SUPERHARMONIC FUNCTIONS IN A DOMAIN OF A RIEMANN SURFACE

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To Professor Kiyoshi Noshiro on the occasion of his 60th birthday

Let R be a Riemann surface. Let G be a domain in R with relative boundary ∂G of positive capacity. Let $U(z)$ be a positive superharmonic function in G such that the Dirichlet integral $D(\min(M, U(z))) < \infty$ for every M. Let D be a compact domain in G. Let ${}_DU^M(z)$ be the lower envelope of superharmonic functions $\{U_n(z)\}$ such that $U_n(z) \geqq \min(M, U(z))$ on $D + \partial G$ except a set of capacity zero, $U_n(z)$ is harmonic in $G-D$ and $U_n(z)$ has M.D.I. (minimal Dirichlet integral) $\leqq D(\min(M, U(z))) < \infty$ over $G-D$ with the same value as $U_n(z)$ on $\partial G + \partial D$. Then ${}_DU^M(z)$ is uniquely determined. Put ${}_DU(z) = \lim_{M=\infty} {}_DU^M(z)$. The mapping from $U(z)$ to ${}_DU(z)$ is clearly linear. Hence there exists a positive measure $\lambda(\xi, z)$[1] such that ${}_DU(z) = \int U(\xi) d\lambda(\xi, z)$ for $z \in G-D$. If for any compact domain D, ${}_DU(z) = U(z)$ or ${}_DU(z) \leqq U(z)$, we call $U(z)$ a *full harmonic* (*F.H.*) or *full superharmonic* (*F.S.H.*) *function in* G respectively. If $U(z)$ is an F.S.H. in G and $U(z) = 0$ on ∂G except at most a set of capacity zero, $U(z)$ is called an F_0.S.H. in G. Let $U(z)$ be an F.S.H. in G. Then ${}_DU(z)\uparrow$ as $D\uparrow$. Put ${}_DU(z) = \lim_n {}_{D\cap G_n}U(z)$ for a non compact domain D, where $\{G_n\}$ is an exhaustion of G with compact relative boundary $\partial G_n (n = 0, 1, \ldots)$.

Functiontheoretic mass $\mathfrak{M}^f(U(z))$ *of an* F_0*.S.H. in* G. Let $U(z)$ be an F_0.S.H. in G. Then $g_M = E[z : U(z) > M]$ is open. Let $\omega(g_M, z, G)$ be a function in G such that $\omega(g_M, z, G)$ is harmonic in $G - g_M$, $= 1$ in g_M and has M.D.I. over $G - g_M$ and further $\omega(g_M, z, G) = 0$ on ∂G, $= 1$ on ∂g_M except a set of cap. zero. Clearly such a function exists by $D(\min(U(z), M))) < \infty$ and $\min(M, U(z)) = M$ on ∂g_M, $= 0$ on ∂G except a set of cap. zero. It is easily seen, $\omega_n(z) \to \omega(g_M, z, G)$ in mean as $n \to \infty$, where $\omega_n(z)$ is a harmonic function in $R_n \cap (G - g_M)$ such that $\omega_n(z) = 0$ on ∂G, $\omega_n(z) = 1$ on ∂g_M except a set of

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capacity zero and $\frac{\partial}{\partial n}\omega_n(z)=0$ on $(G-g_M)\cap\partial R_n$, where $\{R_n\}$ is an exhaustion of R with compact relative boundary ∂R_n. We call $\omega(g_M, z, G)$ C.P. (capacitary potential) of g_M relative to G and define $\mathrm{Cap}(g_M)$ by $D(\omega(g_M, z, G))$. Then there exists a regular niveau[2] C_δ such that

$$D(\omega(g_M, z, G))=\int_{C_\delta}\frac{\partial}{\partial n}\omega(g_M, z, G)ds$$

for almost δ with $0\leqq\delta\leqq 1$.

Since $U(z)$ is an F_0.S.H. in G, $U(z)\geqq {}_{g_{M_1}}U(z)$, whence

$$E[z:{}_{g_{M_1}}U(z)>M_2]=g'_{M_2}\subset g_{M_2}=E[z:U(z)>M_2] \text{ for } M_2<M_1. \tag{1}$$

By the definition ${}_{g_M}U(z)=M\omega(g_M, z, G)$ in Cg_M. On the other hand, $\delta\omega(g_\delta, z, G)=\omega(g_M, z, G)$ in Cg_δ, where $g_\delta=E[z:\omega(g_M, z, G^{[3]})>\delta]$ and

$$D(\omega(g_\delta, z, G))=\frac{1}{\delta}D(\omega(g_M, z, G)) \qquad \text{for any } \delta<1. \tag{2}$$

Let $M_1>M_2$. Then by (1) and (2)

$$\underset{Cg_{M_2}}{D}({}_{g_{M_2}}U(z))=M_2^2D(\omega(g_{M_2}, z, G))\geqq M_2^2D(\omega(g'_{M_2}, z, G)). \tag{3}$$

Put $\delta=\frac{M_2}{M_1}$. Then

$M_2^2D(\omega(g'_{M_2}, z, G))=M_2^2\times\frac{M_1}{M_2}D(\omega(g_{M_1}, z, G))=M_1M_2D(\omega(g_{M_1}, z, G))=\frac{M_2}{M_1}\underset{Cg_{M_1}}{D}({}_{g_{M_1}}U(z))$. Hence by (3) $\left(\frac{1}{M_2}\right)\underset{Cg_{M_2}}{D}({}_{g_{M_2}}U(z))\geqq\left(\frac{1}{M_1}\right)\underset{Cg_{M_1}}{D}({}_{g_{M_1}}U(z))$ for $M_2\leqq M_1$ and $\left(\frac{1}{M}\right)\underset{Cg_M}{D}({}_{g_M}U(z))$ increases as $M\to 0$. Put $\mathfrak{M}^f(U(z))=\frac{1}{2\pi}\lim_{M\to 0}\left(\frac{1}{M}\right)\underset{Cg_M}{D}({}_{g_M}U(z))$ and call $\mathfrak{M}^f(U(z))$ *functiontheoretic mass* of $U(z)$. Then we have the following

LEMMA 1. 1) *Let $U_1(z)$ and $U_2(z)$ be two F_0.S.H.s in G and $U_1(z)\geqq U_2(z)$. Then*

$$\mathfrak{M}^f(U_1(z))\geqq\mathfrak{M}^f(U_2(z)).$$

2) *Let $U_m(z)$ be F_0.S.H.s and $U_m(z)\uparrow U(z)$ as $m\to\infty$. Then*

$$\lim_{m=\infty}\mathfrak{M}^f(U_m(z))=\mathfrak{M}^f(U(z)). \tag{4}$$

(1) is clear by $E[z:U_1(z)>M]\supset E[z:U_2(z)>M]$. At first we suppose $\mathfrak{M}^f(U(z))<\infty$. For any given $\varepsilon>0$, there exists a const. M such that

$\frac{1}{2\pi M} \underset{cg_M}{D} ({}_{g_M}U(z)) = \frac{1}{2\pi} MD(\omega(g_M, z, G)) \geqq \mathfrak{M}^f(U(z)) - \varepsilon$. Since $E[z : U_m(z) > M] = g_{M,m} \uparrow g_M = E[z : U(z) > M]$ as $m \to \infty$, $D(\omega(g_{M,m}, z, G)) \to D(\omega(g_M, z, G))$[4] as $m \to \infty$. Hence $\lim_{m=\infty} \mathfrak{M}^f(U_m(z)) \geqq \frac{1}{2\pi} \lim_{m=\infty} M D(\omega(g_{M,m}, z, G)) \geqq \mathfrak{M}^f(U(z)) - \varepsilon$. Let $\varepsilon \to 0$. Then $\lim_{m=\infty} \mathfrak{M}^f(U_m(z)) \geqq \mathfrak{M}^f(U(z))$.

