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SHARPLY TRANSFERABLE LATTICES

H. GASKILL, G. GRÄTZER, AND C. R. PLATT

1. Introduction and results. In a lecture in 1966 (see [**6**]), the second author considered briefly those first order properties which hold for a lattice $\mathscr{L}$ if and only if they hold for the lattice $I(\mathscr{L})$ of all ideals of $\mathscr{L}$. The best known examples of such properties are those given by identities. The well-known connection between the modular identity ϵ and the five-element nonmodular lattice $\mathscr{N}_5$ transforms the above result for ϵ into the following statement: $\mathscr{N}_5$ is a sublattice of a lattice $\mathscr{L}$ if and only if $\mathscr{N}_5$ is a sublattice of $I(\mathscr{L})$. Furthermore, it can be seen that for any embedding φ of $\mathscr{N}_5$ into $I(\mathscr{L})$, there is an embedding of $\mathscr{N}_5$ into $\mathscr{L}$ which "separates" the image of φ. Generalizing these properties of $\mathscr{N}_5$ we obtain the concepts of transferable and sharply transferable lattices.

1.1 *Definition.* A lattice $\mathscr{K}$ is called *transferable* if and only if, for every embedding φ of $\mathscr{K}$ into the lattice $I(\mathscr{L})$ of ideals of a lattice $\mathscr{L}$, there exists an embedding ψ of $\mathscr{K}$ into $\mathscr{L}$. If, in addition, ψ can always be chosen so as to satisfy the condition: for $x \in \mathscr{K}$,

$$\psi(x) \in \varphi(x), \text{ but if } y < x, \text{ then } \psi(x) \notin \varphi(y),$$

then $\mathscr{K}$ is called *sharply transferable.* (The terminology introduced here differs from that in previously published results. In [**3**] and [**4**], transferable and sharply transferable were called weakly transferable and transferable, respectively. We feel the present terminology is more descriptive and at the same time more appropriate, since the (present) notion of transferability is the more fundamental concept.)

The first general result was already found in 1966 (announced in [**6**]; see [**9**] for a proof): a transferable lattice contains no doubly reducible elements.

In his thesis [**4**], the first author defined similar concepts for semilattices and proved (see [**3**] and [**4**]) that a finite semilattice is sharply transferable if and only if it satisfies the condition (T) (defined in § 2). He then established (see [**4**]) that, if $\langle L; \wedge, \vee\rangle$ is a sharply transferable finite lattice, then both $\langle L; \vee\rangle$ and the dual of $\langle L; \wedge\rangle$ satisfy (T). Finally, he showed that $(T_\vee)$, $(T_\wedge)$, and (W) (defined in § 2) are jointly sufficient for sharp transferability of a finite lattice. The principal result of this paper is the following complete characterization.

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1.2 THEOREM. *A finite lattice is sharply transferable if and only if it satisfies the three conditions* $(T_\vee)$, $(T_\wedge)$, *and* (W).

Observing that $(T_\vee)$ is the dual of $(T_\wedge)$, and that (W) is self-dual, we obtain

1.3 COROLLARY. *If a finite lattice* $\mathscr{L}$ *is sharply transferable, so is the dual of* $\mathscr{L}$.

The sufficiency of the three conditions of 1.2 is established in § 2 below, and the proof follows rather closely that given in [**4**]. In § 3, we give new and simplified proofs of the necessity of $(T_\vee)$ and $(T_\wedge)$. Finally, in § 4 we prove the necessity of (W).

There are a number of recent related results that should be mentioned here. Generalizing the above-mentioned relation between $\mathscr{N}_5$ and the modular identity ϵ, R. McKenzie [**12**] introduced the important concepts of splitting lattices and splitting identities, and characterized splitting lattices. In establishing this theorem, he proved that a finite lattice is embeddable into a free lattice if and only if it satisfies (W) and is a bounded homomorphic image of a free lattice. Utilizing McKenzie's ideas and the present characterization theorem, H. Gaskill and C. R. Platt [**5**] proved that the class of finite sharply transferable lattices coincides with the class of finite sublattices of free lattices. This has the unexpected corollary that a sublattice of a finite sharply transferable lattice is sharply transferable.

R. Freese has communicated to us his unpublished result that a finite lattice satisfies (W) if and only if it is a retract of the lattice of ideals of the lattice of dual ideals of some finitely generated free lattice, $I(D(F(n)))$ (here the dual ideals are ordered by reverse inclusion). This, together with 1.2 and 1.3 above, yields another proof of one direction of the result of Gaskill and Platt.

Finally, we mention the paper of K. Baker and A. Hales [**1**] in which they give a number of interesting results concerning first order properties of the sort mentioned in the first paragraph of this paper.

In conclusion, we summarize in the form of a theorem what we know about transferability.

1.4 THEOREM. *Let* $\mathscr{L}$ *be a finite lattice.*

(i) *If* $\mathscr{L}$ *is transferable, then* $\mathscr{L}$ *has no doubly reducible elements.*

(ii) *If* $\mathscr{L}$ *is transferable, then* $\mathscr{L}$ *satisfies the semi-distributive laws* $(SD_\wedge)$ *and* $(SD_\vee)$.

(iii) *If* $\mathscr{L}$ *is transferable, then* $\mathscr{L}$ *can be embedded into a finite partition lattice.*

(iv) *If* $\mathscr{L}$ *satisfies* (W), *and if* $\mathscr{L}$ *and its dual are transferable, then* $\mathscr{L}$ *is embeddable into a free lattice.*

Statement (i) is proved in [**9**], (ii) in § 3 below, and (iii) in [**1**] and [**8**]. Statement (iv) follows from R. Freese's result quoted above.

The notation used in this paper is that of [**7**]. We mention specifically only the following: if $\mathscr{L}$ is a lattice, $I(\mathscr{L})$ is the lattice of all ideals of $\mathscr{L}$; if X is a finite subset of a lattice, $\bigwedge X$ and $\bigvee X$ denote the greatest lower bound and the least upper bound of X, respectively; if a and b are elements of a partially ordered set, then

$$(a] = \{x : x \leqq a\}, [b) = \{x : x \geqq b\}, \text{ and } [a, b] = \{x : a \leqq x \leqq b\};$$

if $f : A \to B$ is a function and $X \subseteq A$, then $f(X) = \{f(x) : x \in X\}$.

2. Sufficient conditions for sharp transferability. We begin with several definitions.

2.1 *Definition.* Let $\langle P; \leqq \rangle$ be a partially ordered set. For $X, Y \subseteq P$, define $X \prec Y$ to hold if and only if for every $x \in X$ there exists $y \in Y$ such that $x \leqq y$.

