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A MODULAR ANALOG OF A THEOREM OF R. STEINBERG ON COINVARIANTS OF COMPLEX PSEUDOREFLECTION GROUPS

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Abstract. Let $\rho : G \hookrightarrow \mathrm{GL}(n, \mathbb{F})$ be a representation of a finite group over the field $\mathbb{F}$, $V = \mathbb{F}^n$ the corresponding G-module, and $\mathbb{F}[V]$ the algebra of polynomial functions on V. The action of G on V extends to $\mathbb{F}[V]$, and $\mathbb{F}[V]^G$, respectively $\mathbb{F}[V]_G$, denotes the ring of invariants, respectively coinvariants. The theorem of Steinberg referred to in the title says that when $\mathbb{F} = \mathbb{C}$, $\dim_{\mathbb{C}}$ $(\mathrm{Tot}(\mathbb{C}[V]_G)) = |G|$ if and only if G is a complex reflection group. Here $\mathrm{Tot}(\mathbb{F}[V]_G)$ denotes the direct sum of all the homogeneous components of the graded algebra $\mathbb{F}[V]_G$ and $|G|$ is the order of G. Chevalley's theorem tells us that the ring of invariants of a complex pseudoreflection representation $G \hookrightarrow \mathrm{GL}(n, \mathbb{C})$ is polynomial algebra, and the theorem of Shephard and Todd yields the converse. Combining these results gives: $\dim_{\mathbb{F}}(\mathrm{Tot}(\mathbb{C}[V]_G) = |G|$ if and only if $\mathbb{C}[V]^G$ is a polynomial algebra. The purpose of this note is to show that the two conditions

(i) $\dim_{\mathbb{F}}(\mathrm{Tot}(\mathbb{F}[V]_G)) = |G|$,
(ii) $\mathbb{F}[V]^G$ is a polynomial algebra

are equivalent regardless of the ground field; in particular in the modular case.

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Let $\rho : G \hookrightarrow \mathrm{GL}(n, \mathbb{F})$ be a representation of a finite group over the field $\mathbb{F}$, $V = \mathbb{F}^n$ the corresponding G-module, and $\mathbb{F}[V]$ the algebra of polynomial functions on V. The action of G on V extends to $\mathbb{F}[V]$, and we denote by $\mathbb{F}[V]^G$, respectively $\mathbb{F}[V]_G$, the ring of invariants, respectively coinvariants. As a general reference for invariant theoretic matters we use [**2**]. The theorem of Steinberg referred to in the title is the equivalence $(a) \Longleftrightarrow (e')$ in Theorem 1.3 of the amazing paper [**3**]. It says that when $\mathbb{F} = \mathbb{C}$, $\dim_C(\mathrm{Tot}(\mathbb{C}[V]_G)) = |G|$ if and only if G is a complex reflection group. Here $\mathrm{Tot}(\mathbb{F}[V]_G)$ denotes the direct sum of all the homogeneous components of the graded algebra $\mathbb{F}[V]_G$ and $|G|$ is the order of G. Chevalley's theorem tells us that the ring of invariants of a complex pseudoreflection group $G \hookrightarrow \mathrm{GL}(n, \mathbb{C})$ is a polynomial algebra, and the theorem of Shephard and Todd yields the converse, [**2**, Theorem 7.4.1]. The purpose of this note is to prove the following theorem, which is a characteristic free analog of Steinberg's theorem.

THEOREM. *Let* $G \overset{\rho}{\hookrightarrow} \mathrm{GL}(n, \mathbb{F})$ *be a representation of a finite group over the field* $\mathbb{F}$. *Then the conditions*

(i) $\dim_{\mathbb{F}}(\mathrm{Tot}(\mathbb{F}[V]_G)) = |G|$,
(ii) $\mathbb{F}[V]^G$ *is a polynomial algebra,*

are equivalent.

Proof. Suppose (i) holds. By definition

$$\mathbb{F}[V]_G = \mathbb{F} \otimes_{\mathbb{F}[V]^G} \mathbb{F}[V]$$

is the module of $\mathbb{F}[V]^G$-indecomposable elements of the module $\mathbb{F}[V]$ (see e.g., [**2**, Section 5.1]). Therefore we can find an epimorphism

$$\varphi : \mathbb{F}[V]^G \otimes \mathbb{F}[V]_G \longrightarrow \mathbb{F}[V]$$

of $\mathbb{F}[V]^G$-modules. We claim that φ is actually an isomorphism. To see this let $\mathbb{K} = FF(\mathbb{F}[V]^G)$ be the field of fractions of $\mathbb{F}[V]^G$ and $\mathbb{L} = FF(\mathbb{F}[V])$ that of $\mathbb{F}[V]$ and note that φ induces a map (the classic Cartan–Eilenberg change of rings map)

