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CHIEF SERIES AND RIGHT REGULAR REPRESENTATIONS OF FINITE p-GROUPS

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Dedicated to B. Huppert on the occasion of his sixtieth birthday

Abstract

We study the embeddings of a finite p-group U into Sylow p-subgroups of $\mathrm{Sym}(U)$ induced by the right regular representation $\rho\colon U \to \mathrm{Sym}(U)$. It turns out that there is a one-to-one correspondence between the chief series in U and the Sylow p-subgroups of $\mathrm{Sym}(U)$ containing $U\rho$. Here, the Sylow p-subgroup P_Σ of $\mathrm{Sym}(U)$ corresponding to the chief series Σ in U is characterized by the property that the intersections of $U\rho$ with the terms of *any* chief series in P_Σ form $\Sigma\rho$. Moreover, we see that $\rho\colon U \to P_\Sigma$ are precisely the kinds of embeddings used in a previous paper to construct the non-trivial countable algebraically closed locally finite p-groups as direct limits of finite p-groups.

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1. Introduction

Let $U \leq G$ be finite p-groups. A *chief series* $1 = U_0 < U_1 < \cdots < U_m = U$ in U is said to be *induced* by the chief series $1 = G_0 < G_1 < \cdots < G_n = G$ in G, if $\{U \cap G_j | 0 \leq j \leq n\} = \{U_i | 0 \leq i \leq m\}$. Since chief factors of finite p-groups are cyclic of order p, every chief series in G induces a chief series in U.

In [3, Section 3] we have developed a uniform construction which yields, for any chief series Σ in U, a finite p-group $G_\Sigma \geq U$ such that *every* chief series in G induces Σ in U. In this situation we say that G_Σ *controls* Σ. The construction is a successive application of Frobenius embeddings into wreath products, and

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the group G_Σ obtained in this way is isomorphic to the Sylow p-subgroups of the symmetric group $\mathrm{Sym}(U)$ on U.

Now, it can already be read off from [2, page 487] that there is a close connection between Frobenius embeddings and right regular representations. Thus, the question arises how $\mathrm{id}\colon U \to G_\Sigma$ is related to the embeddings of U into Sylow p-subgroups of $\mathrm{Sym}(U)$ obtained from the right regular representation $\rho\colon U \to \mathrm{Sym}(U)$. It is the aim of the present note to show that this relation is as nice as one can hope for. We will see that, for any Sylow p-subgroup P of $\mathrm{Sym}(U)$ containing $U\rho$, the embedding $\rho\colon U \to P$ is of one of the types "$\mathrm{id}\colon U \to G_\Sigma$", and this will amount to the following result.

THEOREM. *Let $\rho\colon U \to \mathrm{Sym}(U)$ be the right regular representation of a finite p-group U.*

(a) *Every Sylow p-subgroup of* $\mathrm{Sym}(U)$ *containing $U\rho$ controls a chief series in $U\rho$.*

(b) *For every chief series Σ in U there exists precisely one Sylow p-subgroup of* $\mathrm{Sym}(U)$ *which contains $U\rho$ and controls $\Sigma\rho$.*

In particular, the number of chief series in U coincides with the number of Sylow p-subgroups of $\mathrm{Sym}(U)$ *containing $U\rho$.*

In [3, Section 4] we have developed a construction for each of the two (isomorphism types of) non-trivial countable algebraically closed locally finite p-groups as direct limits of finite p-groups G_n, $n \in \mathbb{N}$, which are iterated wreath products of the cyclic group C_p of order p, and where the embeddings $G_n \to G_{n+1}$ are essentially of one of the types "$\mathrm{id}\colon U \to G_\Sigma$". Now, our Theorem says that these embeddings quasi are right regular representations. Therefore, the resemblance of the above constructions to P. Hall's construction of the countable universal locally finite group [1] is even closer than expected then.

