Glasgow Math. J. **61** (2019) 487–500. © Glasgow Mathematical Journal Trust 2018.
doi:10.1017/S0017089518000319.

EXTREME POINTS FOR COMBINATORIAL BANACH SPACES

KEVIN BEANLAND, NOAH DUNCAN, MICHAEL HOLT
and JAMES QUIGLEY

Department of Mathematics, Washington and Lee University, Lexington, VA 24450.
e-mail: beanlandk@wlu.edu, naduncan16@gmail.com, holtm11493@gmail.com,
jamesbquigley93@gmail.com

(Received 13 April 2017; revised 26 February 2018; accepted 10 April 2018; first published online 27 July 2018)

Abstract. A norm $\|\cdot\|$ on c_{00} is called combinatorial if there is a regular family of finite subsets $\mathcal{F}$, so that $\|x\| = \sup_{F\in\mathcal{F}} \sum_{i\in F} |x(i)|$. We prove the set of extreme points of the ball of a combinatorial Banach space is countable. This extends a theorem of Shura and Trautman. The second contribution of this article is to exhibit many new examples of extreme points for the unit ball of dual Tsirelson's original space and give an explicit construction of an uncountable collection of extreme points of the ball of Tsirelson's original space. We also prove some stability properties of the intermediate norms used to define Tsirelson's space and give a lower bound of the stabilization function for these intermediate norms.

2010 *Mathematics Subject Classification.* 48B20.

1. Introduction. In this paper we consider problems related to the cardinalities of the set of extreme points of certain Banach spaces. The first type of Banach space we consider are *combinatorial* Banach spaces. A norm $\|\cdot\|$ on the vector space of all finitely supported scalar sequences c_{00} is called combinatorial if there is a regular (i.e., compact, spreading and hereditary) family of finite subsets $\mathcal{F}$, so that for $x \in c_{00}$

$$\|x\|_{\mathcal{F}} = \sup_{F\in\mathcal{F}} \sum_{i\in F} |x(i)|.$$

The Banach space $X_{\mathcal{F}}$ is the completion of c_{00} with respect to the above norm. A basic example of a regular family of finite subsets is the Schreier family $\mathcal{S}_1 = \{F \subset \mathbb{N} : |F| \leqslant \min F\}$. The space $X_{\mathcal{S}_1}$ is the famous space of J. Schreier. Let X be a Banach space and $B(X)$ unit ball. We denote by $E(X)$ the set of all extreme points of the $B(X)$. Recall that $x \in E(X)$, if $x \in B(X)$ and whenever $x = 1/2(y+z)$ for $y, z \in B(X)$, we have $x = y = z$. Shura and Trautman [**9**] proved that $E(X_{\mathcal{S}_1})$ is countably infinite. The first main result of the current paper is that whenever X is a combinatorial Banach space, $E(X)$ is countable.

Our second set of results concerns extreme points of Tsirelson's original space [**10**], which, in this paper, we denote by T^*. Recall that Tsirelson's space [**10**] was the first example of a reflexive space not containing any ℓ_p for $1 < p < \infty$. The norm is defined using an inductive procedure and satisfies an implicit formula. By a result of Lindenstrauss and Phelps [**8**], since Tsirelson's space T^* is reflexive, $E(T^*)$ and

Noah Duncan, Michael Holt and James Quigley were undergraduate students at Washington and Lee University when the main results of this paper were obtained.

$E(T)$ are uncountable. The proof of this statement is non-constructive as it uses the Baire Category Theorem. In [**6**], Casazza and Shura exhibit a countable collection of elements of $E(T)$. In the final section of this paper, we given many more new examples of elements of $E(T)$ and give an explicit construction of uncountably many elements of $E(T^*)$. Our final set of results deal with the stabilisation of the intermediate norms used to define the norm of T. First, we verify a conjecture in [**6**] by showing that these intermediate norms can stabilise for arbitrarily long periods before achieving the norm of a vector. After this we establish a lower bound on the quantity $j(n)$, which is the minimum integer value so that if x has a maximum support of n, then $\|x\|_{j(n)} = \|x\|_T$. Here. $(\|\cdot\|_n)_{n=1}^{\infty}$ are the intermediate norms and $\|x\|_T = \max \|x\|_n$ is norm of the dual of Tsirelson's space.

2. Extreme points for combinatorial spaces. Let X be a Banach space and $B(X)$, $S(X)$ and $E(X)$ denote the unit ball of X, the unit sphere of X and the set of extreme point of $B(X)$, respectively.

A collection $\mathcal{F}$ of finite subsets of $\mathbb{N}$ is called a regular family if it is hereditary ($F \in \mathcal{F}$ and $G \subset F$ implies $G \in \mathcal{F}$), spreading ($\{\ell_1 < \ell_2 \cdots < \ell_n\} \in \mathcal{F}$ and $\ell_i \leqslant k_i$ implies $\{k_1, \ldots, k_n\} \in \mathcal{F}$) and compact as a subset of $\{0, 1\}^{\mathbb{N}}$ (with the natural identification of $\mathcal{P}(\mathbb{N})$ with $\{0, 1\}^{\mathbb{N}}$). We will also assume throughout that $\{1\} \in \mathcal{F}$ for every regular family $\mathcal{F}$. Some examples of regular families of finite subsets include the small families $\mathcal{F}_n = \{F : |F| \leqslant n\}$ and the Schreier families defined as follows. Let $\mathcal{S}_0 = \mathcal{F}_1$. Supposing that $\mathcal{S}_\alpha$ has been defined for some ordinal $\alpha < \omega_1$, we define

$$\mathcal{S}_{\alpha+1} = \{\cup_{i=1}^{n} E_i : n \leqslant E_1 < E_2 < \cdots < E_n \text{ are in } \mathcal{S}_\alpha\}.$$

In the above, we write $E < F$ if $\max E < \min F$ and use the convention that $\emptyset < E < \emptyset$ for each E. If α is a limit ordinal, we fix $\alpha_n \nearrow \alpha$ and define:

$$\mathcal{S}_\alpha = \{F : \exists\, n \leqslant F \in \mathcal{S}_{\alpha_n}\}.$$

For each $\alpha < \omega_1$ the set $\mathcal{S}_\alpha$ is a regular family. These families were introduced in [**1**] and have been extensively used and studied.