Next by Lemma 1.1) $\lim_{m=\infty} \mathfrak{M}^f(U_m(z)) \leqq \mathfrak{M}^f(U(z))$. If $\mathfrak{M}^f(U(z)) = \infty$, we have similarly $\lim_{m=\infty} \mathfrak{M}^f(U_m(z)) = \infty$.

$\mathfrak{M}^f(U(z))$ of an F.S.H. $U(z)$ in G. For a compact domain D in G, suppose that we can define functions $\{U_n(z)\}$ such that $U_n(z)$ is superharmonic in G, $U_n(z)$ is harmonic in $G - D$, $U_n(z) \geqq \min(M, U(z))$ on D, $U_n(z) = 0$ on ∂G except a set of cap. zero and $U_n(z)$ has M.D.I. over $G - D$. Let ${}^0_D U^M(z)$ be the lower envelope of $\{U_n(z)\}$. Put ${}^0_D U(z) = \lim_{M=\infty} {}^0_D U^M(z)$ (clearly ${}^0_D U(z) \leqq {}_D U(z)$). Since ∂D is compact, ${}^0_D U(z) = 0$ on ∂G except a set of cap. zero. For non compact domain, ${}^0_D U(z)$ is defined as ${}_D U(z)$. For $U(z)$, put $\mathfrak{M}^f(U(z)) = \lim_{n=\infty} \mathfrak{M}^f({}_{G_n}U^0(z))$, where $\{G_n\}$ is an exhaustion of G with compact relative boundary.

N-Green's functions of G. Let $N_n(z, p)$ be a positive harmonic function in $(G - p) \cap R_n : p \in G$ such that $N_n(z, p) = 0$ on ∂G except a set of capacity zero, $N_n(z, p)$ has a logarithmic singularity at p and $\frac{\partial}{\partial n} N_n(z, p) = 0$ on $\partial R_n \cap G$. Then $N_n(z, p) \to N(z, p)$ in mean as $n \to \infty$ and $N(z, p)$ has M.D.I. (in this case the Dirichlet integral of $N(z, p)$ is taken with respect to $N(z, p) + \log|z - p|$ in a neighbourhood of p). If ∂G is composed of a finite number of analytic curves in G, we say that ∂G is *completely regular*. Then as case that ∂G is completely regular we see easily[5]

1). $N(z, p) = 0$ on ∂G except at most a set of cap. zero.

2). $D(\min(M, N(z, p))) = 2\pi M$.

3). For any domain D ${}_D N(z, p) = N(z, p)$ if $p \in D$ and ${}_D N(z, p) \leqq N(z, p)$.

4). By 2) and 3) we have $\mathfrak{M}^f(N(z, p)) = 1$.

We show, *for any point z in G and a positive const. d there exists a const. $L(z, d)$ such that $N(z, p) < L(z, d)$ if* $\operatorname{dist}(z, p) > d$.

Case 1. ∂G has a continuum γ. Suppose γ contains a small arc C' with endpoints p_1 and p_2. Let C'' be also an arc in G connecting p_1 and p_2 so that $C' + C''$ may enclose a simply connected domain D of R. Let C''' be a subarc in C'' such that $\operatorname{dist}(C''', \partial G) > 0$. Let $w(z)$ be a harmonic

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function in D such that $w(z)=1$ on C''', $w(z)=0$ on $\partial D - C'''$. Then $w(z)=0$ on C' and $\infty > \int_{C'} \frac{\partial}{\partial n} w(z)ds > \delta > 0$. Without loss of generality we can suppose $\operatorname{dist}(p, D) > d >$ diameter of D. Let $N^*(z, p)$ be an N-Green's function of $G + (CG \cap D)$. Then $N^*(z, p) \geqq N(z, p)$ and $N^*(z, p)$ is harmonic in a neighbourhood of C'''. Hence by Harnack's theorem, there exists a const.K such that $\max_{z \in C'''} N^*(z, p) \leqq K \min_{z \in C'''} N^*(z, p)$. Let $L = \max_{z \in C'''} N^*(z, p)$. Then $N^*(z, p) \geqq \frac{L}{K} w(z)$ in D and $2\pi > \int_{C'} \frac{\partial}{\partial n} N^*(z, p)ds \geqq \frac{L\delta}{K}$, whence $L \leqq \frac{2\pi K}{\delta}$. Hence also by Harnack's theorem, for any point z, there exists a const. $L(z, d)$ such that $N(z, p) \leqq L(z, d)$ if $\operatorname{dist}(z, p) \geqq d$. If G has a continuum γ (is not an analytic curve). Map D onto $|\xi| < 1$. Then the image of γ is an analytic curve. Hence even when γ is not analytic curve, we have the same conclusion.

Case 2. ∂G has no continuum. By $N(z, p) \not\equiv \infty$, we can find a point z_0 in ∂G such that $\inf_{z \to z_0} N(z, p) = 0$. Let D be a simply connected domain in R such that ∂D is an analytic curve, $D \ni z_0$ and $(\partial D \cap \partial G) = 0$. Then $\operatorname{dist}(\partial D, \partial G) > 0$. We suppose $p \notin D$, $\operatorname{dist}(p, D) >$ diameter of D and $\operatorname{dist}(\partial G_n, \partial D) > 0$, where $\{G_n\}$ is an exhaustion of G. Let $U_{m,n}(z)$ be a harmonic function in $G_n \cap R_m$ such that $U_{m,n}(z) = 0$ on $\partial G_n \cap R_m$, $\frac{\partial}{\partial n} U_{m,n}(z) = 0$ on $\partial R_m \cap G_n$ and $U_{m,n}(z)$ has a logarithmic singularity at $p : R_m \supset D$. Then $\lim_m \lim_n U_{m,n}(z) = N(z, p)$. Let $w_n(z)$ be a harmonic function in $G_n \cap D$ such that $w_n(z) = 0$ on $\partial G_n \cap D$, $= 1$ on ∂D. Then $\frac{U_{m,n}(z)}{L_{m,n}} \leqq w_n(z)$ and $\frac{2\pi}{L_{m,n}} \geqq \int_{\partial G_n} \frac{1}{L_{m,n}} \frac{\partial}{\partial n} U_{m,n}(z)ds \geqq \int_{\partial G_n} w_n(z)ds > 0$, where $L_{m,n} = \min_{z \in \partial D} U_{m,n}(z)$. Now by $\int_{\partial D} \frac{\partial}{\partial n} U_{m,n}(z)ds = \int_{\partial G_n \cap D} \frac{\partial}{\partial n} U_{m,n}(z)ds > 0$ and $\int_{\partial D} \frac{\partial}{\partial n} w_n(z)ds = \int_{\partial G_n \cap D} \frac{\partial}{\partial n} w_n(z)ds > 0$ we have $\int_{\partial D} \frac{\partial}{\partial n} U_{m,n}(z)ds \geqq L_{m,n} \int_{\partial D} \frac{\partial}{\partial n} w_n(z)ds$. Let $n \to \infty$ and $m \to \infty$. Then since ∂D is compact