As finite nilpotent groups are direct products of finite p-groups, we will also be able to derive the following

COROLLARY. *Let $\rho\colon U \to \mathrm{Sym}(U)$ be the right regular representation of a finite nilpotent π-group U.*

(a) *Every maximal nilpotent subgroup of* $\mathrm{Sym}(U)$ *containing $U\rho$ controls a chief series in each prime component of $U\rho$.*

(b) *For every tuple $(\Sigma_p)_{p\in\pi}$, where Σ_p is a chief series in the p-component of U, there exists precisely one maximal nilpotent subgroup of* $\mathrm{Sym}(U)$ *which contains $U\rho$ and controls $\Sigma_p\rho$ for each $p \in \pi$.*

Note that, in the Corollary, every nilpotent subgroup of $\mathrm{Sym}(U)$ containing $U\rho$ is also a π-group [4, page 61, Lemma 24].

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2. The connection

Concerning wreath products we will adopt the notation introduced in [3, Section 2]. Recall that, for any finite group U of order p^m, the Sylow p-subgroups of $\mathrm{Sym}(U)$ are isomorphic as permutation groups to the iterated wreath product S_{p^m} of m cyclic groups of order p, which is defined recursively by

$$S_{p^0} = 1 \quad \text{and} \quad S_{p^m} = S_{p^{m-1}} \operatorname{wr} C_p.$$

We say that $\phi\colon U \to S_{p^m}$ is an *iterated Frobenius embedding*, if it can be obtained recursively by the following process: Given a normal subgroup V of index p in U and an iterated Frobenius embedding $\phi_0\colon V \to S_{p^{m-1}}$, choose an isomorphism $\gamma\colon U/V \to C_p$ and a transversal $T = \{t_{Vu} | u \in U\}$ of V in U such that $V \cdot t_{Vu} = Vu$; then $\phi\colon U \to S_{p^m} = S_{p^{m-1}} \operatorname{wr} C_p$ is defined by

$$u\phi = ((Vu)\gamma, f_u) \quad \text{for all } u \in U,$$

where

$$f_u((Vu')\gamma) = (t_{Vu'u^{-1}} \cdot u \cdot t_{Vu'}^{-1})\phi_0 \quad \text{for all } u' \in U.$$

It is well known that $U\phi$ is a transitive subgroup of S_{p^m} (see [2, pages 487–488]). Note that, if G_Σ is the finite p-supergroup of U attached to the chief series Σ in U by [3, Construction 3.1], then there exists an isomorphism $G_\Sigma \to S_{p^m}$ whose restriction to U is an iterated Frobenius embedding.

We now come to the key observation.

LEMMA. *Let $\rho\colon U \to \mathrm{Sym}(U)$ be the right regular representation of a finite group U of order p^m. If P is a Sylow p-subgroup of $\mathrm{Sym}(U)$ containing $U\rho$, then there exists an isomorphism $\alpha\colon P \to S_{p^m}$ such that $\rho\alpha\colon U \to S_{p^m}$ is an iterated Frobenius embedding.*

PROOF. Let $m \geq 1$. Choose any permutation group isomorphism

$$\beta\colon S_{p^m} \to P.$$

Let B be the image of the base group of $S_{p^m} = S_{p^{m-1}} \operatorname{wr} C_p$ under β. Then U is the disjoint union of the orbits $\Omega_0, \ldots, \Omega_{p-1}$ under B, and $|\Omega_r| = p^{m-1}$ for $0 \leq r \leq p-1$. Since $U\rho$ acts transitively on U, there exists a normal subgroup V of index p in U such that $V\rho = U\rho \cap B$. Fix $u \in U \backslash V$. Since $V\rho$ acts transitively on each coset Vu^r, we may assume that $\Omega_r = Vu^r$.

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Now, $B = B_0 \times \cdots \times B_{p-1}$ where B_r is the pointwise stabilizer of $U \backslash V u^r$ in B. Let d be the image of an element from the top group of S_{p^m} under β such that $u\rho \in d \cdot B$. Clearly,