For each regular family $\mathcal{F}$, we define the Banach space $X_{\mathcal{F}}$ as the completion of c_{00} with respect to the following norm:

$$\|x\|_{\mathcal{F}} = \sup_{F \in \mathcal{F}} \sum_{i \in F} |x(i)|.$$

The space $X_{\mathcal{S}_1}$ is called Schreier space and is an important example in Banach space theory. The unit vector basis is an example of a weakly null sequence having no Cesaro summable subsequence. Shura and Trautman [**9**] showed that $E(X_{\mathcal{S}_1})$ is countably infinite and that, moreover, $X_{\mathcal{S}_1}$ has the λ-property (and therefore every element in $B(X_{\mathcal{S}_1})$ is a σ-convex combination of extreme points), defined in [**3**]. An interesting unsolved problem is whether $X_{\mathcal{S}_1}$ has the uniform λ-property. In [**4**], Beanland and Chu studied these properties for combinatorial Banach spaces. For each $\alpha < \omega_1$, the spaces $X_{\mathcal{S}_\alpha}$ have also been extensively studied [**7**] and shown to be c_0 saturated. Indeed this follows from the fact that for each regular family $\mathcal{F}$, there this a countable compact set K so that $X_{\mathcal{F}}$ embedds isometrically into $C(K)$. The K is chosen as some ordinal interval $[0, \alpha]$, where α is related to the Cantor–Bendixon index of $\mathcal{F}$. Let us note that $X_{\mathcal{F}_1}$ is isometric to c_0 and so $E(X_{\mathcal{F}_1}) = \emptyset$. For $n \in \mathbb{N}$, $X_{\mathcal{F}_n}$ is easily seen to be isomorphic to c_0.

https://doi.org/10.1017/S0017089518000319 Published online by Cambridge University Press

One of our main results is the following theorem.

THEOREM 2.1. *If $\mathcal{F}$ is a regular family, then $E(X_{\mathcal{F}})$ is countable.*

In 'most' cases, we have that $E(X_{\mathcal{F}})$ is countably infinite. Indeed, in the case the $\mathcal{F}$ contains a two element subset, it is easy to see that there are infinitely many extreme points of the type $e_n + \sum_{i=1}^{n-1} a_i e_i$ for all sufficiently large n.

For $x \in S(X_{\mathcal{F}})$, we let

$$\mathcal{F}_x^1 = \{A \in \mathcal{F} : 1 = \sum_{i \in A} |x(i)| \text{ and for all } B \subsetneq A, \ \sum_{i \in B} |x(i)| < 1\}$$

Note that if $k \in F \in \mathcal{F}_x^1$, then $x(k) \neq 0$. We call the collection $\mathcal{F}_x^1$ the 1-sets of x. Let $\mathcal{M}_x \subset \mathcal{F}$ be the sets in $\mathcal{F} \setminus \mathcal{F}_x^1$ that do not contain any 1-set. We note that for a set $A \notin \mathcal{F}_x^1$, we may have $\sum_{i \in A} |x(i)| = 1$. However, in this case, there will be some $i \in A$ with $x(i) = 0$.

We collect several simple lemmas before proceeding to the proof.

LEMMA 2.2. *Let $\mathcal{F}$ be regular family and $(A_i)_{i=1}^{\infty} \subset \mathcal{F}$. Then, there is a subsequence (A_i'), so that $A = \cap_{i=1}^{\infty} A_i' \in \mathcal{F}$ and $A_i' = A \cup B_i$ (disjoint union) with* $\min B_i \to \infty$.

Proof. Let $(A_i) \subset \mathcal{F}$. Since $\mathcal{F}$ is compact, we can find a subsequence (A_i') that converges to some $A \in \mathcal{F}$. Since A is finite, for sufficiently large i we have that $A \subset A_i'$. Thus, we relabel and assume this holds for all i. Let $B_i = A_i' \setminus A$. Note that for each finite set C with $C \cap A = \emptyset$ there is an I so that for $i \geqslant I$, $C \cap A_i' = \emptyset$. This implies that $\min B_i \to \infty$ as desired. □

REMARK 2.1. From the above lemma we observe that whenever $(A_i) \subset \mathcal{F}$, there is a subsequence A_i', so that for all $x \in X_{\mathcal{F}}$, $\lim_{i\to\infty} \sum_{j \in A_i'} |x(j)| = \sum_{j \in A} |x(j)|$. Indeed, using the previous lemma, we have a subsequence and a disjoint decomposition $A_i' = A \cup B_i$ with $\min B_i \to \infty$. Observe that $\lim_{i\to\infty} \sum_{j \in B_i} |x(j)| = 0$ since $\min B_i \to \infty$.

REMARK 2.2. For each sequence $(A_i) \subset \mathcal{F}$, there is convergent subsequence (A_i') and $A \subset A_i'$ so that for each $i_0 \in \mathbb{N}$, so that

$$\lim_{i\to\infty} \sum_{j \in A_i'} |x(j)| = \sum_{j \in A} |x(j)| \leqslant \sum_{j \in A_{i_0}'} |x(j)|.$$

LEMMA 2.3. *Let $x \in X_{\mathcal{F}}$. The following equivalent statements hold:*

1. *There is no infinite sequence $(A_i) \subset \mathcal{F}$ satisfying*

$$\sup_{i \in \mathbb{N}} \sum_{j \in A_i} |x(j)| = 1 \text{ and } \sum_{j \in A_i} |x(j)| < 1, \forall i \in \mathbb{N}.$$

2. *For each infinite sequence $(A_i) \subset \mathcal{F}$, we have*

$$\sum_{j \in A_i} |x(j)| < 1, \ \forall i \in \mathbb{N} \implies \sup_{i \in \mathbb{N}} \sum_{j \in A_i} |x(j)| < 1.$$

https://doi.org/10.1017/S0017089518000319 Published online by Cambridge University Press

Proof. The statements are clearly equivalent. If such a sequence existed, then by the Remark 2.2 there is a subsequence (A_i') with

$$1 = \lim_{i\to\infty} \sum_{j\in A_i'} |x(j)| = \sum_{j\in A} |x(j)| < 1.$$

□

LEMMA 2.4. *For each $x \in S(X_{\mathcal{F}})$, the set $\mathcal{F}_x^1$ is non-empty and finite.*

Proof. If $\mathcal{F}_x^1$ was empty, there is no $A \in \mathcal{F}$ with $\sum_{i\in A} |x(i)| = 1$; however, since $\sup\{\sum_{i\in A} |x(i)| : A \in \mathcal{F}\} = 1$, there is necessarily a sequence contradicting Lemma 2.3. Therefore, $\mathcal{F}_x^1$ is non-empty.

