CANONICAL TREATMENT OF DISSIPATIVE FORCES BETWEEN EARTH MANTLE AND CORE

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Abstract. Dissipative effects arising from the core–mantle interaction are treated in a Hamiltonian framework, using a simple model. Analytical solutions are obtained for free and forced motions. The first show the persistence or damping of the different components. The latter, frequency dependent changes of amplitude and phase. Preliminary numerical values are in acceptable agreement with observational data.

1. Introduction

The advantages of the Hamiltonian formalism are well known in general Celestial Mechanics, where it has been frequently used. The study of the Earth rotation is not an exception, and the theory of Kinoshita (1977) is the most complete and accurate among those for a rigid Earth model. Recently, the Hamiltonian approach has been followed by the authors in order to extend Kinoshita's theory to general models of deformable Earth through a series of papers, first considering an elastic mantle (Getino and Ferrándiz, 1995), then a liquid core (Getino, 1995a, 1995b). We only point out here that the results obtained are completely analytical, and depend on parameters that are provided by the different Earth models available.

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In this paper we consider the possibilities of including dissipative effects in the previous formulations, like those arising from friction in the boundary layer between mantle and core. As a first step, we have chosen a fairly well known simplified model, in which the dissipation appears through a torque proportional to the difference of angular velocities (Kubo,1979). Using canonical Andoyer–like variables as in Getino (1995b), it is only necessary to introduce the generalized forces in a straightforward way. By integrating the unperturbed Hamiltonian equations, solutions for the free wobbles are obtained, which in our model represent the CW and the NDFW, using the standard terminology for conservative free oscillations. It is remarkable that only the component terms corresponding to the NDFW exhibit a damping, but not the component associated to CW. That is in good agreement with the difficulties found in the experimental determination of the free nutation itself (Mathews and Shapiro, 1992), and on the other hand is a common feature of general dissipative systems (Birkhoff, 1927), that are to tend towards conservative systems of lower dimension.

Besides the solutions for free wobbles and polar motion, the forced nutations are also obtained. Together with the frequency dependent amplitude variations due to non–rigidity, phase changes caused by the dissipation appear. This last effect is not obtained in other theories like Wahr (1981). As in our previous developments, the analytical expressions depend on parameters provided by quite general Earth models.

2. Canonical expression of the kinetic energy

We consider an Earth model composed of a rigid mantle and a liquid core. Let $OXYZ$ be a non rotating inertial frame, $Oxyz$ the frame of the principal axes of the total Earth rotating with an angular velocity $\vec{\omega}$ with respect to the inertial frame, and $Ox_cy_cz_c$ a core fixed frame rotating with angular velocity $\delta\vec{\omega}$ with respect to the mantle. To formulate canonically the kinetic energy let us use the set of canonical variables $\lambda, \mu, \nu, \Lambda, M, N$ for the total Earth, and $\lambda_c, \mu_c, \nu_c, \Lambda_c, M_c, N_c$ for the core, described in Getino (1995a,b).

We can write the components of M and M_c in the $Oxyz$ frame in terms of the canonical variables as follows

$$\mathrm{M} = \begin{pmatrix} A\omega_1 + A_c\delta\omega_1 = K\sin\nu \\ A\omega_2 + A_c\delta\omega_2 = K\cos\nu \\ C\omega_3 + C_c\delta\omega_3 = N \end{pmatrix}, \ \mathrm{M}_c = \begin{pmatrix} A_c\omega_1 + A_c\delta\omega_1 = K_c\sin\nu_c \\ A_c\omega_2 + A_c\delta\omega_2 = -K_c\cos\nu_c \\ C_c\omega_3 + C_c\delta\omega_3 = N_c \end{pmatrix} \quad (1)$$

where we have put

$$K = (M^2 - N^2)^{1/2} = M\sin\sigma\,, \qquad K_c = (M_c^2 - N_c^2)^{1/2} = M_c\sin\sigma_c\ . \quad (2)$$

Then, the canonical expression of the kinetic energy is (Getino, 1995b):

$$T = \frac{1}{2(A-A_c)}\left(K^2 + \frac{A}{A_c}K_c^2\right) + \frac{1}{2(C-C_c)}\left(N^2 - 2\,N\,N_c + \frac{C}{C_c}N_c^2\right) + \frac{K\,K_c}{A-A_c}\cos(\nu+\nu_c)\,, \tag{3}$$

3. Generalized forces and equations of motion

The frictional torque,including electromagnetic coupling and the effects of the viscosity, is proportional to the difference of the angular velocities of the mantle and the core. According to Kubo (1979) this torque is given by

$$\begin{array}{lllll} \overrightarrow{t_m} & = & \Gamma\,(\overrightarrow{\omega_c} - \overrightarrow{\omega_m}) & = & \Gamma\,\overrightarrow{\delta\omega} \\ \overrightarrow{t_c} & = & \Gamma\,(\overrightarrow{\omega_m} - \overrightarrow{\omega_c}) & = & -\,\Gamma\,\overrightarrow{\delta\omega} \end{array}\,, \text{ with } \Gamma = \begin{pmatrix} \lambda_\perp & 0 & 0 \\ 0 & \lambda_\perp & 0 \\ 0 & 0 & \lambda_\| \end{pmatrix}, \tag{4}$$