$$\Phi : \mathbb{K} \otimes \mathbb{F}[V]_G = \mathbb{K} \otimes_{\mathbb{F}[V]^G} (\mathbb{F}[V]^G \otimes \mathbb{F}[V]_G) \longrightarrow \mathbb{L} \otimes_{\mathbb{F}[V]} \mathbb{F}[V] = \mathbb{L}$$

of $\mathbb{K}$-vector spaces. We claim Φ is an epimorphism. For, if $f/h \in \mathbb{L}$, then by multiplying numerator and denominator by $\prod_{1 \neq g \in G} gh$ we may suppose that h is G-invariant, so $1/h \in \mathbb{K}$. Since φ is an epimorphism we may write

$$f = \varphi\left(\sum_{i \in \mathcal{I}} f_i \otimes u_i\right),$$

where $\mathcal{I}$ is a finite index set, $f_i \in \mathbb{F}[V]^G$, and $u_i \in \mathbb{F}[V]_G$. Then

$$\frac{f}{h} = \frac{1}{h}\varphi\left(\sum_{i \in \mathcal{I}} f_i \otimes u_i\right) = \Phi\left(\frac{1}{h} \otimes \varphi\left(\sum_{i \in \mathcal{I}} f_i \otimes u_i\right)\right),$$

showing Φ is an epimorphism.

Since (i) holds, $\mathbb{K} \otimes \mathbb{F}[V]_G$ has dimension $|G|$ as a $\mathbb{K}$-vector space, and since $\mathbb{K} \hookrightarrow \mathbb{L}$ is Galois with Galois group G so does $\mathbb{L}$. Hence Φ is an isomorphism.

The functor $\mathbb{K} \otimes_{\mathbb{F}[V]^G}$—is an exact functor, so $\ker(\mathbb{K} \otimes \varphi) = \mathbb{K} \otimes_{\mathbb{F}[V]^G} \ker(\varphi)$. Since $\ker(\varphi)$ is $\mathbb{F}[V]^G$-torsion free $\ker(\varphi) \neq 0$ implies that $\mathbb{K} \otimes_{\mathbb{F}[V]^G} \ker(\varphi) \neq 0$ and this implies $\ker(\Phi) \neq 0$ contrary to what was just shown. Hence $\ker(\varphi) = 0$ and φ is an isomorphism. Therefore $\mathbb{F}[V]$ is a free $\mathbb{F}[V]^G$-module and the result follows from [**2**, Theorem 6.4.4] or [**1**].

The implication (ii) $\Rightarrow$ (i) follows from the Degree theorem, [**2**, Theorem 5.5.3]. Specifically, if $\mathbb{F}[V]^G = \mathbb{F}[f_1, \ldots, f_n]$ then $f_1, \ldots, f_n \in \mathbb{F}[V]$ are a system of parameters since $\mathbb{F}[V]^G \hookrightarrow \mathbb{F}[V]$ is a finite extension. Since $\mathbb{F}[V]$ is Cohen–Macaulay $f_1, \ldots, f_n \in \mathbb{F}[V]$ is a regular sequence so $\mathbb{F}[V]$ is a free $\mathbb{F}[V]^G$ module. Let $\deg(f_i) = d_i$ for $i = 1, \ldots, n$. Then the Poincaré series of $\mathbb{F}[V]_G$ is

$$P(\mathbb{F}[V]_G, t) = \frac{P(\mathbb{F}[V], t)}{P(\mathbb{F}[V]^G, t)} = \frac{\frac{1}{(1-t)^n}}{\frac{1}{\prod_{i=1}^n (1-t^{d_i})}} = \prod_{i=1}^n (1 + t + \cdots + t^{d_i - 1})$$

and evaluating $P(\mathbb{F}[V]_G, t)$ at $t = 1$ then gives

$$\dim_{\mathbb{F}}(\mathrm{Tot}(\mathbb{F}[V]_G)) = d_1 \cdots d_n = |G|$$

by a corollary to the Degree theorem [**2**, Corollary 5.5.4]. □

REMARK. The argument in the proof of the theorem that shows that Φ is an epimorphism is valid in general. It implies the following elementary fact which shows that the condition (i) is an extremal condition, namely a minimum condition.

COROLLARY. *Let* $G \overset{\rho}{\hookrightarrow} \mathrm{GL}(n, \mathbb{F})$ *be a faithful representation of a finite group over the field* $\mathbb{F}$. *Then* $\dim_{\mathbb{F}}(\mathbb{F}[V]_G) \geq |G|$.

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