2.2 *Definition.* Let $\langle S; \vee \rangle$ be a join-semilattice, $p \in S$, and $J \subseteq S$. We say $\langle p, J \rangle$ is a *minimal pair* if and only if the following three conditions hold:
(i) $p \notin J$;
(ii) $p \leqq \bigvee J$;
(iii) if $J' \subseteq S$, $p \leqq \bigvee J'$, and $J' \prec J$, then $J \subseteq J'$

2.3 *Definition.* A semilattice $\langle S; \vee \rangle$ is said to satisfy (T) if and only if there exists a linear order relation R on S such that if $\langle p, J \rangle$ is a minimal pair then $p \, R \, x$ holds for all $x \in J$.

2.4 *Definition.* A lattice $\langle L; \wedge, \vee \rangle$ is said to satisfy $(T_\vee)$ if and only if $\langle L; \vee \rangle$ satisfies (T), and to satisfy $(T_\wedge)$ if and only if the dual of $\langle L; \wedge \rangle$ satisfies (T).

The following remarks will help clarify the connection between these definitions and the concepts in [**3**] and [**4**].

2.5 *Remark.* If $\langle p, J \rangle$ is a minimal pair, then clearly J is an antichain; i.e., any two members of J are incomparable. Furthermore, $p \nleqq q$ for all $q \in J$.

2.6 *Remark.* If $\langle p, J \rangle$ is a minimal pair, then every element $x \in J$ is join-irreducible. Indeed, if $x \in J$ and $x = y \vee z$, let $J' = (J - \{x\}) \cup \{y, z\}$. Then $p \leqq \bigvee J'$ and $J' \prec J$, so $J \subseteq J'$ by (iii) of 2.2. Thus, $x = y$ or $x = z$.

2.7 *Remark.* If $\langle p, J \rangle$ is a minimal pair, then $|J| \geqq 2$.

2.8 *Remark.* If p is join-irreducible, then the present concept of minimal pair coincides with that of [**3**] and [**4**].

2.9 *Remark.* $\langle S; \vee \rangle$ fails to satisfy (T) if and only if for some $n \in \omega$ there exist minimal pairs $\langle p_i, J_i \rangle$, $i = 1, 2, \cdots, n$, such that $p_{i+1} \in J_i$ for $1 \leqq i < n$ and $p_1 \in J_n$. Therefore, in view of 2.6–2.8, $\langle S; \vee \rangle$ satisfies (T) if and only if it is *strictly transferable* in the sense of [**3**].

The next lemma illustrates the significance of minimal pairs.

2.10 LEMMA. *Let* $\langle S; \vee \rangle$ *and* $\langle S'; \vee \rangle$ *be semilattices, S finite, and let* $\varphi : S \to S'$ *be an order-preserving function. Then the following are equivalent*:

(i) φ *is join-preserving.*

(ii) *For each minimal pair* $\langle p, J \rangle$ *of* $\langle S; \vee \rangle$, $\varphi(p) \leqq \bigvee \varphi(J)$.

(iii) *For each minimal pair* $\langle p, J \rangle$ *of* $\langle S; \vee \rangle$ *satisfying* $p = \bigvee J$, $\varphi(p) \leqq \bigvee \varphi(J)$.

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are trivial, so assume (iii). Let $x, y \in S$ be incomparable. Since φ is order-preserving, $\varphi(x \vee y) \geqq \varphi(x) \vee \varphi(y)$ is clear. Since S is finite there must exist $J \subseteq S$ such that $J \prec \{x, y\}$ and $\langle x \vee y, J \rangle$ is a minimal pair. Since $\bigvee J = x \vee y$, (iii) implies that

$$\varphi(x \vee y) \leqq \bigvee \varphi(J) \leqq \varphi(x) \vee \varphi(y),$$

the latter inequality holding because $J \prec \{x, y\}$. This completes the proof.

The next definition introduces some useful terminology.

2.11 *Definition.* Let $\mathscr{L} = \langle L; \wedge, \vee \rangle$ and $\mathscr{L}' = \langle L'; \wedge, \vee \rangle$ be lattices, and let $I(\mathscr{L}')$ denote the lattice of all ideals of $\mathscr{L}'$. Let $\varphi : \mathscr{L} \to I(\mathscr{L}')$ be a lattice embedding. A function $\psi : L \to L'$ is called φ*-normal* if and only if it satisfies the following condition:

for $x, y \in L$, $\psi(x) \in \varphi(y)$ if and only if $x \leqq y$.

If ψ is also a lattice homomorphism, it is called a φ*-transfer*.

The following remarks are immediate from the definition.

2.12 *Remark.* The condition for ψ to be φ-normal is equivalent to:

$$\text{for all } x \in L,\ \psi(x) \in \varphi(x) - \bigcup_{y<x} \varphi(y).$$

2.13 *Remark.* If ψ is φ-normal, then ψ is one-to-one. Consequently, $\mathscr{L}$ is sharply transferable if and only if, for every embedding $\varphi : \mathscr{L} \to I(\mathscr{L}')$, there is a φ-transfer $\psi : \mathscr{L} \to \mathscr{L}'$.

2.14 *Remark.* If ψ is φ-normal and $\psi' : L \to L'$ satisfies $\psi(x) \leqq \psi'(x) \in \varphi(x)$ for all $x \in L$, then ψ' is φ-normal.

Our last condition for sharp transferability is the following.

2.15 *Definition.* A lattice $\mathscr{L} = \langle L; \wedge, \vee \rangle$ is said to satisfy (W) if and only if for every $x, y, u, v \in L$,

$$(W)\ x \wedge y \leqq u \vee v \text{ implies } [x \wedge y, u \vee v] \cap \{x, y, u, v\} \neq \emptyset.$$

The condition (W) was formulated by B. Jónsson [**10**] on the basis of P. M. Whitman's characterization of free lattices [**13**]. By induction we have the following

2.16 *Remark.* $\mathscr{L}$ satisfies (W) if and only if for any non-empty finite sets X,

$Y \subseteq L$, if $\bigwedge X \leqq \bigvee Y$, then $\bigwedge X \leqq y$ for some $y \in Y$ or else $x \leqq \bigvee Y$ for some $x \in X$.

We are now ready to prove the main result of this section.