$$B_0^{(d^r)} = B_r \quad \text{for } 0 \leq r \leq p-1.$$

Put

$$(u\rho)^r = d^r \cdot b_{r,0} b_{r,1}^d \cdots b_{r,p-1}^{(d^{p-1})}$$

where

$$b_{r,s} \in B_0 \quad \text{for } 0 \leq s \leq p-1.$$

Next, if we identify $\mathrm{Sym}(V)$ canonically with the pointwise stabilizer of $U \backslash V$ in $\mathrm{Sym}(U)$, then the right regular representation $\rho_0 \colon V \to \mathrm{Sym}(V)$ embeds V into the Sylow p-subgroup B_0 of $\mathrm{Sym}(V)$. Hence, proceeding by induction over m, we may assume that there does already exist an isomorphism $\alpha_0 \colon B_0 \to S_{p^{m-1}}$ such that $\rho_0 \alpha_0 \colon V \to S_{p^{m-1}}$ is an iterated Frobenius embedding. Let $C_p = \langle c \rangle$. Define an isomorphism $\eta \colon P \to S_{p^m} = S_{p^{m-1}} \operatorname{wr} C_p$ via

$$\eta \colon d^r \cdot \left[\prod_{s=0}^{p-1} b_s^{(d^s)} \right] \mapsto (c^r, f)$$

where $b_s \in B_0$ and $f(c^s) = b_s \alpha_0$ for $0 \leq s \leq p-1$. Now, let $\alpha \colon P \to S_{p^m}$ be conjugation with

$$x = \left[\prod_{s=1}^{p-1} b_{s,s}^{(d^s)} \right]^{-1} \in B,$$

followed by η. Let us calculate $\rho\alpha \colon U \to S_{p^m}$ to show that it is an iterated Frobenius embedding.

In the following, w will be an element from V. Observe that

$$wu^r = [w](u^r \rho) = [wd^r](b_{r,r}^{(d^r)}) = [wd^r]x^{-1} \quad \text{for } 1 \leq r \leq p-1,$$

while $wu^0 = w = [w]x^{-1} = [wd^0]x^{-1}$. Thus,

$$(*) \qquad [wu^r]x = [w]d^r \quad \text{for } 0 \leq r \leq p-1.$$

For every $v \in V$, we conclude that

$$\begin{aligned} [wd^r](x^{-1} \cdot v\rho \cdot x) &\overset{(*)}{=} [wu^r](v\rho \cdot x) = [w \cdot v^{(u^{-r})} \cdot u^r]x \\ &\overset{(*)}{=} [w \cdot v^{(u^{-r})}]d^r = [w](v^{(u^{-r})}\rho_0 \cdot d^r). \end{aligned}$$

Hence,

$$v\rho^x = \prod_{s=0}^{p-1} (v^{(u^{-s})}\rho_0)^{d^s} \in B,$$

and thus

$$v\rho\alpha = v\rho^x \eta = (1, f_v) \quad \text{where } f_v(c^s) = v^{(u^{-s})}\rho_0\alpha_0.$$

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Furthermore,

$$[wd^r](x^{-1}\cdot u\rho\cdot x) \overset{(*)}{=} [wu^r](u\rho\cdot x) = [wu^{r+1}]x$$
$$\overset{(*)}{=} [w]d^{r+1} \quad \text{for } 0 \le r \le p-2,$$

while

$$[wd^{p-1}](x^{-1}\cdot u\rho\cdot x) \overset{(*)}{=} [wu^{p-1}](u\rho\cdot x) = [wu^p]x$$
$$= wu^p = [w](u^p\rho_0).$$

Hence, $u\rho^x = d\cdot u^p\rho_0$, and thus $u\rho\alpha = u\rho^x\eta = (c, f_u)$ where

$$f_u(c^s) = \begin{cases} u^p\rho_0\alpha_0 & \text{if } s = 0, \\ 1 & \text{else.} \end{cases}$$

Straightforward calculations yield that the iterated Frobenius embedding $\phi\colon U \to S_{p^m}$ obtained from choosing $\phi_0 = \rho_0\alpha_0$, $\gamma\colon Vu \mapsto c$ and $T = \{u^r | 0 \le r \le p-1\}$ satisfies $u\phi = u\rho\alpha$ and $v\phi = v\rho\alpha$ for every $v \in V$.

3. Proof of the Theorem

Part (a) is a consequence of the Lemma and of [3, Theorem 3.3].

