-\int_{-\pi}^{\pi} \log (1 - ze^{-it})d\alpha(t)\right\}$$

where α is increasing on $[-\pi, \pi]$ with

$$\int_{-\pi}^{\pi} d\alpha(t) = 2, \int_{-\pi}^{\pi} e^{-it}d\alpha(t) = 2(k-2)/(k+2).$$

Then

$$\omega^{\frac{-4}{k+2}} = \exp\left\{-\int_{-\pi}^{\pi} \log |1 - e^{-it}|d\alpha(t)\right\}.$$

Proof. Since $\omega > 0$ and $s_2(z) = z/(1 - z)^2$, the condition

$$\int_{-\pi}^{\pi} e^{-it}d\alpha(t) = 2(k-2)/(k+2)$$

is equivalent to $(k + 2)s_1''(0) = (k - 2)s_2''(0)$. Also, the $d\alpha$-measure of the point $t = 0$ is zero. Therefore, with $h_r(t) = -\log |1 - re^{-it}|$, $\lim_{r\to 1} h_r(t) = h_1(t)$ $d\alpha$ – a.e.; in addition,

$$\lim_{r\to 1} \int_{-\pi}^{\pi} h_r(t)d\alpha(t) = -\frac{4}{k+2}\log \omega < \infty.$$

Fatou's lemma thus implies that $h_1(t)$ is $d\alpha$-integrable. In order to complete the proof, we need only note that $h_r(t) \leqq h_1(t) + \log 2\, d\alpha$ — a.e. and apply the Lebesgue dominated convergence theorem.

Lemma 5 and the fact that the step functions are dense in the class of increasing functions will allow us to conclude that F_k is the unique solution (up to rotation) of the constrained optimization problem: $\max \{\omega : f \in \Lambda_k\}$. We first give a rather awkward preliminary technical lemma.

LEMMA 6. *With $k > 2$, $N \geqq 2$, $X = (x_1, \ldots, x_N)$, and $A = (a_1, \ldots, a_N)$, set*

$$h(X, A) = \prod_{j=1}^{N} (1 - x_j)^{a_j/2}.$$

Let $\epsilon > 0$ be given. Suppose that $X^* = (x_1^*, \ldots, x_N^*)$ and $A^* = (a_1^*, \ldots, a_N^*)$ are such that $h(X^*, A^*) = \max h(X, A)$, where the maximum is taken subject to the conditions $-1 \leqq x_j \leqq 1$ and $a_j \geqq 0 \quad (1 \leqq j \leqq N)$,

$$|x_1 - (k-2)/(k+2)| \geqq \epsilon, \quad a_1 \geqq \epsilon,$$

and

$$\sum_{j=1}^{N} a_j = 2, \quad \sum_{j=1}^{N} a_j x_j = 2(k-2)/(k+2).$$

Then there exists $\epsilon_1 > 0$ depending only on ϵ and k (in particular, ϵ_1 is independent of N) such that $h(X^*, A^*) \leqq 4/(k+2) - \epsilon_1$.

The proof of Lemma 6 consists of a tedious but straightforward application of the Lagrange multiplier theorem. Since the arguments required are similar in nature to the argument given in [**5**, Lemma 3.2], we omit the details. The fact that ϵ_1 is independent of N follows from the fact that, in the course of applying the Lagrange multiplier theorem, one shows that at the maximum point (X^*, A^*), the variables x_j^*, $1 \leqq j \leqq N$, can assume at most four distinct values.

We now complete the proof of Theorem 3. First note that F_k has a representation of the form (2) with $S_2(z) = z/(1-z)^2$ and

$$S_1(z) = z \exp \left\{ - \int_{-\pi}^{\pi} \log (1 - ze^{-it}) d\alpha(t) \right\},$$

where α is a step function having jumps of magnitude 1 at each of the points $t = \pm\delta$; here $\delta = \arccos (k-2)/(k+2)$.

Suppose that $f_1 \in \Lambda_k$, $f_1 \neq F_k$. We shall show $\omega(f_1) < \omega^*$. Clearly we may assume

$$0 < \omega(f_1) = \lim_{r \to 1} (1 - r)^{k/2-1} |f_1'(r)|;$$

we suppose that s_1 and s_2 correspond to f_1 via (2). Since $\omega(f_1) > 0$, we have $s_2(z) = z/(1-z)^2$, and since $f_1 \neq F_k$, we have $s_1 \neq S_1$. Hence α_1, the starlike

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integrator of s_1, is not the same as α, the starlike integrator for S_1. If α_1 were to concentrate all its mass at $t = \pm\delta$, the condition

$$\int_{-\pi}^{\pi} e^{-it} d\alpha_1(t) = 2(k-2)/(k+2)$$

would imply $\alpha_1 = \alpha$. Therefore we can choose $\eta > 0$ (depending only on α_1 and hence only on f_1) such that

$$(7) \quad \int_E d\alpha_1(t) \geqq \eta > 0,$$

where $E = \{t \in [-\pi, \pi] : |t - \delta| \geqq \eta \text{ or } |t + \delta| \geqq \eta\}$.

In view of (7) and Lemma 5, we choose a sequence $\{\mu_N\}_2^{\infty}$ of step functions with the following properties: μ_N has at most N discontinuities,

$$\int_{-\pi}^{\pi} d\mu_N = 2, \int_{-\pi}^{\pi} e^{-it} d\mu_N = 2(k-2)/(k+2),$$

$$\int_E d\mu_N \geqq \eta/2 > 0, \quad \text{and} \quad \lim_{N\to\infty} \omega_N = \omega(f_1),$$

where

$$\omega_N^{-4/(k+2)} = \exp\left\{-\int_{-\pi}^{\pi} \log|1 - e^{-it}| d\mu_N(t)\right\}.$$

Denote by $\{a_j\}_1^N$ and $\{t_j\}_1^N$ respectively the magnitudes and positions of the jumps of μ_N. It follows that

$$\frac{1}{2}\omega_N^{4/(k+2)} = \prod_{j=1}^{N} (1 - x_j)^{a_j/2}$$

where

$$x_j = \cos t_j, \sum_{j=1}^{N} a_j = 2, \text{and} \sum_{j=1}^{N} a_j x_j = 2(k-2)/(k+2).$$

It follows from (7) that there exists $\epsilon > 0$ depending only on f_1 such that $a_1 \geqq \epsilon$, $|x_1 - (k-2)/(k+2)| \geqq \epsilon$ (relabel variables if necessary). By Lemma 6, there exists $\epsilon_1 > 0$ depending only on f_1 such that $\omega_N \leqq (8/(k+2) - 2\epsilon_1)^{(k+2)/4}$. Therefore

$$\omega(f_1) = \lim_{N\to\infty} \omega_N < (8/(k+2))^{(k+2)/4} = \omega^*.$$

This completes the proof of Theorem 3.

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