We will show that $\mathcal{F}_x^1$ is finite. Let $\{A_i : i \in \mathbb{N}\} = \mathcal{F}_x^1$. Pass to an arbitrary subsequence. By Remark 2.2, find a further subsequence so that for any $i_0 \in \mathbb{N}$, we have

$$1 = \lim_{i\to\infty} \sum_{j\in A_i'} |x(j)| = \sum_{j\in A} |x(j)| \leqslant \sum_{j\in A_{i_0}'} |x(j)| = 1.$$

Since $x(j) \neq 0$ for $j \in A_i'$, $A_i' = A$ for all i. We have proved that every subsequence of (A_i) has a further subsequence that is constant. Therefore, $\{A_i : i \in \mathbb{N}\} = \mathcal{F}_x^1$ is finite. □

The next lemma is another restatement of Lemma 2.3.

LEMMA 2.5. *For each $x \in S(X)$, there is an $\varepsilon_x > 0$ (called the ε-gap for x), so that*

$$\sup_{A\in\mathcal{M}_x} \sum_{i\in A} |x(i)| \leqslant 1 - \varepsilon_x.$$

LEMMA 2.6. *If $x \in E(X_{\mathcal{F}})$, then $x \in c_{00}$.*

Proof. Let $x \in E(X_{\mathcal{F}})$ with infinite support and find $\varepsilon_x > 0$ such that

$$\sup_{A\in\mathcal{M}_x} \sum_{i\in A} |x(i)| \leqslant 1 - \varepsilon_x.$$

As the support of x is infinite, each $A \in \mathcal{F}_x^1$ is maximal in $\mathcal{F}$, that is, $\mathcal{M}_x = \mathcal{F} \setminus \mathcal{F}_x^1$. This means that if $\sum_{i\in A} |x(i)| = 1$, then $A \in \mathcal{F}_x^1$. Since there are finitely many such elements, we can find $k \notin \cup\mathcal{F}_x^1$. Consider the vectors $y = x + \varepsilon_x/2e_k$ and $z = x - \varepsilon_x/2e_k$. Then, clearly $x = 1/2(y + z)$ and $\|z\| \leqslant 1$. To see that $\|y\| \leqslant 1$, we fix $A \in \mathcal{F}$. If $A \in \mathcal{F}_x^1$, then $\sum_{i\in A} |y(i)| = \sum_{i\in A} |x(i)| = 1$. Alternatively, $A \notin \mathcal{F}_x^1$ and in that case if $k \in A$,

$$\sum_{i\in A} |y(i)| \leqslant \varepsilon_x/2 + \sum_{i\in A} |x(i)| \leqslant 1 - \varepsilon_x/2.$$

Therefore, $x \notin E(X_{\mathcal{F}})$. □

Proof of Theorem 2.1. Let X_n the span of $\{e_1, \ldots, e_n\}$ in $X_{\mathcal{F}}$. Since $E(X_{\mathcal{F}}) \subset \cup_{n=1}^{\infty} E(X_n)$ is suffices to prove that $E(X_n)$ is finite for each n. Our original proof mimicked the corresponding proof from [**9**]. The following, much simpler proof was pointed out by the referee. Note that for each n, there is an $N > n$, so that X_n embedds

https://doi.org/10.1017/S0017089518000319 Published online by Cambridge University Press

isometrically into ℓ_∞^N. Any such subspace has a polyhedral unit ball and, as such, finitely many extreme points. $\square$

3. Extreme points for Tsirelson's space. Let T denote the dual of Tsirelson's space [**10**]. Let (e_i) and (e_i^*) both denote the standard unit vectors in c_{00}. For $E \subset \mathbb{N}$ and $x = \sum_{i=1}^\infty a_i e_i \in c_{00}$ let $Ex = \sum_{i \in E} a_i e_i$. If E, F are subsets of $\mathbb{N}$, we write $E < F$, if $\max E < \min F$. If $\sum_{i=1}^\infty a_i e_i \in c_{00}$, then $\operatorname{supp} x = \{i : a_i \neq 0\}$. We define the set of norming functionals W_T as the union of the following subsets of c_{00}. A sequence $(f_i)_{i=1}^d \subset c_{00}$ is called admissible, if

$$d \leqslant \operatorname{supp} f_1 < \operatorname{supp} f_2 < \cdots < \operatorname{supp} f_d.$$

Let $W_0 = \{\pm e_i^* : i \in \mathbb{N}\}$ and for $k \geqslant 0$ let

$$W_{k+1} = W_k \cup \left\{ \frac{1}{2} \sum_{i=1}^d Ef_i : d \in \mathbb{N},\ (f_i)_{i=1}^d \subset W_k \text{ is admissible, } E \subset \mathbb{N} \right\}.$$

Then, $W_T = \cup_{k=1}^\infty W_k$.

The intermediate norms are defined by $\|x\|_n = \sup\{f(x) : f \in W_n\}$. Here, $f(x)$ is the usual inner product of f with x. Tsirelson's norm is defined by

$$\|x\|_T = \max_n \|x\|_n = \sup\{f(x) : x \in W_T\}.$$

The space T is the completion of c_{00} with respect to the norm $\|\cdot\|_T$ and (e_i) is a 1-unconditional basis for T.

In [**6**], it is shown that for $2 \leqslant i < j$ and $\varepsilon_1, \varepsilon_2, \varepsilon_i, \varepsilon_j \in \{\pm 1\}$, the vectors $\varepsilon_1 e_1 + \varepsilon_2 e_2 + \varepsilon_i e_i + \varepsilon_j e_j \in E(T)$. Let $E(T)^+$ be the extreme points with positive coefficients. To make the future estimates simpler, we note the following lemma that follows from the fact that the 'sign-changing' operator on a space with an 1-unconditional basis is a surjective isometry, and, as such, maps extreme points to extreme points.

LEMMA 3.1. *Suppose X has a 1-unconditional Schauder basis $(e_i)_{i=1}^\infty$. If $x = \sum_{i=1}^\infty x(i)e_i \in X$ and $\sum_{i=1}^\infty |x(i)|e_i \in E(X)$, then $x \in E(X)$.*

3.1. The set $E(T)$. The next proposition is proved in [**6**], for completeness, we include a different proof here.

PROPOSITION 3.2. *If $x \in E(T)^+$ the $x(1) = x(2) = 1$.*

Proof. Suppose $x \in E(T)^+$. It is clear that $x(1) = 1$; otherwise, we can perturb the first coordinate by an $\varepsilon > 0$, so that $x(1) + \varepsilon < 1$ and write x as the convex combination of vectors whose first coordinates are $x(1) + \varepsilon$ and $x(1) - \varepsilon$, respectively. Suppose $x(2) < 1$. Now, slightly perturb $x(2)$ to make a vector y with $y(2) = x(2) + \varepsilon$ and which agrees with x on all coordinates except 2. Since $x \in E(T)$, we can assume that $\|y\| > 1$. Find $\delta > 0$ with $\|y\| - 2\delta > 1$ and find $f \in W_T$ with $f(y) > \|y\| - \delta$ and $\min \operatorname{supp}(f) = 2$. By definition of $f \in W_T$, there are $f_1 < f_2$ in W_T with $f = 1/2(f_1 + f_2)$. However, since $f(y) > \|y\| - \delta$, we know that $f_2(y) > \|y\| - 2\delta > 1$. But, $f_2(y) = f_2(x) \leqslant \|x\| = 1$. This contradiction yields the result. $\square$

https://doi.org/10.1017/S0017089518000319 Published online by Cambridge University Press

The next remark says that if a vector x is not an extreme point, we can find two distinct norm-one vectors that are as close as we wish, so that x is the midpoint of the two vectors.