To prove part (b), let

$$\Sigma\colon 1 = U_0 < U_1 < \cdots < U_m = U$$

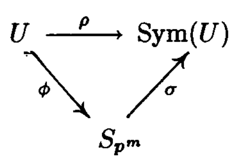
be a fixed chief series in the finite group U of order p^m. Choose an iterated Frobenius embedding $\phi\colon U \to S_{p^m}$. Now, S_{p^m} is a permutation group on the set

$$C_p^{(m)} = C_p \times C_p \times \cdots \times C_p \quad \text{of order } p^m.$$

Since $U\phi$ acts transitively on $C_p^{(m)}$, there exists for each $x \in C_p^{(m)}$ a unique $u \in U$ with $x = [1](u\phi)$. Therefore, an embedding $\sigma\colon S_{p^m} \to \mathrm{Sym}(U)$ is given by

$$[1](([u](g\sigma))\phi) = [1](u\phi\cdot g) \quad \text{for every } g \in S_{p^m}.$$

It is easy to see that the diagram



commutes. And by [3, Theorem 3.3], every chief series of S_{p^m} induces $\Sigma\phi$ in $U\phi$, whence $S_{p^m}\sigma$ is a Sylow p-subgroup of $\mathrm{Sym}(U)$ which contains $U\rho$ and controls $\Sigma\rho$.

Now, for the proof of the uniqueness, let $m \ge 1$, and let P be a Sylow p-subgroup of $\mathrm{Sym}(U)$ which contains $U\rho$ and controls $\Sigma\rho$. From the Lemma and

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[3, Theorem 3.9(a)] we obtain that $U_1\rho = Z(P)$. In particular, P is a Sylow p-subgroup of the centralizer C of $U_1\rho$ in $\mathrm{Sym}(U)$. Denote epimorphic images modulo U_1 by bars. Fix $x \in U_1\backslash 1$. Then $x\rho$ is the product of the p^{m-1} disjoint p-cycles

$$\xi_{\bar{u}} = (u \; ux \; \cdots \; ux^{p-1}) \quad \text{where } u \in U.$$

Define the homomorphism $\gamma\colon C \to \mathrm{Sym}(\bar{U})$ via

$$\xi_{[\bar{u}](\tau\gamma)} = \tau^{-1} \cdot \xi_{\bar{u}} \cdot \tau \quad \text{for all } u \in U \text{ and every } \tau \in C.$$

γ is an epimorphism; for if $\nu \in \mathrm{Sym}(\bar{U})$ and $\xi_{\bar{u}} = (z_{\bar{u},1} \cdots z_{\bar{u},p})$, then the permutation $\tau \in C$, given by $[z_{\bar{u},i}]\tau = z_{\overline{ru\nu},i}$ for all $u \in U$ and $1 \le i \le p$, is a preimage of ν under γ in C.

Clearly,

$$N = \mathrm{Ker}\,\gamma = \bigcap_{u \in U} C_{\mathrm{Sym}(U)}(\xi_{\bar{u}}) = \prod_{\bar{u} \in \bar{U}} \langle \xi_{\bar{u}} \rangle.$$

Let $\bar{\gamma}\colon C/N \to \mathrm{Sym}(\bar{U})$ be the isomorphism induced by γ. Denote by $\bar{\rho}\colon \bar{U} \to \mathrm{Sym}(\bar{U})$ the right regular representation. Since each element of $U\rho\backslash 1$ moves every symbol from U, it is obvious that $U\rho \cap N = U_p\rho$. Hence, ρ induces an embedding $\eta\colon U/U_1 \to C/N$.

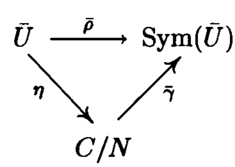
Regard any $g \in U$. Because of $x \in U_1 \le Z(U)$ we have

$$\begin{aligned}(g\rho)^{-1} \cdot \xi_{\bar{u}} \cdot (g\rho) &= (g\rho)^{-1} \cdot (u \; ux \cdots ux^{p-1}) \cdot (g\rho) \\ &= ([u](g\rho) \; [ux](g\rho) \cdots [ux^{p-1}](g\rho)) \\ &= (ug \; ugx \; \cdots \; ugx^{p-1}) = \xi_{\overline{ug}} \quad \text{for every } u \in U.\end{aligned}$$

Therefore, $[\bar{u}](g\rho\gamma) = \overline{ug}$, whence

$$[\bar{u}](\bar{g}\eta\bar{\gamma}) = \overline{ug} = [\bar{u}](\overline{g\rho}) \quad \text{for every } u \in U.$$

Thus, the diagram



commutes.