$\lambda_\perp$ and $\lambda_\|$ being the coefficients for perpendicular and parallel directions to the rotational axis. The generalized forces due to the dissipative torque are obtained through the dot product:

$$\overrightarrow{t}\cdot\overrightarrow{d\phi} = \overrightarrow{t_m}\cdot\overrightarrow{d\phi_m} + \overrightarrow{t_c}\cdot\overrightarrow{d\phi_c} = \overrightarrow{t_c}\cdot\left(\overrightarrow{d\phi_c} - \overrightarrow{d\phi_m}\right)\,, \tag{5}$$

where $\overrightarrow{d\phi_m}$ and $\overrightarrow{d\phi_c}$ are the infinitesimal rotations of the reference frames of mantle (total Earth) and core with respect of the inertial frame. Taking into account the meaning of the canonical variables (see Getino, 1995a), the generalized equations of motion will be

$$\dot{q}_i = \partial T/\partial p_i - W_{p_i}\,,\ \ \dot{p}_i = -\partial T/\partial q_i + W_{q_i}\,, \tag{6}$$

where W_{p_i} and W_{q_i} are the components of the generalized forces.

4. Solutions for the free motion

Taking into account the relationships (6), the necessary generalized equations for the free motion problem are:

$$\begin{array}{lllllll} \dot{\nu} & = & \partial T/\partial N\,, & \dot{\nu}_c & = & \partial T/\partial N_c - W_{N_c}\,, \\ \dot{N} & = & -\,\partial T/\partial \nu, & \dot{N}_c & = & -\,\partial T/\partial \nu_c + W_{\nu_c}\,, \\ \dot{M} & = & -\,\partial T/\partial \mu\,, & \dot{M}_c & = & -\partial T/\partial \mu_c + W_{\mu_c}\,, \end{array} \tag{7}$$

where

$$
\begin{aligned}
W_{N_c} &= -\frac{\lambda_\perp}{A_m}\frac{K}{K_c}\sin(\nu+\nu_c) \\
W_{\nu_c} &= \frac{\lambda_\parallel}{C_m}\left(N-\frac{C}{C_c}N_c\right) \\
W_{\mu_c} &= \frac{\lambda_\parallel}{C_m}\cos\sigma_c\left(N-\frac{C}{C_c}N_c\right)
\end{aligned}
$$

As a first result, by means of (1) and (7) we have that $\dot{N}-\dot{N}_c=\lambda_\parallel\,\delta\omega_3$, so that, neglecting second order terms, we obtain $\dot{\omega}_3=\frac{\lambda_\parallel}{C_m}\,\delta\omega_3$, while $\delta\omega_3 = e^{-\frac{\lambda_\parallel}{C_m}\frac{C}{C_c}t}$. That is to say, the relative angular velocity of the core with respect to the mantle is damped by the frictional forces. Thus, we can take $\delta\omega_3 \to 0$, and then ω_3 =constant $=\Omega$. With these simplifications the system of equations (7) can be easily solved performing the change of variables $p=K\sin\nu$, $q=K\cos\nu$, $p_c=K_c\sin\nu_c$ and $q_c=K_c\cos\nu_c$. Notice that these new variables are linear combinations of $\omega_1,\omega_2,\delta\omega_1$ and $\delta\omega_2$ (see Getino, 1995b). The solution can be written in the form:

$$
\begin{aligned}
u &= p+iq &&= \gamma_a\,\alpha\,e^{i m_1 t}+\beta\,e^{-dt}e^{i m_2 t}, \\
v &= p_c-iq_c &&= \gamma_b\,\alpha\,e^{i m_1 t}+\gamma_c\,\beta\,e^{-dt}e^{i m_2 t},
\end{aligned}
\tag{8}
$$

α and β being the arbitrary constants of integration, the coefficients γ_i are

$$
\begin{aligned}
\gamma_a &= \frac{A}{A_m}\left(\frac{C-A}{A}+\frac{C_c}{A_c}\right)-i\,\frac{\lambda_\perp}{\Omega\,A_c}\frac{A}{A_m}, \\
\gamma_b &= \frac{C_c}{A_m}-i\,\frac{\lambda_\perp}{\Omega\,A_m}, \\
\gamma_c &= \frac{C_c}{A_c}-i\,\frac{\lambda_\perp}{\Omega\,A_c},
\end{aligned}
\tag{9}
$$

and the frequencies

$$
\begin{aligned}
m_1 &= \Omega\,\frac{C-A}{A_m} && \to \ \mathbf{CW}\ , \\
m_2 &= -\Omega\left(1+\frac{C_c-A_c}{A_c}\frac{A}{A_m}\right) && \to \ \mathbf{FCN}\ , \\
d &= \frac{\lambda_\perp}{A_c}\frac{A}{A_m} && \to \ \mathbf{Damping}\ .
\end{aligned}
\tag{10}
$$

According to (10), the well known frequencies for an Earth model with liquid core, that is to say, the Chandler Wobble (CW) and the Free Core Nutation (FCN), have been obtained directly from the equations of motion of the Hamiltonian of the system, unlike the majority of approaches which

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obtain these frequencies from the Liouville equations. On the other hand, from (8) we see that the CW is not affected by the dissipation, while the FCN is damped by the coefficient d. This fact would justify the difficulties found in the experimental determination of this free nutation, which "remains elusive" (Mathews and Shapiro, 1992).