REMARK 3.1. Let X be a Banach space. Let $x \in S(X) \setminus E(X)$ and $\varepsilon > 0$. There are $y, z \in S(X)$ not equal to x so that $x = 1/2(y + z)$ and $\max\{\|x - y\|, \|x - z\|\} < \varepsilon$.

One of our main results concerning extreme points of T is the following theorem that gives many new examples of elements of $E(T)$.

THEOREM 3.3. *For each $n \in \mathbb{N}$ and $F \subset \mathbb{N}$ with $\min F = n$ and $|F| = n + 1$, there are coefficients $(a_3, a_4, \ldots, a_{n-1})$, so that*

$$e_1 + e_2 + \sum_{i=3}^{n-1} a_i e_i + \frac{2}{n} \sum_{i \in F} e_i \in E(T).$$

Moreover, by Lemma 3.1 we can put any signs in front of the coordinates.

EXAMPLE 3.1. The proof of the above theorem shows us how to explicitly compute extreme points for any n. However, for even moderately sized n, the computations are cumbersome. Some examples for small values of n are the following:

1. $e_1 + e_2 + \frac{1}{2} \sum_{i=3}^{8} e_i$.
2. $e_1 + e_2 + \frac{3}{5} e_3 + \frac{2}{5} \sum_{i=4}^{10} e_i$.
3. $e_1 + e_2 + \frac{1}{2} e_3 + \frac{1}{3} \sum_{i=4}^{12} e_i$.

LEMMA 3.4. *Let $n \in \mathbb{N}$ and $F \subset \mathbb{N}$ with $\min F = n$ and $|F| = n + 1$. Suppose $x \in c_{00}$ and $x(k) = 2/n$ for $k \in F$ and $x(k) = 0$ for $k \geqslant n$ and $k \notin F$. If $y, z \in B(X)$ with $x = 1/2(y + z)$, then $x(k) = y(k) = z(k)$ for $k \geqslant n$.*

Proof. Let $0 \leqslant \delta_i < 2/n$ and $|\varepsilon_i| = 1$ so that $y(i) = x(i) + \varepsilon_i \delta_i$ and $z(i) = x(i) - \varepsilon_i \delta_i$. There are $n + 1$ many size n subsets of F and for each such set G, since $G \in \mathcal{S}_1$ and $0 \leqslant \delta_i < 2/n$, we know that $\sum_{i \in G} y(i) = 1 = \sum_{i \in G} z(i)$. Therefore, there are $n + 1$ equations of the form $\sum_{i \in G} \varepsilon_i \delta_i = 0$. The corresponding system of equations has only the trivial solution; that is, $\delta_i = 0$ for each $i \in F$.

If $k \geqslant n$ and $k \notin F$, then $G = \{k\} \cup (F \setminus \{n\})$ is admissible, and so, if $y(k) = x(k) + \delta_k$, the fact that $\sum_{i \in G} y(i) = 1$ implies $\delta_k = 0$. This proves the claim. □

Proof of Theorem 3.3. It suffices to construct the sequence $(a_3, a_4, \ldots, a_{n-1})$. We do so recursively starting at a_{n-1}. Let

$$a_{n-1} = \sup\{a : \forall f \in W_T, \min \operatorname{supp}(f) \leqslant n - 1,\ f(a e_{n-1} + \frac{2}{n} \sum_{i \in F} e_i) \leqslant 1\}.$$

Let $x_{n-1} = a_{n-1} e_{n-1} + \frac{2}{n} \sum_{i \in F} e_i$. It follows from the definition that $\|x_{n-1}\| \leqslant 1$. Suppose $x_{n-1} = 1/2(y + z)$. We claim that $y(k) = z(k) = x_{n-1}(k)$ for all $k \geqslant n - 1$. By Lemma 3.4, it suffice to show that $y(n - 1) = z(n - 1) = a_{n-1}$. This is clear by definition. Indeed if $y(n - 1) = a_{n-1} + \delta_{n-1}$, then there is an $f \in W_T$ with $\min \operatorname{supp}(f) = n - 1$ so that

$$\|y\| \geqslant f(y) = f((a_{n-1} + \delta_{n-1}) e_{n-1} + \frac{2}{n} \sum_{i \in F} e_i) > 1.$$

Therefore, $\delta_{n-1} = 0$ which is the desired result. Fix m with $3 \leqslant m < n-1$ and assume a_k has been defined for each $m < k \leqslant n-1$. Let

$$a_m = \sup\{a : \forall f \in W_T, \min \operatorname{supp}(f) \leqslant m,\ f(ae_m + \sum_{i=m+1}^{n-1} a_i e_i + \frac{2}{n}\sum_{i\in F} e_i) \leqslant 1\}.$$

The same argument as before can be applied. The result follows. □

3.2. The set $E(T^*)$**.** Recall that $B(T^*)$ is the point-wise closure of the closed convex hull, $co(W_T)$, of W_T. Let $\overline{W_T}^p$ be the point-wise closure of W_T.

DEFINITION 3.5. Let $n, k \in \mathbb{N}$ and $I_n = [2^{n+1}, 2^{n+2} - 2]$,

$$\phi_n = \frac{1}{2}\sum_{i\in I_n} e_i^* \in W_1 \text{ and } g_k = \sum_{n=k}^{\infty} \frac{1}{2^{n-k}}\phi_n \in B(T^*).$$

Note that $|I_n| = 2^{n+1} - 1$ and so I_n is one element short of being a maximal $\mathcal{S}_1$ set. From this we see that, in particular, $\phi_1 + 1/2\phi_2 \in W_2$. For example, the functional g_1 is simply the point-wise limit of the functionals

$$\phi_1 + \frac{1}{2}\phi_2 + \frac{1}{4}\phi_3 + \cdots + \frac{1}{2^{n-1}}\phi_n,$$

that is, $g_k \in \overline{W_T}^p$ for each $k \in \mathbb{N}$. The main result of this subsection is to prove the following.