Next, observe that N is a normal p-subgroup of C. So, $N \le P$. Further, P/N is a Sylow p-subgroup of C/N which contains $\bar{U}\eta$ and controls the chief series

$$\bar{\Sigma}\colon 1 = \bar{U}_1\eta < \bar{U}_2\eta < \cdots < \bar{U}_m\eta = \bar{U}\eta$$

in $\bar{U}$. Because of $|P\colon N| = |S_{p^{m-1}}|$, this implies that $(P/N)\bar{\gamma}$ is a Sylow p-subgroup of $\mathrm{Sym}(\bar{U})$ which contains $\overline{U\rho}$ and controls $\overline{\Sigma\rho}$. Using induction over

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m, we may assume that $(P/N)\bar{\gamma}$ is uniquely determined by these properties. But then, P/N and P are uniquely determined too.

4. Proof of the Corollary

Every maximal nilpotent subgroup N of $\mathrm{Sym}(U)$ containing $U\rho$ is transitive. Therefore, [4, page 61, Lemma 24] yields that N is a π-group. Let

$$U = \prod_{p\in\pi} U_p \quad \text{and} \quad N = \prod_{p\in\pi} N_p$$

be the decompositions of U and N into their p-components U_p resp. N_p. Since $U_q\rho \le N_q$ for all $q \in \pi$, we have

$$[w_p \cdot w_{p'}]\sigma = [w_p](w_{p'}\rho \cdot \sigma) = [w_p](\sigma \cdot w_{p'}\rho) = w_p\sigma \cdot w_{p'}$$

for every $\sigma \in N_p$ and all $w_p \in U_p$, $w_{p'} \in \prod_{q\neq p} U_q$. Thus, the cosets of U_p in U are precisely the transitivity systems of N_p.

Let $\rho_p\colon U_p \to \mathrm{Sym}(U_p)$ be the right regular representation, and define an embedding $\hat{\rho}\colon U \to \prod_{p\in\pi} \mathrm{Sym}(U_p)$ via

$$\hat{\rho}\colon (u_p)_{p\in\pi} \mapsto (u_p\rho_p)_{p\in\pi}.$$

Further, let $\tau\colon \prod_{p\in\pi} \mathrm{Sym}(U_p) \to \mathrm{Sym}(U)$ be the embedding given by

$$\tau\colon (\sigma_p)_{p\in\pi} \mapsto \sigma \quad \text{where} \quad \sigma\colon (u_p)_{p\in\pi} \mapsto (u_p\sigma_p)_{p\in\pi}.$$

Then the diagram

$$\begin{array}{ccc} U & \xrightarrow{\ \rho\ } & \mathrm{Sym}(U) \\ & {\scriptstyle\hat{\rho}}\searrow \quad \nearrow{\scriptstyle\tau} & \\ & \prod_{p\in\pi} \mathrm{Sym}(U_p) & \end{array}$$

commutes, and the preceding observations show that every maximal nilpotent subgroup of $\mathrm{Sym}(U)$ containing $U\rho$ lies in the image of τ. Therefore, it suffices to prove the Corollary for $\hat{\rho}\colon U \to \prod_{p\in\pi} \mathrm{Sym}(U_p)$ in place of $\rho\colon U \to \mathrm{Sym}(U)$. But this is easily accomplished by applications of the Theorem to the right regular representations $\rho_p\colon U_p \to \mathrm{Sym}(U_p)$, since the p-component of every maximal nilpotent subgroup of $\prod_{p\in\pi} \mathrm{Sym}(U_p)$ containing $U\hat{\rho}$ is a Sylow p-subgroup of $\mathrm{Sym}(U_p)$ (see [4, page 61, Lemma 25]).

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