$$2\pi \geqq \int_{\partial D} \frac{\partial}{\partial n} N(z, p)ds \geqq L \int_{\partial D} \frac{\partial}{\partial n} w(z)ds = L\delta > 0,$$

where $L = \min_{z \in \partial D} N(z, p)$ and $\int_{\partial D} \frac{\partial}{\partial n} w(z)ds = \delta$.

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$\inf\limits_{z\to z_0} N(z, p) = 0$ implies $w(z) \not\equiv 1$ and $\delta > 0$. Hence $L \leqq \dfrac{2\pi}{\delta}$. Whence by Harnack's theorem we have the same conclusion.

Let $\{p_i\}$ in G be a divergent sequence tending to the boundary of R or ∂G. Then $N(z, p_i) \leqq L(z, d) < \infty$ for any point z if $\operatorname{dist}(z, p_i) \geqq d$. Then we can choose a subsequence $\{p_{i'}\}$ such that $N(z, p_{i'})$ converges uniformly to a harmonic function denoted by $N(z, p)$ and we call $\{p_{i'}\}$ a fundamental sequence determining an ideal point p. We denote by B the all the ideal points (p may be on ∂G). We show $N(z, p) = 0$ $(p \in B)$ *for a regular boundary point* z *of* ∂G.

Case 1. ∂G *has a continuum* γ *with endpoints* q_1 *and* q_2. Let $z_0 \in \gamma$, $z_0 \neq q_1$ and $\neq q_2$. Let C be an analytic curve in G connecting q_1 and q_2 so that $C + \gamma$ may enclose a simply connected domain D in R and $D \not\ni p_i$ $(i = 1, 2, \ldots)$, where $\{p_i\}$ is a fundamental sequence determining p. Map D conformally onto $|\xi| < 1$. Then γ and C are mapped onto the images denoted by the same notations for simplicity. $N(z, p_i) = 0$ on $\gamma + (\partial G \cap D)$ except a set of cap. zero. Let $N^*(z, p_i)$ be an N-Green's function of $G + (CG \cap D)$. Then there exists a const. $L^*(t_0)$ such that $\infty > L^*(t_0) \geqq N^*(t_0, p_i) = \dfrac{1}{2\pi}\displaystyle\int_C N^*(\xi, p_i)\frac{\partial}{\partial n}G(\xi, t_0)ds$ for any i, where $G(\xi, t)$ is the Green's function of D. On the other hand, there exists a const. M such that $0 < M < \dfrac{\partial}{\partial n}G(\xi, t_0)$ on C, whence $\displaystyle\int_C N^*(\xi, p_i)ds \leqq \frac{2\pi L^*(t_0)}{M}$. Let $U(\xi)$ be a harmonic function in $|\xi| < 1$ such that $U(\xi) = N^*(\xi, p_i)$ on $|\xi| = 1$. Then

$$N^*(t, p_i) = U(t) = \frac{1}{2\pi}\int_C N^*(\xi, p_i)\frac{1 - r^2}{1 - 2r\cos(\theta - \varphi) + r^2}\,d\varphi : t = re^{i\theta}.$$

Since $\xi_0 = \xi(z_0) \in \gamma$, there exists a neighbourhood $v(\xi_0)$ such that $v(\xi_0) \cap C = 0$. Now there exists a const. M' such that $1 - 2r\cos(\theta - \varphi) + r^2 \geqq M'$ for $e^{i\varphi} \notin \gamma$ and $\xi \in v(\xi_0) : \xi = re^{i\theta}$. Hence $U(\xi) \leqq \dfrac{2\pi L^*(t_0)}{MM'}(1 - r^2)$ for $\xi \in v(\xi_0)$. Then by Fatou's lemma $N^*(\xi, p) \leqq \dfrac{2\pi L^*(t_0)}{MM'}(1 - r^2)$ and by $N(z, p_i) \leqq N^*(z, p_i)$ we have $N(z, p) \to 0$ as $z \to z_0$.

Case 1.2. $z_0 \in$ *endpoint of an arc* γ. Let D be a domain such that $\partial D + \gamma$ encloses a simply connected domain $D - \gamma$. Map $D - \gamma$ onto $|\xi| < 1$. Then the image (z_0) of z_0 is an inner point of the image of γ. Then as case 1.1. we have $N(z, p) \to 0$ as $z \to z_0$.

Case 2. z_0 *is a regular point and* z_0 *is not contained in any continuum.* Let

D be a simply connected domain such that $D \ni z_0$ and $\partial D \cap \partial G = 0$ and $D \not\ni p_i (i = 1, 2, \ldots)$. Then since $(\partial D \cap \partial G) = 0$ implies $\mathrm{dist}(\partial D, \partial G) > 0$, there exist const.s L_1 and L_2 such that $N(z, p_i) \leqq L_1$ and $N(z, p_i) \geqq L_2$ on ∂D. Whence there exists a const. M such that $N(z, p_i) \leqq MN(z, p_1)$ in D. Hence $\lim_{z \to z_0} N(z, p) \leqq M \lim_{z \to z_0} N(z, p_1) = 0$. Thus $N(z, p) = 0$ on ∂G except at most a set of capacity zero. $D(\min(M, N(z, p))) \leqq \underline{\lim}_i D(\min(M, N(z, p_i))) \leqq 2\pi M$. Hence we can define ${}_DN(z, p)$ for any compact domain. $N(z, p_i) \to N(z, p)$ uniformly on ∂D as $i \to \infty$. Hence by ${}_DN(z, p_i) \leqq N(z, p_i)$ we have ${}_DN(z, p) \leqq N(z, p)$. Next we have at once $\mathfrak{M}^f(N(z, p)) \leqq \dfrac{D(\min(M, N(z, p)))}{2\pi M} \leqq 1$. Hence we have the following.