2.17 THEOREM. *Let $\mathscr{L}$ be a finite lattice satisfying $(T_\vee)$, $(T_\wedge)$ and (W). Then $\mathscr{L}$ is sharply transferable.*

Proof. Let $\varphi : \mathscr{L} \to I(\mathscr{L}')$ be an embedding. We will first construct a φ-normal join-preserving function $\psi' : \langle L; \vee \rangle \to \langle L'; \vee \rangle$. Using $(T_\vee)$, choose an ordering $\langle x_1, x_2, \cdots, x_n \rangle$ of all the elements of L such that if $\langle x_i, J \rangle$ is a minimal pair of $\mathscr{L}$ and $x_j \in J$, then $i < j$. Let $\psi_0 : L \to L'$ be an arbitrary φ-normal function. Such a function clearly exists in view of the finiteness of L. We will construct functions $\psi_i : L \to L'$, $1 \leqq i \leqq n$ which are φ-normal, and such that

(1) for $0 \leqq i \leqq n$, if $1 \leqq j \leqq i$ and $\langle x_j, J \rangle$ is a minimal pair, then $\psi_i(x_j) \leqq \bigvee \psi_i(J)$.

This is in fact vacuous for $i = 0$. Suppose $0 \leqq i$ and ψ_i has been defined as required. Let $\langle x_{i+1}, J \rangle$ be a minimal pair. Then $\psi_i(x_{i+1}) \in \varphi(x_{i+1}) \leqq \bigvee \varphi(J)$, so for each $q \in J$ we can choose an element $q^J \in \varphi(q)$ such that $\psi_i(x_{i+1}) \leqq \bigvee \{q^J : q \in J\}$. Doing this for each minimal pair $\langle x_{i+1}, J \rangle$, we then define, for every $q \in L$,

$$\psi_{i+1}(q) = \psi_i(q) \vee \bigvee \{q^J : \langle x_{i+1}, J \rangle \text{ is a minimal pair and } q \in J\}.$$

Observe that ψ_{i+1} is φ-normal by Remark 2.14.

To show that ψ_{i+1} satisfies (1), we begin by noting that $\psi_{i+1}(x_j) = \psi_i(x_j)$ for $j \leqq i + 1$. This is immediate, since if $q = x_j \in J$ and $\langle x_{i+1}, J \rangle$ is a minimal pair, then $i + 1 \leqq j$. Now, let $1 \leqq j \leqq i + 1$ and let $\langle x_j, J \rangle$ be a minimal pair. If $j \leqq i$, then, by the preceding observation,

$$\psi_{i+1}(x_j) = \psi_i(x_j) \leqq \bigvee \psi_i(J) \leqq \bigvee \psi_{i+1}(J),$$

where we have used the inductive hypothesis and the fact that $\psi_i \leqq \psi_{i+1}$. If $j = i + 1$, then we have, by the choice of the q^J,

$$\psi_{i+1}(x_j) \leqq \bigvee \{q^J : q \in J\} \leqq \bigvee \psi_{i+1}(J).$$

Thus, (1) holds for ψ_{i+1}.

Now observe that for any minimal pair $\langle p, J \rangle$ of $\mathscr{L}$, we have $\psi_n(p) \leqq \bigvee \psi_n(J)$. Indeed, if $p = x_i$ then for $i = n$, this is (1), and for $i < n$, we have

$$\psi_n(x_i) = \psi_i(x_i) \leqq \bigvee \psi_i(J) \leqq \bigvee \psi_n(J).$$

Define $\psi' : L \to L'$ by

$$\psi'(x) = \bigvee \{\psi_n(y) : y \leqq x\} \quad (x \in L).$$

By Remark 2.14, ψ' is φ-normal. We claim ψ' is join-preserving. Since ψ' is clearly order-preserving, it suffices to establish condition (ii) of Lemma 2.10. If $\langle p, J\rangle$ is a minimal pair, then for each $y \leqq p$ we consider two cases:

(1) If $y \leqq q$ for some $q \in J$, then

$$\psi_n(y) \leqq \psi'(q) \leqq \bigvee \psi'(J).$$

(2) If $y \nleqq q$ for all $q \in J$, then there exists a minimal pair $\langle y, J'\rangle$, where $J' \prec J$. Then

$$\psi_n(y) \leqq \bigvee \psi_n(J') \leqq \bigvee \psi'(J),$$

the latter inequality holding since $\psi_n(J') \prec \psi'(J)$. Since $\psi_n(y) \leqq \bigvee \psi'(J)$ for all $y \leqq p$, we have $\psi'(p) \leqq \bigvee \psi'(J)$, and therefore ψ' is join-preserving.

The next step in the proof is to modify ψ' to obtain a meet-preserving φ-normal mapping. Let us call $\langle p, J\rangle$ a *dual minimal pair* of $\mathscr{L}$ if and only if $\langle p, J\rangle$ is a minimal pair of the dual of $\langle L; \wedge\rangle$. Since $\mathscr{L}$ satisfies $(T_\wedge)$, we can choose an ordering $\langle y_1, y_2, \ldots, y_n\rangle$ of all the elements of L such that if $\langle y_i, J\rangle$ is a dual minimal pair and $y_j \in J$, then $j < i$. We define $\psi : L \to L'$ as follows: for $x \in L$, set

$$\psi(x) = \psi'(x) \vee \bigvee\{\bigwedge \psi(J) : \langle x, J\rangle \text{ is a dual minimal pair}\}.$$

Observe that the formula for $\psi(y_i)$ depends only on values $\psi(y_j)$ satisfying $j < i$, so ψ is well-defined.

First we show ψ is order-preserving. We will prove the following statement by induction on j:

$$\text{if } y \in L \text{ and } y_j \leqq y, \text{ then } \psi(y_j) \leqq \psi(y).$$

Indeed, given i, $1 \leqq i \leqq n$, suppose this holds for all j such that $j < i$. If $y_i \leqq y$, then since ψ' is order-preserving, we have $\psi'(y_i) \leqq \psi'(y) \leqq \psi(y)$. Let $\langle y_i, J\rangle$ be a dual minimal pair. If $y_j \leqq y$ holds for some $y_j \in J$, then $j < i$, so by inductive hypothesis,

$$\bigwedge \psi(J) \leqq \psi(y_j) \leqq \psi(y).$$

On the other hand, if $y_j \nleqq y$ for all $y_j \in J$, then there exists $J' \subseteq L$ such that $\langle y, J'\rangle$ is a dual minimal pair and $J' \prec J$ holds in the dual of $\mathscr{L}$; that is, for every $u \in J'$ there exists $y_j \in J$ such that $y_j \leqq u$. Since $j < i$, we then have by inductive hypothesis that $\psi(y_j) \leqq \psi(u)$. From this it follows that $\bigwedge \psi(J) \leqq \bigwedge \psi(J')$. However, by definition of ψ, $\bigwedge \psi(J') \leqq \psi(y)$. Since $\psi(y_i)$ is a join of terms each of which we have shown to be less than or equal to $\psi(y)$, we have proved $\psi(y_i) \leqq \psi(y)$.