THEOREM 3.6. *Let* $k \in \mathbb{N}$. *Then* $g_k \in E(T^*)$. *Moreover this implies that* $E(T^*)$ *is uncountable.*

While the proof that $g_k \in E(T^*)$ is technical, the idea is pretty straightforward. Indeed, we show that g_k is the unique norming functional of a vector called z_k. It is easy to see that among the elements of $g \in \overline{W_T}^p$ not equal to g_k, $g(z_k) < g_k(z_k)$. The technical complication arises when we have to prove that the same for an arbitrary elements of $B(T^*)$.

REMARK 3.2. Let $\ell < k$ in $\mathbb{N}$. Then,

$$g_\ell = \sum_{i=\ell}^{k-1} \frac{1}{2^{i-\ell}}\phi_i + \frac{1}{2^{k-\ell}} g_k.$$

Below we define the vectors z_k. Note that for each k, there are many vectors that are uniquely normed by g_k.

DEFINITION 3.7. Let $m_i \nearrow \infty$ be positive integers so that for each $n \in \mathbb{N}$, the following is satisfied:

$$\sum_{i=n+1}^{\infty} \frac{1}{m_i} < \frac{1}{m_n 2^{n+2}}. \tag{1}$$

For each $k \in \mathbb{N}$, let

$$z_k = \sum_{i=k}^{\infty} x_i \text{ where } x_i = \frac{1}{m_i 2^{i+1}} \sum_{j \in I_i} e_j.$$

REMARK 3.3. For each $k \in \mathbb{N}$ we have

$$\|z_k\|_{\ell_1} = \sum_{i=k}^{\infty} \frac{1}{m_i}\left(1 - \frac{1}{2^{i+1}}\right) \text{ and } g_k(z_k) = \sum_{n=k}^{\infty} \frac{1}{m_n 2^{n-k+1}}\left(1 - \frac{1}{2^{n+1}}\right). \tag{2}$$

Let us, roughly, explain why g_k uniquely norms z_k. First, note that the supports are the equal. The vector z_k can be understood as a 'cascade' of flat vectors whose coefficients are chosen inductively to rapidly decrease in modulus. In addition, the coefficients of g_k are maximal in the following sense: for each $n \in \mathbb{N}$, $1/2^n$ is a coefficient and for each $n \in \mathbb{N}$, the coefficient $1/2^n$ appears the maximum number of times. Therefore, if $g \in W_T \setminus W_0$ and not equal to g_k, the first non-zero coordinate of g_k that g disagrees with must be larger than the corresponding coefficient of g or else g is identically 0 after that coefficient. This smaller coefficient for g enables larger coefficients that must occur further into the support. However, since the coefficients in subsequent blocks of the vector z_k are chosen to be very small depending of previous blocks, the larger coefficient later on cannot make up for an earlier loss. This allows us to conclude that $g(z_k) < g_k(z_k)$ for $g \in \overline{W_T}^p$ not equal to g_k. The technical step of proving that g_k is the unique norming functional when considering elements in the point-wise closure of the convex hull of W_T is dealt with in Lemmas 3.11 and 3.12.

As a last remark before the proof, we would like to point out that it would be convenient to decompose each $f \in B(T^*)$ as an infinite convex combination of elements in $\overline{W_T}^p$. However, we were not able to show that elements of $B(T^*)$ have this kind of decomposition. Indeed, whether such a decomposition holds is an open question to us.

REMARK 3.4. Let $h \in \overline{W_T}^p$ with $d = \min \operatorname{supp} h$. Let $F = \{i : h(e_i) = 1/2\}$ and suppose $|F| = d - 1$. Then, every decomposition of h is of the form

$$h = \frac{1}{2}\Big(\sum_{i \in F} e_i^* + g_0\Big)$$

for some $g_0 \in \overline{W_T}^p$.

Proof. Fix $h \in \overline{W_T}^p$. Note that the statement is trivial for $h \in W_T$ and that, in this case, $g_0 \in W_T$. Let $E_N = [1, N]$. Then, $E_N h \in W_T$ for $N \in \mathbb{N}$. Thus, $E_N g_0 \in W_T$. Since $E_N g_0 \to g_0$ as $N \to \infty$ point-wise, we know that $g_0 \in \overline{W_T}^p$. □

LEMMA 3.8. *Let $\ell < k$ in $\mathbb{N}$. If $g \in \overline{W_T}^p$ with* $\min \operatorname{supp} g \geqslant 2^{k+1} - 1$ *and*

$$\sum_{i=\ell}^{k-1} \frac{1}{2^{i-\ell}} \phi_i + g \in \overline{W_T}^p,$$

then $2^{k-\ell} g \in \overline{W_T}^p$.

Proof. Fix ℓ and proceed by induction. The base case $k = \ell + 1$. In this case we assume $g \in \overline{W_T}^p$ and

$$h = \frac{1}{2}\sum_{j\in I_\ell} e_j^* + g \in \overline{W_T}^p.$$

By applying Remark 3.4, we conclude that

$$h = \frac{1}{2}(\sum_{i\in I_\ell} e_i^* + g_0)$$

Thus, $g_0 = 2g \in \overline{W_T}^p$ as desired.

Assume the statement holds for some $k \geqslant \ell + 1$ and we shall prove the statement for $k + 1$. Using our induction hypothesis find g so that

$$\sum_{i=\ell}^{k-1} \frac{1}{2^{i-\ell}}\phi_i + \frac{1}{2^{k-\ell}}\phi_k + g \in \overline{W_T}^p.$$

By our induction hypothesis,

$$h_k = \frac{1}{2}\sum_{i\in I_k} e_i^* + 2^{k-\ell}g \in \overline{W_T}^p.$$

Using Remark 3.4, we know that

$$h_k = \frac{1}{2}(\sum_{i\in I_k} e_i^* + g_0) \in \overline{W_T}^p.$$

Thus, $2^{k-\ell+1}g = g_0 \in \overline{W_T}^p$ as desired. □

The next remark is a consequence of the maximality of the coefficients of g_ℓ and can be proved by a simple counting argument.

REMARK 3.5. Let $\ell \in \mathbb{N}$ and $g \in \overline{W_T}^p$. Suppose that for $k \geqslant \ell$, there is an i with $g(e_i) \neq g_\ell(e_i)$ and $i < \min I_k$. Then, there is an $i_0 \in I_k$, so that $g(e_{i_0}) \neq g_\ell(e_{i_0})$.

REMARK 3.6. Fix $g \in W_T$ and $\ell \in \mathbb{N}$. Suppose there is a minimum $k \geqslant \ell$, so that $|g(e_i)| > g_\ell(e_i)$ for some $i \in I_k$, then, $g|_{[\min I_k - 1,\infty)} = \frac{1}{2^{k-\ell}} e_i^*$.