LEMMA 2. *$N(z, p)$ is an F_0.S.H. in G such that $D(\min(M, N(z, p))) \leqq 2\pi M$ and $\mathfrak{M}^f(N(z, p)) \leqq 1$ for $p \in G + B$.*

N-Martin topology in G. Let D be a compact disc in G and p_0 be a fixed point in D. we define the *distance between two points* p_1 and p_2 of $G + B$ as

$$\delta(p_1, p_2) = \sup_{z \in D} \left| \frac{N(z, p_1)}{1 + N(z, p_1)} - \frac{N(z, p_2)}{1 + N(z, p_2)} \right|.$$

Then the topology induced by this metric is homeomorphic to the original topology in G. In the following we use this topology. $\delta(p, p_i) \to 0$ if and only if $N(z, p_i) \to N(z, p)$. Put $G_\delta = E[z : N(z, p_0) > \delta]$. Then the distance between G_δ and $CG_{\delta'} = E[z : N(z, p) \leqq \delta']$ is not less than $\dfrac{\delta - \delta'}{4}$, if $0 < \delta' < \delta < 1$. In fact, by the symmetry of $N(p, q)$ we have at once

$$\delta(q_1, q_2) \geqq \left| \frac{N(p_0, q_1)}{1 + N(p_0, q_1)} - \frac{N(p_0, q_2)}{1 + N(p_0, q_2)} \right| \geqq \frac{\delta - \delta'}{4} : q_1 \in G_\delta \text{ and } q_2 \in CG_{\delta'}.$$

Also we easily see $B \cap \bar{G}_\delta$ *is compact for every* $\delta > 0$.

Potentials. Let $\mu > 0$ be a positive mass distribution on $G + B$ such that $\int d\mu(p) < \infty$ and put $U(z) = \int N(z, p) d\mu(p)$. If a potential $U(z) = 0$ on ∂G except at most a set of cap. zero, we call $U(z)$ a *regular potential.* Then we have the following

THEOREM 1. 1). $\qquad D(\min(M, U(z))) \leqq 2\pi M \int d\mu$.

2). *Let D be a compact or non compact domain. Then ${}_DU(z) = \int {}_DN(z, p) d\mu(p)$.*

3). *Let μ_ε be the restriction of μ on $G_\varepsilon = E[z : N(z, p_0) > \varepsilon]$. Then*

$$U(z) = \lim_{\varepsilon \to 0} \int N(z, p) d\mu_\varepsilon(p).$$

4). *If* $U(z)$ *is a regular potential,* $U(z)$ *is an* F_0*.S.H. in* G *with* $\mathfrak{M}^f(U(z)) \leqq \int d\mu$.

Proof of 1). For any number $\varepsilon > 0$ we can find a compact set K in $H = E[z : U(z) \leqq M]$ such that $D(\min(M, U(z))) < \underset{K}{D}(U(z)) + \varepsilon$. Since $N(z, p)$ is a continuous function of p for fixed z, $U(z)$ can be approximated on K by a sequence of linear forms : $U_i(z) = \sum_{j=1}^{j(i)} \lambda_{ij} N(z, p_j), \lambda_{ij} \geqq 0, \ p_j \in G$, $\int d\mu = \sum_j \lambda_{ij} : i = 1, 2, \ldots$. Hence $\underset{K}{D}(U(z)) \leqq \varliminf_i \underset{K}{D}(U_i(z))$. Also $U_{i,n}(z) \to U_i(z)$ in mean as $n \to \infty$, where $U_{i,n}(z) = \sum \lambda_{ij} N_n(z, p_j)$ and $N_n(z, p_j)$ is a harmonic function in $G \cap R_n$ such that $N_n(z, p_j) = 0$ on $\partial G \cap R_n$ except a set of cap. zero, $\frac{\partial}{\partial n} N_n(z, p_j) = 0$ on $\partial R_n \cap G$ and $N_n(z, p_j)$ has logarithmic singularity at p_j. Put $\underset{\varepsilon, i, n}{H} = E[z : U_{i,n}(z) < M + \varepsilon]$. Then $\underset{\varepsilon, i, n}{H} \supset K$ for $n \geqq n_0$, where n_0 is a sufficiently large number. We can prove (with some modefication to the fact $N_n(z, p_j) = 0$ on ∂G except a set of cap. zero instead of $N_n(z, p_j) = 0$ on ∂G) that $\underset{H_{\varepsilon, i, n}}{D}(U_{i,n}(z)) = 2\pi(M + \varepsilon) \int d\mu(p)$. Let $n \to \infty$, $i \to \infty$ and then $\varepsilon \to 0$. Then $D(\min(M, U(z)) \leqq 2\pi M \int d\mu(p)$.

Proof of 2). Put $D_n = D \cap G_n$. Then D_n is compact. Put $N^M(z, p) = \min(M, N(z, p))$. Then $N^M(z, p)$ is uniformly continuous with respect to p on D_n. Hence $\int N^M(z, p) d\mu(p)$ can be approximated uniformly on D_n by a sequence of linear forms: $U_i(z) = \sum_{j=1}^{j(i)} \lambda_{i,j} N^M(z, p_j)$, $\lambda_{i,j} \geqq 0$. Clearly ${}_{D_n}U_i(z) = \sum \lambda_{ij} {}_{D_n}N^M(z, p_j)$. Let $i \to \infty$. Then ${}_{D_n}\left(\int N^M(z, p) d\mu(p)\right) = \int {}_{D_n}N^M(z, p) d\mu(p)$. Now by $\int N^M(z, p) d\mu(p) \uparrow U(z)$ as $M \to \infty$, we have ${}_{D_n}\left(\int N^M(z, p) d\mu(p)\right) \uparrow {}_{D_n}\left(\int N(z, p) d\mu(p)\right) = {}_{D_n}U(z)$ and $\int {}_{D_n}N^M(z, p) d\mu(p) \uparrow \int {}_{D_n}N(z, p) d\mu(p)$ as $M \uparrow \infty$. Hence ${}_{D_n}U(z) = \int {}_{D_n}N(z, p) d\mu(p)$. Since ${}_{D_n}N(z, p) \uparrow {}_{D}N(z, p)$ and ${}_{D_n}U(z) \uparrow {}_{D}U(z)$ as $n \to \infty$, ${}_{D}\left(\int N(z, p) d\mu(p)\right) = \lim_n \left({}_{D_n}\int N(z, p) d\mu(p)\right) = \int \lim_n {}_{D_n}N(z, p) d\mu(p) = \int {}_{D}N(z, p) d\mu(p)$.

Proof of 3). Suppose $p \notin G_\varepsilon$. Let $\{p_i\}$ be a fundamental sequence determining $p \in B$ (if $p \in G$, put $p_i = p$). Then $\delta(p, p_i) \to 0$ as $i \to \infty$. Then by $\operatorname{dist}(CG_\varepsilon, G_{2\varepsilon}) \geqq \frac{\varepsilon}{4}$, (if $2\varepsilon < 1$), $p_i \notin G_{2\varepsilon}$ for $i \geqq i_0$, where i_0 is a number. Hence $N(p_0, p_i) \leqq 2\varepsilon$ and $N(p_0, p) \leqq 2\varepsilon$ for $p \in (B+G) - G_{2\varepsilon}$. Let $\mu'_\varepsilon = \mu - \mu_\varepsilon$. Then ${}_\varepsilon U'(p_0) = \int N(p_0, p) d\mu'_\varepsilon(p) \leqq 2\varepsilon \int d\mu'$. $U'_\varepsilon(z)$ is harmonic in $G - G_{2\varepsilon}$. Hence by Harnack's theorem $U'_\varepsilon(z) \to 0$ as $\varepsilon \to 0$ at every point z. Hence we have 3).