5. Forced nutations

Once the solution for the unperturbed Hamiltonian has been obtained, we can undertake the problem of the perturbed case, taking into account the disturbing potential of Moon and Sun, following the same analytical procedure described by Kinoshita (1977) for the rigid Earth, by Getino and Ferrándiz (1995) for a deformable Earth with elastic mantle, and by Getino (1995b) for a rigid mantle–liquid core conservative Earth model. The most remarkable fact is that, when obtaining the generating function of the canonical transformation corresponding to the analytical perturbation method, a delay appears due to the imaginary coefficients γ_i (9) presents in the solution (8) of the unperturbed case. Thus, when obtaining the nutation series we have in–phase and out–of–phase terms. For the Oppolzer terms of obliquity and longitude of the figure plane, we have respectively the final expressions:

$$\begin{aligned}
\Delta(I_f - I) &= k \sum_i \sum_{\tau=\pm 1} \frac{C_i}{n_\mu - \tau\, n_i} F_a(\tau n_i) \cos\Theta_i - \\
&\quad - \frac{k}{\sin I} \sum_i \sum_{\tau=\pm 1} \frac{\tau\, C_i}{n_\mu - \tau\, n_i} F_b(\tau n_i) \sin\Theta_i \,, \\
\Delta(\lambda_f - \lambda) &= \frac{k}{\sin I} \sum_i \sum_{\tau=\pm 1} \frac{\tau\, C_i}{n_\mu - \tau\, n_i} F_a(\tau n_i) \sin\Theta_i + \\
&\quad + \frac{k}{\sin I} \sum_i \sum_{\tau=\pm 1} \frac{C_i}{n_\mu - \tau\, n_i} F_b(\tau n_i) \cos\Theta_i \,.
\end{aligned} \tag{11}$$

In these expressions, the effect of dissipative forces as well as the liquid core is included in the corrector factors F_a and F_b, which are written as follows:

$$\begin{aligned}
F_a(\tau n_i) &= \frac{\frac{C}{A}\Omega - \tau n_i}{f_1(\tau n_i)\ f_2(\tau n_i)} \left(f_2(\tau n_i) - \Omega\, \frac{A_c}{A_m} \right), \\
F_b(\tau n_i) &= \frac{\frac{C}{A}\Omega - \tau n_i}{f_1(\tau n_i)\ f_2(\tau n_i)} \frac{\lambda_\perp}{A_c} \frac{A^2}{A_m^2} \frac{f_2(\tau n_i) - \Omega\, \frac{A_c}{A}}{f_2(\tau n_i)} \,,
\end{aligned} \tag{12}$$

with

$$f_1(\tau n_i) = \Omega + m_1 - \tau\, n_i \,, \quad f_2(\tau n_i) = \Omega + m_2 - \tau\, n_i \,.$$

Note that the corrector factor F_b is responsible of the out–of–phase terms, while the in–phase terms depend on F_a. If we neglect the dissipation ($\lambda_\perp = 0$), the out–of–phase terms disappears, and we get similar expressions as those obtained in Getino (1995b) for the conservative case. Even more, if we particularize to the rigid case ($A_c = 0$), then $F_a = 1$, and we obtain the same expressions as in Kinoshita's theory.

Thus, we can conclude that it is possible to develop a canonical theory for the rotation of an Earth model with liquid core, including the effect of dissipative forces, in a way very similar to that of Kinoshita for the rigid case. Neglecting the presence of liquid core and dissipation, our approach coincides exactly with that of Kinoshita, so that we can say that this model is a more general one, including the theory of Kinoshita as a particular case.

6. Final remarks

The Hamiltonian treatment of dissipative effects, like those arising in the mantle–core boundary, provides a fairly simple, completely analytical solution for the Earth rotation, including free motion as well as forced nutations. In spite of the simplicity of the dissipative model, the solution is qualitatively well–fitted to the experimental evidence, providing amplitude and phase–frequency dependent forced nutations, and a clear theoretical derivation of the free motion showing the damping of NDFW and the persistence of CW. Numerical values of in–phase and out–of–phase nutations are also compatible in magnitude with experimental data. Nevertheless, we insist that the main purpose of this report is not to obtain accurate series for the real Earth, but to show how the canonical approach can be applied to study dissipative phenomena in Earth rotation.

In future steps we expect to progress in the development of a complete unified Hamiltonian theory for the Earth rotation adapted to more accurate Earth models.

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