Proof. We use Lemma 3.8; since

$$g = \sum_{i=\ell}^{k-1} \frac{1}{2^{i-\ell}}\phi_i + g|_{[\min I_k-1,\infty)} \in W_T,$$

we have $2^{k-\ell}g|_{[\min I_k-1,\infty)} \in W_T$. By assumption,

$$|g(e_i)| > g_\ell(e_i) = \frac{1}{2^{k-\ell+1}}.$$

Therefore, $|g(e_i)| \geqslant \frac{1}{2^{k-\ell}}$ and $2^{k-\ell}g|_{[\min I_k-1,\infty)} \in W_T$. This implies the desired result. □

https://doi.org/10.1017/S0017089518000319 Published online by Cambridge University Press

LEMMA 3.9. *Let $k \in \mathbb{N}$ and $g \in W_T$. Suppose there is an $i_0 \in I_k$, so that $g(e_{i_0}) \neq g_k(e_{i_0})$. Then,*

$$|g(z_k)| \leqslant \frac{1}{2m_k}\left(1 - \frac{1}{2^{k+1}}\right). \tag{3}$$

Proof. Fix $k \in \mathbb{N}$, $g \in W_T$ and i_0 satisfying the assumptions. If $g(e_j) = 1$, then $g = \pm e^*_{i_0}$ and so $|g(z_k)| = 1/(m_k 2^{k+1})$. Therefore, we assume that $|g(e_{i_0})| \leqslant 1/4$. Thus, $|\{i \in I_k : |g(e_i)| = 1/2\}| \leqslant 2^{k+1} - 2$. Also, since $g \notin W_0$, $g(x) \leqslant 1/2\|x\|_{\ell_1}$ for each $x \in T$. These observations combined with (2) and (1) yield

$$\begin{aligned}
g(z_k) &= g\left(\frac{1}{m_k 2^{k+1}}\sum_{i\in I_k} e_i\right) + g(\sum_{n=k+1}^{\infty} x_n) \\
&< \frac{1}{2m_k}\left(\frac{2^{k+1}-2}{2^{k+1}}\right) + \frac{1}{4m_k}\left(\frac{1}{2^{k+1}}\right) + \frac{1}{2}\|\sum_{n=k+1}^{\infty} x_n\|_{\ell_1} \\
&\leqslant \frac{1}{2m_k}\left(1 - \frac{2}{2^{k+1}}\right) + \frac{1}{4m_k}\left(\frac{1}{2^{k+1}}\right) + \frac{1}{2}\sum_{i=k+1}^{\infty}\frac{1}{m_i}\left(1 - \frac{1}{2^{i+1}}\right) \\
&\overset{\text{by (1)}}{<} \frac{1}{2m_k}\left(1 - \frac{1}{2^{k+1}}\right).
\end{aligned}$$

□

LEMMA 3.10. *Let $\ell \in \mathbb{N}$ and $f \in W_T$ not equal to g_ℓ. Suppose that $k \geqslant \ell$ is minimum, so that there is an $i_0 \in I_k$ with $f(e_{i_0}) \neq g_\ell(e_{i_0})$. Then,*

$$f|_{[\min I_k - 1,\infty)}(z_\ell) < \frac{1}{2^{k-\ell+1}m_k}\left(1 - \frac{1}{2^{k+1}}\right).$$

Moreover, there is an $\delta_{\ell,k}$ so that for any f satisfying the above hypothesis $f(z_\ell) < g_\ell(z_\ell) - \delta_{\ell,k}$.

Proof. Note that by Remark 3.5, such a k exists. Considering our set-up, we have

$$f = \sum_{i=\ell}^{k-1}\frac{1}{2^{i-\ell}}\phi_i + f|_{[\min I_k - 1,\infty)}.$$

Applying Lemma 3.8, we know that $2^{k-\ell}f|_{[\min I_k-1,\infty)} \in W_T$. Let $f_0 = 2^{k-\ell}f|_{[\min I_k-1,\infty)}$. Then, $\frac{1}{2^{k-\ell}}f_0(e_j) \neq g_\ell(e_j) = \frac{1}{2^{k-\ell}}g_k(e_j)$. Therefore, we can apply Lemma 3 to conclude that $f_0(z_k) < 1/(2m_k)(1 - 1/2^{k+1})$. Combining these equations, we have

$$f|_{[\min I_k-1,\infty)}(z_\ell) = \frac{1}{2^{k-\ell}}f_0(z_k) < \frac{1}{2^{k-\ell}}\frac{1}{2m_k}(1 - 1/2^{k+1}).$$

This is the desired inequality. The 'moreover' statement follows from comparing the above with the quantity $g_\ell(z_\ell)$ from (2). □

LEMMA 3.11. *Let $\ell \in \mathbb{N}$ and $f \in B(T^*)$. Suppose there exists $k \in \mathbb{N}$ minimum with $k \geqslant \ell$ so that $|f(e_{i_0})| > g_\ell(e_{i_0}) + \eta$ for some $\eta > 0$ and $i_0 \in I_k$. Then,*

$$|f(\sum_{i \in I_k} e_i)| \leqslant \frac{|I_k|}{2^{k-\ell+1}} - \eta = g_\ell(\sum_{i \in I_k} e_i) - \eta.$$

Proof. Suppose first that $f \in W_T$. Using Remark 3.6, we know $f|_{[\min I_k - 1, \infty)} = \pm \frac{1}{2^{k-\ell}} e^*_{i_0}$ and the conclusion follows easily since $|I_k| \geqslant 4$ for all $k \in \mathbb{N}$.

Next, we consider the case $f \in co(W_T)$. Let $f = \sum_{j=1}^d \lambda_j f_j$ with $\sum_{j=1}^d \lambda_j = 1$ and $(f_j)_{j=1}^d \subset W_T$. Let $J \subset \{1, \dots, d\}$, so that for $j \in J$, $f_j = \pm \frac{1}{2^{k-\ell}} e^*_{i_0}$. Note that $J \neq \emptyset$. Let $\lambda := \sum_{j \in J} \lambda_j$. Using Remark 3.6, for $j \in J' := \{1, \dots, d\} \setminus J$, $|f_j(e_{i_0})| \leqslant 1/2^{k-\ell+1}$. Note the following easy inequality

$$\frac{1}{2^{k-\ell+1}} + \eta < |f(e_{i_0})| \leqslant \sum_{j \in J'} \lambda_j |f_j(e_{i_0})| + \sum_{j \in J} \lambda_j \frac{1}{2^{k-\ell}} \leqslant (1-\lambda)\frac{1}{2^{k-\ell+1}} + \lambda \frac{1}{2^{k-\ell}}.$$

Therefore, $\eta < 1/2^{k-\ell+1}\lambda$. This yields

$$|\sum_{j=1}^d \lambda_j f_j(\sum_{i \in I_k} e_i)| \leqslant (1-\lambda)\frac{|I_k|}{2^{k-\ell+1}} + \frac{1}{2^{k-\ell}}\lambda \leqslant \frac{|I_k|}{2^{k-\ell+1}} - \frac{\lambda}{2^{k-\ell+1}} < \frac{|I_k|}{2^{k-\ell+1}} - \eta.$$

The penultimate inequality uses the fact that $(|I_k|/2 - 1) > 1/2$. This finishes the case of $f \in co(W_T)$.