Proof of 4). $\underset{cg_M}{D}({}_{g_M}U(z)) \leqq D(\min(M, U(z))) \leqq 2\pi M \int d\mu$ by (1). Hence by definition $\mathfrak{M}^f(U(z)) = \lim\limits_{M=0} \dfrac{\underset{cg_M}{D}({}_{g_M}U(z))}{2\pi M} \leqq \int d\mu$, where $g_M = E[z : U(z) > M]$. By (2) ${}_DU(z) \leqq U(z)$, hence $U(z)$ is an F_0.S.H. in G with $\mathfrak{M}^f(U(z)) \leqq \int d\mu$.

THEOREM 2. *Let $U(z)$ be an F_0.S.H. in G with $\mathfrak{M}^f(U(z)) < \infty$. Then $U(z)$ can be represented by a positive mass distribution μ on $G+B$ such that $\int d\mu \leqq \mathfrak{M}^f(U(z))$.*

Let D and D' be compact domains in G with finite number of analytic curves as their relative boundaries such that dist $(D, \partial D') > 0$ and $D' \supset D$. Let M be a number. Put $U^M(z) = \min(M, U(z))$. Then $U^M(z)$ is also an F_0.S.H. in G and $\mathfrak{M}^f(U^M(z)) \leqq \mathfrak{M}^f(U(z))$. Let δ be a positive const. such that $\delta < \min\limits_{z \in \bar{D}'} {}_DU^M(z)$. Then $\partial H_\delta \cap \bar{D}' = 0$, where $H_\alpha = E[z : {}_DU^M(z) > \alpha]$. Then ${}_DU^M(z) = \delta\omega(H_\delta, z, G)$ in CH_δ. Hence we can find a const. δ' such that $\delta' < \delta$ and $\partial H_{\delta'}$ is a regular niveau of $\omega(H_\delta, z, G)$. Hence

$$\mathfrak{M}^f(U^M(z)) \geqq \mathfrak{M}^f({}_DU^M(z)) = \lim_{M' \to 0} \frac{D(\min(M', {}_DU^M(z)))}{2\pi M'} = \frac{D(\min(\delta, {}_DU^M(z)))}{2\pi\delta} = \frac{1}{2\pi} \int_{\partial H_{\delta'}} \frac{\partial}{\partial n} {}_DU^M(z) ds.$$

Put

$$U(\delta', z) = {}_DU^M(z) - \delta' + \delta'\omega(D, z, H_{\delta'}) \text{ in } H_{\delta'} - D. \tag{4}$$

Then $U(\delta', z)$ is a harmonic function in $H_{\delta'} - D$ such that $U(\delta', z) = {}_DU^M(z)$ on ∂D, $= {}_DU^M(z) - \delta' = 0$ on $\partial H_{\delta'}$ except a set of cap. zero and $U(\delta', z)$ has M.D.I. over $H_{\delta'} - D$, because both ${}_DU^M(z)$ and $\omega(D, z, H_{\delta'})$ have M.D.I.s over $H_{\delta'} - D$. Now by the regularity of $\partial H_{\delta'}$

$$0 \leqq \int_{\partial H_{\delta'}} \frac{\partial}{\partial n} \omega(D, z, H_{\delta'}) ds = -\int_{\partial D} \frac{\partial}{\partial n} \omega(D, z, H_{\delta'}) ds = a(\delta').$$

Since $\int_{\partial H_{\delta'}} \frac{\partial}{\partial n} \omega(D, z, H_{\delta'}) ds \downarrow$ as $\delta' \downarrow$,

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$$\delta' \int_{\partial H_{\delta'}} \frac{\partial}{\partial n} \omega(D, z, H) ds \downarrow 0 \text{ as } \delta' \to 0. \tag{5}$$

Hence $2\pi \mathfrak{M}^f({}_D U^M(z)) = \lim_{\delta' \to 0} \int_{\partial H_{\delta'}} \frac{\partial}{\partial n} {}_D U^M(z) ds = \lim_{\delta' \to 0} \int_{\partial H_{\delta'}} \frac{\partial}{\partial n} (U(\delta', z)$

$$- \delta' \omega(D, z, H_{\delta'})) ds = \lim_{\delta' \to 0} \int_{\partial H_\delta} \frac{\partial}{\partial n} U(\delta', z) ds.$$

Hence for any $\varepsilon > 0$ we can find a const. δ^* such that ∂H_{δ^*} is regular and

$$2\pi \mathfrak{M}^f({}_D U^M(z)) \geqq \int_{\partial H_{\delta^*}} \frac{\partial}{\partial n} U(\delta^*, z) ds - \varepsilon. \tag{6}$$

Since ${}_D U^M(z)$ is an F_0.S.H. in G, there exists a uniquely determined positive mass distribution μ on $\bar{D}$ such that

$${}_D U^M(z) = \int N(z, p) d\mu(p).$$

Let $N'(z, p)$ be an N-Green's function of $H_{\delta^*} + D$ with pole at p. Then $N'_n(z, p)$ is uniformly bounded on $\partial D'$ for $p \in D''$ and $N'_n(z, p) \to N'(z, p)$ in mean as $n \to \infty$, where D'' is another domain such that $D \subset D'' \subset D'$ and dist$(\partial D, \partial D'') > 0$ and dist$(\partial D'', \partial D') > 0$ and $N'_n(z, p)$ is a harmonic function in $((H_{\delta^*} + D) \cap R_n) - p$ such that $N'_n(z, p)$ has a logarithmic singularity at p, $N'_n(z, p) = 0$ on ∂H_{δ^*} and $\frac{\partial}{\partial n} N'_n(z, p) = 0$ on $\partial R_n \cap (H_{\delta^*} + D)$. Then by the regularity of ∂H_{δ^*}[5]

$$\int_{\partial H_{\delta^*}} \frac{\partial}{\partial n} N'(z, p) ds = \lim_{n = \infty} \int_{\partial H_{\delta^*} \cap R_n} \frac{\partial}{\partial n} N'_n(z, p) ds = 2\pi. \tag{7}$$

N'-Martin topology induced by $N'(z, p)$ is homeomorphic to N-Martin topology on $G + B$ in $(D + H_{\delta^*}) \cap G$. Hence μ can be approximated by a sequence of points masses: $\sum_{j=1}^{i(j)} \lambda_{ij}(p_{ij})$ uniformly in D, i.e. both $\int N(z, p) d\mu(p)$ and $\int N'(z, p) d\mu(p)$ can be approximated by sequences of linear forms $U_i(z) = \sum_{j=1}^{j(i)} \lambda_{ij} N(z, p_{ij})$ and $U'_i(z) = \sum_{j=1}^{j(i)} \lambda'_{ij} N'(z, p_{ij})$, where $\lambda_{ij} = \lambda'_{ij} \geqq 0$ and $p_{ij} \in D''$. Hence $\int N(z, p) d\mu(p) - \int N'(z, p) d\mu(p)$ is full harmonic in $H_{\delta^*} + D$ and $= \delta^*$ on ∂H_{δ^*}. Hence by the maximum principle $\int N(z, p) d\mu(p) - \int N'(z, p) d\mu(p) \equiv \delta^*$ in $H_{\delta^*} + D$.