It is clear from the definition that if $\langle p, J\rangle$ is a dual minimal pair, then $\psi(p) \geqq \bigwedge \psi(J)$. Thus, by the dual of Lemma 2.10, ψ is meet-preserving.

We next show that ψ is φ-normal. Since $\psi' \leqq \psi$. it suffices to prove $\psi(y_i) \in \varphi(y_i)$ for all $y_i \in L$. The proof is again by induction on i. Suppose then that $\psi(y_j) \in \varphi(y_j)$ for all $j < i$. We have $\psi'(y_i) \in \varphi(y_i)$ by the φ-normality of ψ'. If

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$\langle y_i, J\rangle$ is a dual minimal pair, then, by inductive hypothesis,

$$\bigwedge\psi(J) \in \bigcap\varphi(J) = \varphi(\bigwedge J) \subseteq \varphi(x).$$

Thus, $\psi(y_i)$, being a finite join of elements of the above types, is in $\varphi(x)$.

Finally, we conclude the proof of the theorem by using (W) to establish condition (iii) of Lemma 2.10, proving that ψ is join-preserving. Let $\langle p, J\rangle$ be a minimal pair with $p = \bigvee J$. Since ψ' is join-preserving, we have $\psi'(p) \leqq \bigvee\psi'(J) \leqq \bigvee\psi(J)$. If $\langle p, J'\rangle$ is a dual minimal pair, then $\bigvee J = p \geqq \bigwedge J'$. Since, by the definition of a dual minimal pair, we have $p \ngeqq q$ for all $q \in J'$, it follows by (W) (Remark 2.16) that $\bigwedge J' \leqq q$ for some $q \in J$. Then we have

$$\bigwedge\psi(J') = \psi(\bigwedge J') \leqq \psi(q) \leqq \bigvee\psi(J).$$

Thus, we have $\psi(x) \leqq \bigvee\psi(J)$, completing the proof of the theorem.

3. Necessity of $(T_\vee)$ and $(T_\wedge)$. In this section we will prove that every sharply transferable lattice satisfies $(T_\vee)$ and $(T_\wedge)$. We begin with the following construction.

3.1 *Definition.* Let $\mathscr{L} = \langle L; \wedge, \vee\rangle$ be a finite lattice. Let $\mathbf{Z}$ denote the integers with their usual linear ordering. Let $\pi_1 : L \times \mathbf{Z} \to L$ and $\pi_2 : L \times \mathbf{Z} \to \mathbf{Z}$ be the projections onto the two factors. Then we define $\hat{L}$ to be the collection of all non-empty subsets H of $L \times \mathbf{Z}$ which satisfy the following two conditions:

(i) for some $n \in \mathbf{Z}$, $\pi_2(H) \subseteq (n]$;

(ii) if $\emptyset \neq X \subseteq H$, X is finite, $z \leqq \bigvee\pi_1(X)$, and $i < \bigwedge\pi_2(X)$, then $\langle z, i\rangle \in H$.

Observe that for each $n \in \mathbf{Z}$, $L \times (n] \in \hat{L}$. Therefore we can make the following definition.

3.2 *Definition.* If $n \in \mathbf{Z}$, $X \subseteq L \times (n]$, and $X \neq \emptyset$, then we define

$$[X] = \bigcap\{H \in \hat{L} : X \subseteq H\}.$$

If $u \in L \times \omega$, we write $[u]$ for $[\{u\}]$.

3.3 *Remark.* Clearly $[X] \in \hat{L}$. Furthermore, there exists a finite set $X_0 \subseteq X$ such that $[X_0] = [X]$.

3.4 *Definition.* If $H, K \in \hat{L}$, define $H \vee K = [H \cup K]$ and $H \wedge K = H \cap K$.

The proof of the following is straightforward and is left to the reader.

3.5 Lemma. *$\hat{\mathscr{L}} = \langle \hat{L}; \wedge, \vee\rangle$ is a lattice with set inclusion as its partial order.*

The following remark is easily verified.

3.6 *Remark.* If $\emptyset \neq X \subseteq L \times (n]$, $n \in \mathbf{Z}$, then

$$[X] = X \cup \{u \in L \times \mathbf{Z} : \text{there exists } Y \subseteq X,\ Y \text{ finite}, \\ \pi_1(u) \leqq \bigvee \pi_1(Y),\ \pi_2(u) < \bigwedge \pi_2(Y)\}.$$

In particular, if X is finite, then $[X] \subseteq (\bigvee \pi_1(X)] \times (n]$.

3.7 LEMMA. *If* $X \subseteq L \times (n]$, $n \in \mathbf{Z}$, *and* $\langle x, i\rangle \in [X] - X$, *then*

$$\langle x, i\rangle \in [X \cap (L \times [i + 1))].$$

Proof. Obvious by Remark 3.6.

3.8 LEMMA. *For a finite lattice* $\mathscr{L}$, *define, for each* $x \in L$,

$$\varphi(x) = \{H \in \hat{L} : H \subseteq (x] \times \mathbf{Z}\}.$$

Then $\varphi(x)$ *is an ideal of* $\mathscr{L}$, *and* $\varphi : \mathscr{L} \to I(\mathscr{L})$ *is an embedding.*

Proof. The proofs that $\varphi(x)$ is an ideal and that φ is one-to-one and meet-preserving are straightforward, so we prove φ is join-preserving. Since φ is order-preserving it suffices to show that $\varphi(x \vee y) \leqq \varphi(x) \vee \varphi(y)$ for $x, y \in L$. Let $H \in \varphi(x \vee y)$, and put $i = \bigvee \pi_2(H)$. Then $H \subseteq [\langle x \vee y, i + 1\rangle]$. Also, $[\langle x, i + 2\rangle] \in \varphi(x)$, $[\langle y, i + 2\rangle] \in \varphi(y)$, and $\langle x \vee y, i + 1\rangle \in [\{\langle x, i + 2\rangle, \langle y, i + 2\rangle\}]$. Therefore,

$$H \subseteq [\langle x \vee y, i + 1\rangle] \subseteq [\langle x, i + 2\rangle] \vee [\langle y, i + 2\rangle] \in \varphi(x) \vee \varphi(y),$$

so $H \in \varphi(x) \vee \varphi(y)$. Thus $\varphi(x \vee y) \subseteq \varphi(x) \vee \varphi(y)$, completing the proof.