In the final case, we assume $f \in \overline{co(W_T)}^p = B(T^*)$. In this case, the result follows since f is the point-wise limit of a sequence in $co(W_T)$ for which the desired estimate holds. □

The next lemma provides the desired upper bound in the general case.

LEMMA 3.12. *Let $\ell \in \mathbb{N}$ and $f \in B(T^*)$ not equal to g_ℓ. Then, $f(z_\ell) < g_\ell(z_\ell)$.*

Proof. Fix $f \in B(T^*)$ not equal to g_ℓ. Then, using Remark 3.5, we can find $k \geqslant \ell$ minimum, so that there is some $i_0 \in I_k$ with $|f(e_{i_0})| \neq g_\ell(e_{i_0})$. First, we claim that there is an $\eta > 0$, so that

$$|f(\sum_{i \in I_k} e_i)| < g_\ell(\sum_{i \in I_k} e_i) - \eta.$$

Since $f(e_{i_0}) \neq g_\ell(e_{i_0})$ for some $i_0 \in I_k$, either there is an i_0, so that $|f(e_{i_0})| > 1/2^{k-\ell+1} + \eta$ for some $\eta > 0$, or $|f(\sum_{i \in I_k} e_i)| < |I_k|/2^{k-\ell+1}$. In this latter case we have the above claim trivially. In the former case the claim follows from Lemma 3.11. Consider the following claim.

Claim: If $\sum_{j=1}^d \lambda_j = 1$ and $(f_j)_{j=1}^d \subset W_T$, so that

$$|(f - \sum_{j=1}^d \lambda_j f_j) \sum_{i \in I_k} e_i)| < (1 - 1/2^{k-\ell+2})\eta.$$

Then, $\eta < \sum_{j \in B} \lambda_j$, where $B = \{j \in \{1 \dots d\} : \exists\, i \in I_k,\ f_j(e_i) \neq g_\ell(e_i)\}$.

We prove the above claim. Let $B' = \{1, \ldots, d\} \setminus B$. Also, by Remark 3.6, for each $j \in B$, there is an $i \in I_k$, so that either $|f_j(e_i)| = 1/2^{k-\ell}$ or $|f_j(e_i)| \leqslant 1/2^{k-\ell+2}$. Either way

$$|(g_\ell - f_j)(\sum_{i \in I_k} e_i)| \leqslant 1/2^{k-\ell+2}$$

for each $j \in B$. Using these observations

$$\begin{aligned}
\eta &\leqslant |(g_\ell - f)(\sum_{i \in I_k} e_i)| < \sum_{j=1}^{d} \lambda_j |(g_\ell - f_j)(\sum_{i \in I_k} e_i)| + (1 - 1/2^{k-\ell+2})\eta \\
&= \sum_{j \in B} \lambda_j |(g_\ell - f_j)(\sum_{i \in I_k} e_i)| + \sum_{j \in B'} \lambda_j |(g_\ell - f_j)(\sum_{i \in I_k} e_i)| + (1 - 1/2^{k-\ell+2})\eta \\
&= \sum_{j \in B} \lambda_j |(g_\ell - f_j)(\sum_{i \in I_k} e_i)| + (1 - 1/2^{k-\ell+2})\eta \leqslant \sum_{j \in B} \lambda_j \frac{1}{2^{k-\ell+2}} + (1 - 1/2^{k-\ell+2})\eta.
\end{aligned} \tag{4}$$

This finishes the proof of the claim.

Recall the definition of $\delta_{\ell,k}$ from Lemma 3.10. To finish the proof, find a convex combination $\sum_{j=1}^{d} \gamma_j = 1$ in and $(f_j)_{j=1}^{d} \subset W_T$ satisfying

$$|(f - \sum_{i=1}^{d} \gamma_i f_j)(\sum_{j \in I_k} e_j)| < \eta \delta_{\ell,k}.$$

Define B and B' as before. For $j \in B'$, we have the trivial estimate $|f_j(z_\ell)| \leqslant g_k(z_k)$ and for $j \in B$, we have $|f_j(z_\ell)| < g_\ell(z_\ell) - \delta_{\ell,k}$.

$$\begin{aligned}
|f(z_\ell)| &< \sum_{j=1}^{d} \gamma_j |f_j(z_\ell)| + \eta\delta_{\ell,k} = \sum_{i \in B} \gamma_j |f_j(z_\ell)| + \sum_{i \in B'} \gamma_j |f_j(z_\ell)| + \eta\delta_{\ell,k} \\
&\leqslant \sum_{j \in B} \gamma_j (g_\ell(z_\ell) - \delta_{\ell,k}) + \sum_{j \in B'} \gamma_j g_\ell(z_\ell) + \eta\delta_{\ell,k} \\
&< g_\ell(z_\ell) - \sum_{j \in B} \gamma_j \delta_k + \eta\delta_{\ell,k} \\
&< g_\ell(z_\ell) - \eta\delta_{\ell,k} + \eta\delta_{\ell,k} = g_\ell(z_\ell).
\end{aligned} \tag{5}$$

In the above, we use the estimate $\eta < \sum_{j \in B} \gamma_j$ from the claim at the beginning. This is the desired result. □

Proof of Theorem 3.6. By Lemma 3.12, g_k uniquely norms z_k for each k. This easily yields that g_k is in $E(T^*)$. Indeed if $g_k = 1/2(f + g)$ for $f, g \in B(T^*)$ and different from g_k, we have that $|1/2(f + g)(z_k)| < 1/2g_k(z_k) + 1/2g_k(z_k) = g_k(z_k)$. To see that $E(T^*)$ is uncountable, note that for any sequence of signs and any $k \in \mathbb{N}$, $(\varepsilon_i)_{i=1}^{\infty}$ so that $\sum_{i=1}^{\infty} \varepsilon_i g_k(e_i) e_i^* \in E(T^*)$ by Lemma 3.1. □

In their book on Tsirelson's space, Casazza and Shura conjectured [**6**, p. 170] that for each $m < n$, there is a vector x, so that

$$\|x\|_{m-1} < \|x\|_m = \|x\|_n < \|x\|_{n+1}. \tag{6}$$

https://doi.org/10.1017/S0017089518000319 Published online by Cambridge University Press

The last result of this paper is a proof of this fact. We need the following finite truncations of the vectors z_1 and g_1.