Now $\omega(D, z, H_{\delta^*})$ is represented by a positive mass distribution μ^* on $\bar{D}$. Hence by (4) $U(\delta^*, z) = \int N'(z, p) d(\mu + \delta^* \mu^*)(p)$ in $H_{\delta^*} + D$. Whence by (7) $\int\limits_{\partial H_{\delta^*}} \frac{\partial}{\partial n} U(\delta^*, z) ds = \int\limits_{\partial H_{\delta^*}} \frac{\partial}{\partial n} \int N'(z, p) d(\mu + \delta^* \mu^*)(p) ds = \int\limits_{\partial H_{\delta^*}} \int \frac{\partial}{\partial n} N'(z, p) ds$ $d(\mu + \delta^* \mu^*)(p) = 2\pi \int d(\mu + \delta^* \mu^*) \geqq 2\pi \int d\mu$.

Hence by (6) $2\pi \mathfrak{M}^f({}_D U^M(z)) \geqq 2\pi \int d\mu - \varepsilon$.

Let $\varepsilon \to 0$. Then $\mathfrak{M}^f(U(z)) \geqq \mathfrak{M}^f({}_D U(z)) \geqq \mathfrak{M}^f({}_D U^M(z)) \geqq \int d\mu(p)$ and ${}_D U^M(z)$ is representable by a mass distribution μ on $\bar{D}$ of total mass $\leqq \mathfrak{M}^f(U(z))$ for every M. Let $\{G_n\}$ be an exhaustion of G. Then ${}_{G_n} U^M(z)$ is representable by mass distribution μ_n^M on $\bar{G}_n$. Then $\{\mu_n^M\}$ has a weak limit μ_n on $\bar{G}_n$ as $M \uparrow \infty$. Also $\{\mu_n\}$ has a weak limit μ on $G + B$ such that $U(z) = \int N(z, p) d\mu(p)$ and $\int d\mu(p) \leqq \mathfrak{M}^f(U(z))$ as $n \to \infty$. Thus we have the theorem.

Corollary. *Let μ be a positive mass distribution on a compact set F in G. Then $U(z) = \int N(z, p) d\mu(p)$ is an F_0.S.H. in G with $\mathfrak{M}^f(U(z)) = \int d\mu$.*

Let $D_1 \supset D_2$ be two domains in G such that $D_1 \supset D_2 \supset F$, $\mathrm{dist}(D_2, \partial D_1) > 0$ and $\mathrm{dist}(F, \partial D_2) > 0$. Then $N(z, p) : p \in F$ is uniformly bounded on ∂D_1. Put $L = \max\limits_{p \in F} (\max\limits_{z \in \partial D_1} N(z, p))$ and $L' = \min\limits_{z \in \partial D_1} N(z, p_0)$, where p_0 is a fixed point in F. Then $N(z, p_0) \geqq \frac{L'}{L} N(z, p)$ in $G - D_2$ for any $p \in F$. Hence $U(z) = 0$ on ∂G except at most a set of cap. zero. Also ${}_{D_1} U(z) = U(z)$ and $\mathfrak{M}^f(U(z)) = \mathfrak{M}^f({}_{D_1} U(z))$. By Theorem 2 $U(z)$ is representable by a mass distribution μ^* on $\bar{D}_1$ such that $\mathfrak{M}^f(U(z)) \geqq \int d\mu^*$. But since D_1 is compact, by the uniqueness of distribution, $\mu = \mu^*$ and $\mathfrak{M}^f(U(z)) \geqq \int d\mu$. On the other hand, by Theorem 1.2. $\mathfrak{M}^f(U(z)) \leqq \int d\mu$. Hence $\mathfrak{M}^f(U(z)) = \int d\mu$ and $U(z)$ is an F_0.S.H. in G.

THEOREM 3. *Let $U(z)$ be an F.S.H. in G with $\mathfrak{M}^f(U(z)) < \infty$. Then $U(z)$ is representable by a positive mass distribution μ with $\int d\mu \leqq \mathfrak{M}^f(U(z))$. Conversely a potential $U(z) = \int N(z, p) d\mu(p)$ is an F.S.H. in G with $\mathfrak{M}^f(U(z)) \leqq \int d\mu$.*

Let $\{G_n\}$ be an exhaustion of G. Suppose $U(z)$ is an F.S.H. in G. Then ${}_{G_n} U(z)$ can be defined and there exists a mass distribution μ_n on $\bar{G}_n$ such that ${}_{G_n} U(z) = \int N(z, p) d\mu_n(p)$ and $\int d\mu_n \leqq \mathfrak{M}^f({}_{G_n} U(z)) \leqq \mathfrak{M}^f(U(z))$. Hence $\{\mu_n\}$ has a weak limit μ such that $U(z) = \lim\limits_n {}_{G_n} U(z) = \int N(z, p) d\mu(p)$. Let $U(z)$ be a potential. Let D be a compact domain. Then ${}_D U(z) =$

$\int_D N(z,p)d\mu(p)$ and ${}_DU(z)$ can be defined and ${}_DU(z) \leqq U(z)$. Now $N(z,p)$ is an F_0.S.H. in G with $\mathfrak{M}^f(N(z,p)) \leqq 1$ by Theorem 1, whence by the corollary ${}_DN(z,p) = \int_D N(z,q)d\mu_p(q)$: $\int d\mu_p(q) \leqq 1$. Hence ${}_DU(z) = \iint N(z,q)d\mu_p(q)d\mu(p) = \int N(z,q)d\mu(q)$: $\mu(q) = \int \mu_p(q)d\mu(p)$ and $\mathfrak{M}^f({}_DU(z)) = \int d\mu(q) \leqq \int d\mu$ for any D. Hence $U(z) = \lim_n {}_{G_n}U(z)$ is an F.S.H. in G with $\mathfrak{M}^f(U(z)) \leqq \int d\mu$.

Remark. Let $U(z)$ be an F.S.H. in G with $\mathfrak{M}^f(U(z)) < \infty$. Then $V_M = E[z : U(z) > M]$ is so thinly distributed in a neighbourhood of ∂G. In fact, $D(\omega(V_M, z, G)) \leqq 2\pi M \mathfrak{M}^f(U(z))$. This means V_M is thin. If V_M is very thik, $D(\omega(V_M \cap G_n, z, G)) \uparrow \infty$ as $n \to \infty$.

Let D be a domain. Then by Theorems 1 and 2 we can consider the mass distribution of ${}_{v_n(p)}N(z,p)$, where $v_n(p) = E\left[z \in G + B : \text{ dist}(z,p) < \frac{1}{n}\right]$. As case that ∂G is completely regular we have the following[6]

LEMMA 3. *Let $U(z)$ be an F_0.S.H. (or F.S.H.) in G with $\mathfrak{M}^f(U(z)) < \infty$. Let F be a closed set. We define ${}_FU(z)$ by $\lim_{n=\infty} {}_{F_n}U(z)$, where $F_n = E\left[z \in G + B :\right.$ $\left.\text{dist}(z,F) \leqq \frac{1}{n}\right]$. Then*

1). ${}_F({}_FU(z)) = {}_FU(z)$, *if* $\omega(F,z,G) = 0$. (8)

2). $\omega(F,z,G) = {}_F\omega(F,z,G)$, *if* $\omega(F,z,G) > 0$. (9)

$\mathfrak{M}^f(N(z,p)) \leqq 1$ *for* $p \in G + B$. *If ∂G is completely regular* $\mathfrak{M}^f(N(z,p)) = \frac{1}{2\pi}\int_{\partial G} \frac{\partial}{\partial n} N(z,p)ds = 1$.