Now we prove the necessity of $(T_\vee)$. The stronger statement in Theorem 3.9 will be used in a forthcoming paper [**5**], and also yields a new proof of the main result of [**3**]. It is only necessary to observe that the construction of $\mathscr{L}$ and the proof of Lemma 3.8 remain valid when $\mathscr{L}$ is a join semilattice.

3.9 THEOREM. *If a finite lattice* $\mathscr{L}$ *is sharply transferable, then* $\mathscr{L}$ *satisfies* $(T_\vee)$. *In fact, if for the embedding* $\varphi : \mathscr{L} \to I(\hat{\mathscr{L}})$ *there exists a* φ*-normal join-preserving function, then* $\mathscr{L}$ *satisfies* $(T_\vee)$.

Proof. Let $\psi : \mathscr{L} \to \hat{\mathscr{L}}$ be φ-normal and join-preserving. If $x \in L$, we claim that $x \in \pi_1(\psi(x))$. Indeed, if $z = \bigvee \pi_1(\psi(x))$, then for some $i \in \mathbf{Z}$, $\langle z, i\rangle \in \psi(x)$. Since $\psi(x) \in \varphi(x)$, $z \leqq x$. But clearly $\psi(x) \in \varphi(z)$, so since ψ is φ-normal, $x \leqq z$. Thus, $\langle x, i\rangle = \langle z, i\rangle \in \psi(x)$.

Now, for each $x \in L$, we define

$$\rho(x) = \bigvee\{j \in \mathbf{Z} : \langle x, j\rangle \in \psi(x)\}.$$

Claim: If $\langle p, J\rangle$ is a minimal pair of $\mathscr{L}$ and $q \in J$, then $\rho(p) < \rho(q)$.

Given this, $(T_\vee)$ follows immediately, letting R be any total ordering such that $\rho(p) < \rho(q)$ implies $p\ R\ q$.

To prove the claim, let $j = \rho(p)$. Then

$$\langle p, j\rangle \in \psi(p) \leqq \bigvee \psi(J) = [\bigcup \psi(J)].$$

By Remark 3.3, we can choose a finite set $T \subseteq \bigcup \psi(J)$ which is minimal with respect to $\langle p, j\rangle \in [T]$. Now $\langle p, j\rangle \notin T$, for otherwise $\langle p, j\rangle \in \psi(q) \in \varphi(q)$ for some $q \in J$, hence $p \leqq q$, contrary to Remark 2.5. Thus, by Lemma 3.7 and the minimality of T, if $\langle x, i\rangle \in T$, then $i > j$. Let $J' = \pi_1(T)$. By Remark 3.6, $p \leqq \bigvee J'$. If $\langle x, i\rangle \in T$, then for some $q \in J$, $\langle x, i\rangle \in \psi(q) \in \varphi(q)$, so $x \leqq q$. Thus, $J' \prec J$, so by definition of minimal pair, $J \subseteq J'$. Now let $q \in J$ be given. Then $q \in J'$, so let $\langle q, i\rangle \in T$, hence $\langle q, i\rangle \in \psi(q_0)$ for some $q_0 \in J$. Then $q \leqq q_0$, whence $q = q_0$ by Remark 2.5. Thus, $\langle q, i\rangle \in \psi(q)$, so $\rho(q) \geqq i > j = \rho(p)$, proving the claim and the theorem.

Turning now to $(T_\wedge)$, we wish to dualize the construction of $\hat{\mathscr{L}}$. Let $\mathscr{L}^d$ denote the dual of a lattice $\mathscr{L}$; we define

$$\mathscr{L}^* = \widehat{(\mathscr{L}^d)}^d.$$

Thus, L^* is the set of all non-empty subsets H of $L \times \mathbf{Z}$ with $H \subseteq L \times (n]$ for some $n \in \mathbf{Z}$ and satisfying

(iii) if $\emptyset \neq X \subseteq H$, X is finite, $z \geqq \bigwedge \pi_1(X)$, and $i < \bigwedge \pi_2(X)$, then $\langle z, i\rangle \in H$.

The ordering of $\mathscr{L}^*$ is the dual of set inclusion. If the notation $[X]$ is defined as before, then for $H, K \in L^*$, $H \vee K = H \cap K$ and $H \wedge K = [H \cap K]$ in $\mathscr{L}^*$.

We again have a natural embedding of $\mathscr{L}$ into $I(\mathscr{L}^*)$.

3.10 Lemma. *For a finite lattice $\mathscr{L}$, define, for each $x \in L$,*

$$\varphi^*(x) = \{H \in L^* : x \in \pi_1(H)\}.$$

Then $\varphi^*(x)$ is an ideal of $\mathscr{L}^*$, and $\varphi^* : \mathscr{L} \to I(\mathscr{L}^*)$ is an embedding.

Proof. If $H \in \varphi^*(x)$, $K \in L^*$, and $K \leqq H$, i.e., $H \subseteq K$, then $x \in \pi_1(H) \subseteq \pi_1(K)$, so $K \in \varphi^*(x)$. If $H, K \in \varphi^*(x)$, choose $\langle x, i\rangle \in H$ and $\langle x, j\rangle \in K$. Then $\langle x, i \wedge j\rangle \in H \cap K = H \vee K$, so $H \vee K \in \varphi^*(x)$. Thus $\varphi^*(x)$ is an ideal of $\mathscr{L}^*$.

Suppose $x, y \in L$ and $x \leqq y$. If $H \in \varphi^*(x)$, let $\langle x, i\rangle \in H$. Then $\langle y, i - 1\rangle \in H$, so $H \in \varphi^*(y)$. Thus, $\varphi^*(x) \subseteq \varphi^*(y)$, and φ^* is order-preserving.

Next, assume $\varphi^*(x) \subseteq \varphi^*(y)$. Let $H = [\langle x, 1\rangle]$. Then $H \in \varphi^*(y)$, so $y \in \pi_1(H)$. But by (the dual of) Remark 3.6, $\pi_1(H) \subseteq [x)$, so $x \leqq y$. Hence φ^* is an order embedding.

To show φ^* is meet-preserving, it suffices to show $\varphi^*(x) \cap \varphi^*(y) \subseteq \varphi^*(x \wedge y)$ for $x, y \in L$. Let $H \in \varphi^*(x) \cap \varphi^*(y)$. Then there exist $\langle x, i\rangle$, $\langle y, j\rangle \in H$, so $\langle x \wedge y, i \wedge j - 1\rangle \in H$. Thus, $H \in \varphi^*(x \wedge y)$.