DEFINITION 3.13. For each $k \in \mathbb{N}$, define the following vector and functional:

$$y_k = \sum_{i=1}^{k} x_i + \frac{1}{m_k 2^{k+1}} e_{2^{k+2}-1},$$

$$h_k = \sum_{i=1}^{k} \frac{1}{2^{i-1}} \phi_i + \frac{1}{2^k} e_{2^{k+2}-1}.$$

Let $E_k := \operatorname{supp} y_k = \operatorname{supp} h_k$.

Note that $\phi_i(x_i) = \frac{1}{2m_i}(1 - \frac{1}{2^{i+1}})$. Therefore,

$$h_k(y_k) = \sum_{i=1}^{k-1} \frac{1}{2^i m_i}(1 - \frac{1}{2^{i+1}}) + \frac{1}{m_k 2^{k+1}}.$$

PROPOSITION 3.14. *Let $k \in \mathbb{N}$. Then, $h_k(y_k) = \|y_k\|$ and there is an $\varepsilon > 0$ so that if $f \in W_T$ with $f \neq h_k$, then $f(y_k) < \|y_k\| - \varepsilon$.*

The existence of the $\varepsilon > 0$ above follows from the fact $y_k \in c_{00}$ and the coefficients of elements of W_T lie in the set $\{\pm 2^{1-k} : k \in \mathbb{N}\}$.

PROPOSITION 3.15. *For each $m < n$ in $\mathbb{N}$, there is a vector x so that*

$$\|x\|_{m-1} < \|x\|_m = \|x\|_n < \|x\|_{n+1}. \tag{7}$$

Proof. Fix $m < n$ in $\mathbb{N}$. Using Proposition 3.14, find $\varepsilon > 0$ so that if $f \in W_T$ with $f \neq h_m$, then $f(y_m) < \|y_m\|_m - \varepsilon$. Note that $\|y_m\|_{m-1} < \|y_m\|_m = \|y_m\|$.

Using the repeated average hierarchy from [**2**, p. 17], there is a vector w_n, so that $\min \operatorname{supp} w_n > \max \operatorname{supp} y_m$ and so that if $f \in W_n$, then $f(w_n) < \varepsilon/2$, but $\|w_n\|_{n+1} \geqslant 1$. Let $x = y_m + w_n$.

Let $f \in W_n$ with $f \neq h_m$. Then, it follows that

$$f(x) = f(y_m + w_n) \leqslant \|y_m\|_m - \varepsilon/2 < h_m(x).$$

As $h_m \in W_m \setminus W_{m-1}$, this implies that $\|x\|_{m-1} < \|x\|_m = \|x\|_n$.

Also, note that $\|x\|_n = h_m(y_m) < 1$. Thus,

$$\|x\|_{n+1} \geqslant \|w_n\|_{n+1} \geqslant 1 > \|x\|_n.$$

This is the desired result. □

For our final result, we use the above vectors to establish a lower bound on a quantity $j(n)$ defined below an first defined in [**6**].

https://doi.org/10.1017/S0017089518000319 Published online by Cambridge University Press

DEFINITION 3.16. For n a positive integer, $j(n)$ is the smallest non-negative integer such that for all $x \in c_{00}$ with max supp $x \leqslant n$, we have

$$\|x\|_{j(n)} = \max_{m \in \mathbb{N}} \|x\|_m.$$

In [6] they state that $j(n) \leqslant \lfloor (n+1)/2 \rfloor$, admitting this is likely not a sharp upper bound for $j(n)$. In a recent work [5], Beanland et al. improved this upper bound by showing that $j(n)$ is $O(\sqrt{n})$. Here, we give a lower bound on $j(n)$.

THEOREM 3.17. *For each* $k \in \mathbb{N}$, $j(k) \geqslant \log_2(k+2) - 3$.

Proof. For each $n \in \mathbb{N}$, the vector y_n is uniquely normed and max supp $y_n = 2^{n+2} - 2$. Therefore, $j(2^{n+2}-2) \geqslant n$. Now, let $k \in \mathbb{N}$ and find $n \in \mathbb{N}$ with $2^{n+1} \leqslant k+2 < 2^{n+2}$. Then,

$$j(k) \geqslant j(2^{n+1}-2) \geqslant n-1 \geqslant \log_2(k+2) - 3.$$

This is the desired result. □

ACKNOWLEDGEMENTS. The authors would like to thank the referee for several improvements and simplifications to the paper. In particular, the current proofs of Theorem 2.1 and Lemma 3.1 are due to the referee. He/she also pointed out a missing case in our original proof of Theorem 3.6 and made several other useful comments. The authors would also like to acknowledge the support from Washington & Lee University Summer Research Scholars program.

REFERENCES

1. D. E. Alspach and S. A. Argyros, Complexity of weakly null sequences, *Dissertationes Math. (Rozprawy Mat.)*, **321**(44) (1992).

2. S. A. Argyros and A. Tolias, Methods in the theory of hereditarily indecomposable Banach spaces, *Mem. Amer. Math. Soc.* **170**(806) (2004), vi+114, 2004.

3. R. M. Aron and R. H. Lohman, A geometric function determined by extreme points of the unit ball of a normed space, *Pac. J. Math.* **127**(2) (1987), 209–231.

4. K. Beanland and H. Chu, The λ-property for combinatorial Banach spaces. Preprint.

5. K. Beanland, N. Duncan and M. Holt, Time stopping for Tsirelson's norm. *Involve* **11**(5) (2018), 857–866.

6. P. G. Casazza and T. J. Shura, *Tsirel'son's space*, Lecture notes in mathematics, vol. 1363 (Springer-Verlag, Berlin, 1989). With an appendix by J. Baker, O. Slotterbeck and R. Aron.

7. I. Gasparis and D. H. Leung, On the complemented subspaces of the Schreier spaces, *Stud. Math.* **141**(3) (2000), 273–300.

8. J. Lindenstrauss and R. R. Phelps, Extreme point properties of convex bodies in reflexive Banach spaces, *Isr. J. Math.* **6** (1968), 39–48.

9. T. J. Shura and D. Trautman, The λ-property in Schreier's space S and the Lorentz space $d(a, 1)$, *Glasg. Math. J.* **32**(3) (1990), 277–284.

10. B. S. Tsirelson, It is impossible to imbed ℓ_p of c_0 into an arbitrary Banach space, *Funkcional. Anal. Priložen* **8**(2) (1974), 57–60.