But in the present case $\mathfrak{M}^f(N(z,p))$ is not necessarily equal to 1. Then we shall prove the following

THEOREM 4. 1). *Put $\mathfrak{M}(p) = \mathfrak{M}^f(N(z,p))$. Then $\mathfrak{M}(p) = 1$ for $p \in G$ and $\mathfrak{M}(p)$ is lower semicontinuous.*

2). *Put $\phi(v_n(p)) = \mathfrak{M}^f({}_{v_n(p)}N(z,p))$. Then $\phi(v_n(p)) = 1$ for $p \in G$ and $\phi(v_n(p))$ is lower semicontinuous. Clearly $\phi(v_n(p)) \downarrow$ as $n \to \infty$. Put $\phi(p) = \lim_{n=\infty} \phi(v_n(p))$. Then $\phi(p) = 1$* or 0.

Proof of 1). Let $p \in G$. Then clearly $D(\min(M, N(z,p))) = 2\pi M$ and $\mathfrak{M}(p) = 1$ for $p \in G$. By definition $\left(\frac{1}{2\pi M}\right) D(\omega(V_M(p), z, G)) \uparrow \mathfrak{M}(p)$ as $M \downarrow 0$, where $V_M(p) = E[z : N(z,p) > M]$. Hence for any given $\varepsilon > 0$, there exists a number M such that $\mathfrak{M}(p) < \left(\frac{1}{2\pi M}\right) D(\omega(V_M(p), z, G)) + \varepsilon$ and we can find

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a compact set K in $V_M(p)$ such that $D(\omega(V_M(p), z, G)) < D(M\,\omega(K, z)) + 2\varepsilon$, because, if $F_m \uparrow F$, $D(\omega(F_m, z, G)) \uparrow D(\omega(F, z, G))$[7]. Since $\delta(p, p_i) \to 0$ implies $N(z, p_i) \to N(z, p)$ in every compact set, we can find a number i_0 such that $V_{M-\varepsilon}(p_i) \supset K$ for $i \geqq i_0$, whence $\dfrac{D(\omega(K, z, G))}{2\pi M} \leqq \dfrac{D(\omega(V_{M-\varepsilon}(p_i, z, G)))}{2\pi(M-\varepsilon)}$ $\leqq \mathfrak{M}(p_i) : i \geqq i_0$. Let $\varepsilon \to 0$. Then $\mathfrak{M}(p) \leqq \varliminf_i \mathfrak{M}(p_i)$.

Proof of 2). If $p \in G$, clearly ${}_{v_n(p)}N(z, p) = N(z, p)$ and $\phi(v_n(p)) = 1$ for every n. Put $g_M(p) = E[z : {}_{v_n(p)}N(z, p) > M]$. Then by the definition of $\phi(v_n(p))$, for any given $\varepsilon > 0$, there exists a number $M_0 < 1$ such that $\phi(v_n(p)) \leqq \dfrac{D(M\,\omega(g_M, z, G))}{2\pi} + \dfrac{\varepsilon}{2\pi} = \dfrac{M}{2\pi} D(\omega(g_M, z, G)) + \dfrac{\varepsilon}{2\pi}$ for $M \leqq M_0$. Also we can find a compact set K in $Cg_M(p)$ such that $D(\omega(g_M(p), z, G)) \leqq D(\omega(K, z, G)) + \varepsilon$. Now ${}_{v_n(p)}N(z, p) = \lim_{m=\infty} {}_{v_n(p)\cap G_m}N(z, p)$, where $\{G_m\}$ is an exhaustion of G. Hence there exists a number m_0 such that

$$ {}_{v_n(p)}N(z, p) \leqq \frac{M\varepsilon}{2} + {}_{v_n(p)\cap G_m}N(z, p) \qquad \text{on } K \text{ for } m \geqq m_0. $$

Now $N(z, q)$ is continuous in $G - q$, whence ${}_{v_n(q)\cap G_m}N(z, q)$ is continuous on K and there exists a number i_0 such that

$$ {}_{v_n(p_i)}N(z, p_i) \geqq {}_{v_n(p_i)\cap G_m}N(z, p_i) \geqq {}_{v_n(p)\cap G_m}N(z, p) - \frac{M\varepsilon}{2} \geqq {}_{v_n(p)}N(z, p) - M\varepsilon \quad \text{on } K \text{ for} $$

$i \geqq i_0$. This implies $E[z : {}_{v_n(p_i)}N(z, p) \geqq M - M\varepsilon] \supset K$ and

$$ D(\omega(g_{M-M\varepsilon}(p_i), z, G)) \geqq D(\omega(K, z, G)) \geqq D(\omega(g_M(p), z, G)) - \varepsilon \quad \text{for } i \geqq i_0. $$

Thus $2\pi\phi(v_n(p_i)) \geqq M(1-\varepsilon)D(\omega(g_{M-M\varepsilon}(p_i), z, G)) \geqq M\,D(\omega(g_M(p), z, G))\left(\dfrac{M(1-\varepsilon)}{M}\right)$ $- M(1-\varepsilon)\varepsilon \geqq 2\pi(\phi(v_n(p)) - \varepsilon)(1-\varepsilon) - M\varepsilon$ for $i \geqq i_0$.
Let $i \to \infty$ and then $\varepsilon \to 0$. Then $\varliminf_i \phi(v_n(p_i)) \geqq \phi(v_n(p))$.

By Lemma 1, 2, $\mathfrak{M}^f({}_{v_n(p)}N(z, p)) = \lim_{m=\infty} \mathfrak{M}^f({}_{v_n(p)\cap G_m}N(z, p))$. Since $v_n(p) \cap G_m$ is compact, by the corollary of Theorem 2 ${}_{v_n(p)\cap G_m}N(z, p)$ is representable by $\mu_{n,m}$ on $\overline{v_n(p) \cap G_m}$ with $\mathfrak{M}^f({}_{v_n(p)\cap G_m}N(z, p)) = \int d\mu_{n,m}$. Next $\{\mu_{n,m}\}$ has a weak limit μ_n as $m \to \infty$ such that $\mathfrak{M}^f({}_{v_n(p)}N(z, p)) = \int d\mu_n$ on $\bar{v}_n(p)$. Let $n \to \infty$. Then $\{\mu_n\}$ has also a weak limit μ at $p = \bigcap_{n>0} \overline{v_n(p)}$ such that $\int d\mu = \lim_n \mathfrak{M}^f({}_{v_n(p)}N(z, p)) = \phi(p)$. Thus ${}_pN(z, p) = \phi(p)N(z, p)$.