Finally, we show that $\varphi^*(x \vee y) \subseteq \varphi^*(x) \vee \varphi^*(y)$, proving φ^* is join-preserving. Given $H \in \varphi^*(x \vee y)$, let $\langle x \vee y, i\rangle \in H$. Define $K_x = [\langle x, i\rangle]$ and $K_y = [\langle y, i\rangle]$. Since $K_x \in \varphi^*(x)$ and $K_y \in \varphi^*(y)$, it suffices to show that $H \leqq K_x \vee K_y$ in $\mathscr{L}^*$, i.e., that $K_x \cap K_y \subseteq H$. Suppose then that $\langle z, j\rangle \in$

$K_x \cap K_y$. By (the dual of) Remark 3.6, $z \geqq x \vee y$ and $j \leqq i$. Thus, if $j < i$, we have $\langle z, i\rangle \in [\langle x \vee y, i\rangle] \subseteq H$. On the other hand, by Lemma 3.7, $K_x \cap (L \times \{i\}) = \{\langle x, i\rangle\}$ and $K_y \cap (L \times \{i\}) = \{\langle y, i\rangle\}$, so if $j = i$, then $z = x = y$, so again $\langle z, i\rangle \in H$. This completes the proof.

We now have the dual of Theorem 3.9.

3.11 THEOREM. *If a finite lattice $\mathscr{L}$ is sharply transferable, then $\mathscr{L}$ satisfies $(T_\vee)$. In fact, if for the embedding $\varphi^* : \mathscr{L} \rightarrow I(\mathscr{L}^*)$ there exists a φ^*-normal meet-preserving function, then $\mathscr{L}$ satisfies $(T_\wedge)$.*

Proof. Let $\psi^* : \mathscr{L} \rightarrow \mathscr{L}^*$ be φ^*-normal and meet-preserving. We begin by observing that for $x \in L$ and $i \in \mathbf{Z}$,

(2) if $\langle x, i\rangle \in \psi^*(y)$, then $x \geqq y$.

Indeed, if $\langle x, i\rangle \in \psi^*(y)$, let $\langle y, j\rangle \in \psi^*(y)$. Then $\langle x \wedge y, i \wedge j - 1\rangle \in \psi^*(y)$, so $\psi^*(y) \in \varphi^*(x \wedge y)$. By φ^*-normality, $y \leqq x \wedge y \leqq x$.

Now for $x \in L$, define

$$\sigma(x) = \bigvee\{j \in \mathbf{Z} : \langle x, j\rangle \in \psi^*(x)\}.$$

The proof will be completed by establishing the following.

Claim: If $\langle p, J\rangle$ is a dual minimal pair in $\mathscr{L}$, then for all $q \in J$, $\sigma(p) < \sigma(q)$.

Thus, let $\sigma(p) = j$. We have $p \geqq \bigwedge J$, so $\psi^*(p) \geqq \psi^*(\bigwedge J)$, i.e.

$$\psi^*(p) \subseteq \psi^*(\bigwedge J) = \bigwedge \psi^*(J) = [\bigcup \psi^*(J)].$$

Since $\langle p, j\rangle \in \psi^*(p)$, we can choose a finite set $T \subseteq \bigcup \psi^*(J)$, minimal with respect to $\langle p, j\rangle \in [T]$. Now $\langle p, j\rangle \notin T$, since $\langle p, j\rangle \in \psi^*(q)$, $q \in J$ would imply $p \geqq q$, contrary to Remark 2.5. Thus, by Lemma 3.7, $i > j$ for all $i \in \pi_2(T)$. Let $J' = \pi_1(T)$. By (the dual of) Remark 3.6, $p \geqq \bigwedge J'$, and by (2), if $q \in J'$, then $q \geqq q'$ for some $q' \in J$. Thus, by the definition of dual minimal pair, $J \subseteq J'$. Now let $q \in J$ be given. Then $q \in J'$, so $\langle q, i\rangle \in T$ for some i. But then $\langle q, i\rangle \in \psi^*(q_0)$ for some $q_0 \in J$, implying $q \geqq q_0$, so $q = q_0$ by Remark 2.5. Thus, $\langle q, i\rangle \in \psi^*(q)$, so $\sigma(q) \geqq i > j = \sigma(p)$, proving the claim and the theorem.

In the remainder of this section, we will use the constructions $\hat{\mathscr{L}}$ and $\mathscr{L}^*$ to establish the semi-distributivity of transferable lattices. This will be needed in Section 4. We first define the concept.

3.12 *Definition*. A lattice $\mathscr{L}$ is said to satisfy the *semi-distributive law* $(S\,D_\vee)$ if and only if for $x, y, z \in L$, $x \vee y = x \vee z$ implies $x \vee y = x \vee (y \wedge z)$. The *semi-distributive law* $(S\,D_\wedge)$ is defined dually.

There is a related property which for finite lattices is equivalent to $(S\,D_\vee)$.

3.13 *Definition*. A join-semilattice $\langle S; \vee\rangle$ is said to have *canonical join representations* if and only if for every $x \in S$, there is a set $Q_x \subseteq S$ such that

$\bigvee Q_x = x$, and if $J \subseteq S$ with $x = \bigvee J$, then $Q_x \prec J$. (Canonical join representations are usually required to be antichains. However it is clear that the set Q_x in Definition 3.13 contains a unique antichain which will also satisfy the definition.)

We will need the following observation due to B. Jónsson and J. Kiefer [**11**]. For completeness, we include a proof.

3.14 LEMMA. *If a lattice $\mathscr{L} = \langle L; \wedge, \vee \rangle$ is such that $\langle L; \vee \rangle$ [respectively, the dual of $\langle L; \wedge \rangle$] has canonical join representations, then $\mathscr{L}$ satisfies $(S\,D_\vee)$ [respectively, $(S\,D_\wedge)$].*

Proof. Let $x, y, z \in L$ and suppose $x \vee y = x \vee z$. Let Q be the canonical join representation for $x \vee y$. Put $R = \{q \in Q : q \leqq x\}$, and $S = Q - R$. Then $Q \prec \{x, y\}$, hence $S \prec \{y\}$. Similarly, $S \prec \{z\}$, so $S \prec \{y \wedge z\}$. Since $R \prec \{x\}$, we have $Q \prec \{x \vee (y \wedge z)\}$, so that

$$x \vee y = \bigvee Q \leqq x \vee (y \wedge z) \leqq x \vee y.$$

Thus, equality holds, proving the lemma.