Case 1. $p \in G$. Then $\phi(p) = \lim_n \phi(v_n(p)) = 1$.

Case 2. $\omega(p, z, G) > 0$. In this case $\omega(p, z, G) = \lim_n {}_{v_n(p)}\omega(p, z, G) = \lim_n \int_{\overline{v_n(p)}} N(z, p) d\mu(p) = KN(z, p)$. Now by (9), ${}_p\omega(p, z, G) = \omega(p, z, G)$, i.e. $N(z, p) = {}_pN(z, p)$, whence $\phi(p) = 1$.

Case 3. $\omega(p, z, G) = 0$. By (8) $\phi(p)N(z, p) = {}_pN(z, p) = {}_p({}_pN(z, p)) = \phi^2(p) N(z, p)$. Hence $\phi(p) = 0$ or 1.

N-minimal function and N-minimal points. Let $U(z)$ be an F_0.S.H. in G. If $V(z) = \lambda U(z) : 0 \leqq \lambda \leqq 1$ for any F.S.H. $V(z)$ such that both $V(z)$ and $U(z) - V(z)$ are F.S.H.s in G, we call $U(z)$ *an N-minimal function.* Then as the case that ∂G is completely regular we have the following

THEOREM 5. 1).[7] *Let A be a closed set in $G + B$. Then $\omega(A, z, G) = \int_A N(z, p) d\mu(p)$.*

2). *$\omega(p, z, G) = 0$ for $p \in G$. If $\omega(p, z, G) > 0$, $\omega(p, z, G) = KN(z, p) : K > 0$. We call such a point a singular point and denote by B_s the set of singular points. By Theorem 2 we have*

3). *${}_pN(z, p) = \phi(p)N(z, p)$ and $\phi(p) = 1$ for p with $\omega(p, z, G) > 0$ and $\phi(p) = 1$ or 0. Denote by B_0 and B_1 sets of points of B for which $\phi(p) = 0$ and $\phi(p) = 1$ respectively. Then by* (2) *$B_s \subset B_1$ and $B = B_0 + B_1$.*

4). *B_0 is an F_σ set of capacity zero, whence $B_s \subset B_1$.*

5). *If $U(z) = \int_{B_0} N(z, p) d\mu(p)$, ${}_{B_0}U(z) = 0$.*

6). *Let $U(z)$ be an N-minimal function such that $U(z) = \int_A N(z, p) d\mu(p)$. Then $U(z) = KN(z, p) : p \in (G + B_1) \cap A$.*

7). *$N(z, p)$ is N-minimal or not according as $\phi(p) = 1$ or 0.*

8). Let *$V_M(p) = E[z : N(z, p) > M]$ and suppose $p \in G + B_1$. Then $N(z, p) = {}_{V_M(p)}N(z, p) = {}_{V_M(p) \cap v_n(p)}N(z, p)$ for $M < \sup_{z \in G} N(z, p)$ and for every n, whence $N(z, p) = M\omega(V_M(p), z, G)$ in $G - V_M(p)$.*

9). *Every potential $U(z) = \int N(z, p) d\mu(p)$ can be represented by another distribution μ on $G + B_1$ without any change of $U(z)$. This distribution is called canonical.*

If ∂G is completely regular $\mathfrak{M}^f(p) = 1$ for $p \in G + B$. But in general cases $\mathfrak{M}(p)$ is not necessarily $= 1$. We shall prove

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LEMMA 4. $\mathfrak{M}(p) = \mathfrak{M}^f(N(z, p)) = 1$ *for* $p \in G + B_1$.

Let $\{G_m\}$ be an exhaustion of G. By $p \in G + B_1$ ${}_{v_n(p)}N(z, p) = N(z, p)$. Assume $\mathfrak{M}^f(N(z, p)) \leqq \delta < 1$. Then $\mathfrak{M}^f({}_{v_n(p)\cap G_m}N(z, p)) \leqq \mathfrak{M}^f(N(z, p)) \leqq \delta$. By Theorem 2 ${}_{v_n(p)\cap G_m}N(z, p)$ is represented by a mass $\mu_{n,m}$ on $\overline{v_n(p) \cap G_m}$ with $\int d\mu_{n,m} \leqq \delta$. Let $m \to \infty$ and then $n \to \infty$. Then ${}_pN(z, p) \leqq \delta N(z, p)$. This contradicts ${}_pN(z, p) = N(z, p)$. Hence $\mathfrak{M}(p) = 1$.

THEOREM 6. *Let* $U(z) = \int\limits_{G+B_1} N(z, p)d\mu(p)$. *Then*

$$\mathfrak{M}^f(U(z)) = \int d\mu\,,$$

where $U(z)$ *is not necessarily an* F_0*.S.H. in* G (*clearly for an F.S.H. in* G).

This is an extension of the corollary of Theorem 2.

Put $\phi(p, n, m) = \mathfrak{M}^f({}_{v_n(p)\cap G_m}N(z, p))$. Then by Theorem 4 and by $p \in G + B_1$ $\phi(p, n, m) \uparrow \phi(p, n) = \mathfrak{M}^f({}_{v_n(p)}N(z, p)) = \mathfrak{M}^f(N(z, p)) = 1$ as $m \to \infty$. Put $U_m(z) = \int {}_{v_n(p)\cap G_m}N(z, p)d\mu(p)$. Then

$$U(z) = \int \lim_{m=\infty} {}_{v_n(p)\cap G_m}N(z, p)d\mu(p) = \lim_{m=\infty} \int {}_{v_n(p)\cap G_m}N(z, p)d\mu(p) = \lim_{m=\infty} U_m(z)\,.$$

Now ${}_{v_n(p)\cap G_m}N(z, p) = \int\limits_{\overline{v_n(p)\cap G_m}} N(z, q)d\mu_p(q)$ and since $\mu_p(q) > 0$ only on a compact set $\bar{G}_m$, we have $\int d\mu_p(q) = \phi(p, n, m)$ by the corollary of Theorem 2. Hence $U_m(z) = \iint\limits_{\bar{G}_m} N(z, q)d\mu_p(q)d\mu(p)$ and $\mathfrak{M}^f(U_m(z)) = \int \phi(p, n, m)d\mu(p)$. It is easily verified that Lemma 1. 2. holds for F.S.H.s and $\mathfrak{M}^f(U_m(z)) \uparrow \mathfrak{M}^f(U(z))$, if $U_m(z) \uparrow U(z)$. Now $\mathfrak{M}^f(U_m(z)) \uparrow \mathfrak{M}^f(U(z))$ and $\phi(p, n, m) \uparrow \phi(p, n) = 1$ as $m \to \infty$ for $p \in G + B_1$. Hence $\mathfrak{M}^f(U(z)) = \int d\mu(p)$.

REFERENCES

[1] If ∂G and ∂D are compact and smooth, $d(\lambda, z)$ is given as $\frac{\partial N}{\partial n}(\zeta, z)ds$, where $N(\zeta, z)$ is the N-Green's function of $G - D$ with pole at z.

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[3] See [2].

[4] See [2].

[5] See [2].
[6] See [2].
[7] See [2].
[8] See [2].