3.15 LEMMA. *For any finite lattice $\mathscr{L}$, $\hat{\mathscr{L}}$ has canonical join representations.*

Proof. For $H \in \hat{\mathscr{L}}$, define

$$H_0 = \{u \in H : u \notin [H - \{u\}]\}.$$

We claim that

(i) $[H_0] = H$, and

(ii) if $X \subseteq H$ and $[X] = H$, then $H_0 \subseteq X$.

For (i) it suffices to prove $H \subseteq [H_0]$. If $u \in H$, we prove $u \in [H_0]$ by downward induction on $\pi_2(u)$. Thus, suppose that $v \in H$ and $\pi_2(v) > \pi_2(u)$ imply $v \in [H_0]$. If $u \in H_0$, then of course $u \in [H_0]$. If $u \notin H_0$, then, by definition, $u \in [H - \{u\}]$. By Lemma 3.7, $u \in [(H - \{u\}) \cap (L \times [\pi_2[u] + 1))] \subseteq [H_0]$, the latter inclusion following by inductive hypothesis.

To establish (ii), let $[X] = H$. If $u \in H_0$ but $u \notin X$, then $u \in [X] = [X - \{u\}] \subseteq [H - \{u\}]$ contradicting the definition of H_0.

Now let $Q_H = \{[u] : u \in H_0\}$. Then $\bigvee Q_H = [H_0]$. If $J \subseteq \hat{L}$ and $H = \bigvee J = [\bigcup J]$, (ii) implies $H_0 \subseteq \bigcup J$, hence $Q_H \prec J$. This completes the proof.

3.16 THEOREM. *A finite transferable lattice $\mathscr{L}$ satisfies $(S\,D_\vee)$ and $(S\,D_\wedge)$.*

Proof. First, by 3.14 and 3.15, $\hat{\mathscr{L}}$ satisfies $(S\,D_\vee)$. By Lemma 3.8 there is an embedding $\varphi : \mathscr{L} \to I(\hat{\mathscr{L}})$, so by transferability, there is an embedding $\psi : \mathscr{L} \to \hat{\mathscr{L}}$. Since $(S\,D_\vee)$ is clearly hereditary, $\mathscr{L}$ satisfies $(S\,D_\vee)$.

Now, by Lemma 3.10, $\mathscr{L}$ is embeddable into $I(\mathscr{L}^*)$, where $\mathscr{L}^* = (\widehat{\mathscr{L}^d})^d$. By transferability, $\mathscr{L}$ is embeddable in $(\widehat{\mathscr{L}^d})^d$, which satisfies $(S\,D_\wedge)$. Again by heredity, $\mathscr{L}$ satisfies $(S\,D_\wedge)$. This completes the proof.

3.17 *Remark.* A direct proof that, in a semilattice $\langle S;\ \vee\rangle$, (T) implies existence of canonical join representations can be obtained as follows: if x is join-reducible, define $Q_x = \{y \in S : y < x$ and there exists no minimal pair $\langle y, J\rangle$ with $J \prec \{x\}\}$. Using induction on an ordering given by (T), one can show Q_x is the canonical join representation of x.

3.18 *Remark.* The lattice in Figure 1 shows that $(S\,D_\vee)$ does not imply $(T_\vee)$.

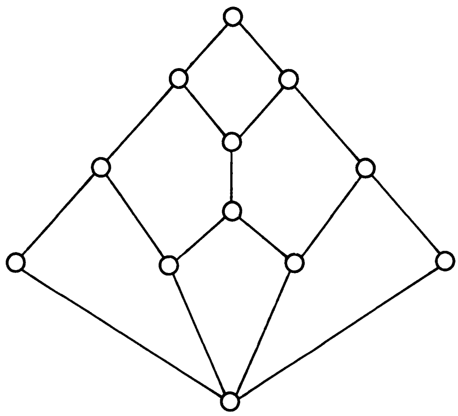

FIGURE 1

In view of the results of [**5**], the assertion that $(S\,D_\vee)$, $(S\,D_\wedge)$, and (W) jointly imply $(T_\vee)$ is equivalent to Jónsson and Kiefer's conjecture [**11**] that $(S\,D_\vee)$, $(S\,D_\wedge)$, and (W) characterize finite sublattices of free lattices.

4. Necessity of (W)**.** The "splitting" of an element of a lattice was introduced to facilitate a proof in [6] (see [**9**]). This was extended by A. Day [**2**] to the "splitting" of an interval. We begin here with a generalization of this construction. Let $\mathscr{L} = \langle L;\ \wedge,\ \vee\rangle$ be a lattice and let $\{C_\lambda : \lambda \in \Lambda\}$ be a collection of pairwise disjoint convex subsets of $\mathscr{L}$. Put $S = \bigcup\{C_\lambda : \lambda \in \Lambda\}$, and form the set

$$L^{\star} = ((L - S) \times \{0\}) \cup (S \times \{0, 1\}),$$

where $\{0, 1\}$ forms a chain with $0 < 1$.

Let $\pi_1 : L \times \{0, 1\} \rightarrow L$ and $\pi_2 : L \times \{0, 1\} \rightarrow \{0, 1\}$

be the projections. Then if $u, v \in L^{\star}$, define $u \leqq^{\star} v$ to hold if and only if the following two conditions are satisfied:

(i) $\pi_1(u) \leqq \pi_1(v)$

(ii) if $\pi_1(u)$ and $\pi_1(v)$ are both elements of C_λ for some $\lambda \in \Lambda$, then $\pi_2(u) \leqq \pi_2(v)$.

4.1 LEMMA. *The relation* $\leqq^*$ *is a partial ordering under which* L^* *becomes a lattice* $\mathscr{L}^* = \langle L^*; \wedge, \vee \rangle$. *The projection* π_1 *is a lattice homomorphism of* $\mathscr{L}^*$ *onto* $\mathscr{L}$.

Proof. It is trivial to check that $\leqq^*$ is reflexive and antisymmetric. To establish transitivity, suppose $u \leqq^* v$ and $v \leqq^* w$ hold. Then $\pi_1(u) \leqq \pi_1(v) \leqq \pi_1(w)$ so $\pi_1(u) \leqq \pi_1(w)$. Furthermore, if $\pi_1(u), \pi_1(w) \in C_\lambda$ for some $\lambda \in \Lambda$, then, by convexity, $\pi_1(v) \in C_\lambda$ so by (ii), $\pi_2(u) \leqq \pi_2(v) \leqq \pi_2(w)$. Thus, $u \leqq^* w$.

In order to describe joins and meets in $\mathscr{L}^*$, we define, for each $\lambda \in \Lambda$, two functions, f_λ and f^λ, from L^* to $\{0, 1\}$ as follows: if $u \in L^*$ and $\pi_1(u) \in C_\lambda$, then $f_\lambda(u) = f^\lambda(u) = \pi_2(u)$; if $\pi_1(u) \notin C_\lambda$ then $f_\lambda(u) = 0$ and $f^\lambda(u) = 1$. Now, given $u, v \in L^*$, define j by:

$$j = \begin{cases} f_\lambda(u) \vee f_\lambda(v), & \text{if } \pi_1(u) \vee \pi_1(v) \in C_\lambda,\ \lambda \in \Lambda; \\ 0 & , \text{if } \pi_1(u) \vee \pi_1(v) \notin S. \end{cases}$$

Then we claim that $w = \langle \pi_1(u) \vee \pi_1(v), j \rangle$ is the least upper bound of u and v, relative to $\leqq^*$. To prove $u \leqq^* w$, we need only establish (ii). But if, say, $\pi_1(u)$ and $\pi_1(u) \vee \pi_1(v)$ are in C_λ, then

$$\pi_2(u) = f_\lambda(u) \leqq f_\lambda(u) \vee f_\lambda(v) = j.$$

Similarly, $v \leqq^* w$.

Next, suppose $z \in L^*$, $u \leqq^* z$, and $v \leqq^* z$. Then clearly $\pi_1(w) = \pi_1(u) \vee \pi_1(v) \leqq \pi_1(z)$. Suppose $\pi_1(w)$ and $\pi_1(z)$ are in C_λ. Then certainly $f_\lambda(u) \leqq \pi_2(z)$, for if $f_\lambda(u) = 1$, then $\pi_1(u) \in C_\lambda$, whence by (ii), $f_\lambda(u) = \pi_2(u) \leqq \pi_2(z)$. Similarly, $f_\lambda(v) \leqq \pi_2(z)$, hence

$$\pi_2(w) = j = f_\lambda(u) \vee f_\lambda(v) \leqq \pi_2(z).$$

This proves $w \leqq^* z$ and our claim.

In a similar manner, we can show that the greatest lower bound of u and v is $\langle \pi_1(u) \wedge \pi_1(v), k \rangle$, where

$$k = \begin{cases} f^\lambda(u) \wedge f^\lambda(v), & \text{if } \pi_1(u) \wedge \pi_1(v) \in C_\lambda,\ \lambda \in \Lambda; \\ 0 & , \text{if } \pi_1(u) \wedge \pi_1(v) \notin S. \end{cases}$$

The last statement of the lemma is obvious.

From the proof, we extract for future reference two observations.

4.2 *Remark.* Suppose $u, v \in L^*$, $\pi_1(u) \notin C_\lambda$ and $\pi_1(v) \notin C_\lambda$. Then if $\pi_1(u) \vee \pi_1(v) \in C_\lambda$, then $\pi_2(u \vee v) = 0$ and if $\pi_1(u) \wedge \pi_1(v) \in C_\lambda$, then $\pi_2(u \wedge v) = 1$. In any case, if $u, v \in L^*$ and $\pi_2(u) = \pi_2(v) = 0$, then $\pi_2(u \vee v) = 0$.

We now introduce a weak form of condition (W).

4.3 *Definition.* A lattice $\mathscr{L}$ is said to satisfy (W') if and only if for every $x, y, u, v \in L$,

$$(W')\ u \leqq x \wedge y \leqq u \vee v \text{ implies } [x \wedge y, u \vee v] \cap \{x, y, u, v\} \neq \emptyset.$$

We next show that sharp transferability implies (W'). We remark that the following theorem does not require $\mathscr{L}$ to be finite.

4.4 THEOREM. *A sharply transferable lattice satisfies* (W').

Proof. Let $\mathscr{L}$ be a lattice which does not satisfy (W'). We will show that $\mathscr{L}$ is not sharply transferable. Choose $a, b, c, d \in L$ such that $c \leqq a \wedge b \leqq c \vee d$, and $\{a, b, c, d\} \cap [a \wedge b, c \vee d] = \emptyset$. Let $J = (d]$, and form the set

$$L' = (J \times \{0\}) \cup ((L - J) \times \boldsymbol{\omega}),$$

where $\boldsymbol{\omega}$ denotes the natural numbers. Let L' have the partial ordering inherited from the direct product $\mathscr{L} \times \boldsymbol{\omega}$, where $\boldsymbol{\omega}$ is given its usual ordering. It is a trivial exercise to show that L' with this ordering forms a lattice $\mathscr{L}'$ with joins and meets as follows: if $\langle x, i\rangle, \langle y, j\rangle \in L'$, then

$$\langle x, i\rangle \vee \langle y, j\rangle = \langle x \vee y, i \vee j\rangle,$$
$$\langle x, i\rangle \wedge \langle y, j\rangle = \begin{cases} \langle x \wedge y, i \wedge j\rangle, & \text{if } x \wedge y \notin J; \\ \langle x \wedge y, 0\rangle & \text{, otherwise.} \end{cases}$$

Now take $\Lambda = \boldsymbol{\omega}$, and for $n \in \boldsymbol{\omega}$, $C_n = [a \wedge b, c \vee d] \times \{n\}$. Then construct the lattice $\mathscr{L}^*$ from $\mathscr{L}'$ and $\{C_n : n \in \boldsymbol{\omega}\}$ as in Lemma 4.1. Let $\pi_1 : L^* \to L'$ and $\pi_2 : L^* \to \{0, 1\}$ be the projections as before, and also introduce the projections $\sigma_1 : L' \to L$ and $\sigma_2 : L' \to \boldsymbol{\omega}$. Observe that $\sigma_1 \circ \pi_1 : \mathscr{L}^* \to \mathscr{L}$ is an onto lattice homomorphism, and $\sigma_2 \circ \pi_1 : \mathscr{L}^* \to \boldsymbol{\omega}$ is a join-preserving function.

For $x \in L$, define

$$\varphi(x) = \{z \in L^* : \sigma_1 \circ \pi_1(z) \leq x\}.$$

Claim: $\varphi(x)$ is an ideal of $\mathscr{L}^*$, and $\varphi : \mathscr{L} \to I(\mathscr{L}^*)$ is an embedding.

Indeed, $\varphi(x)$ is the inverse image under $\sigma_1 \circ \pi_1$ of the principal ideal $(x]$. It is, therefore, immediate that $\varphi(x)$ is an ideal and that φ is one-to-one and meet-preserving.

To see that φ preserves joins, let $x, y \in L$. Since φ is order-preserving, it is enough to show $\varphi(x \vee y) \subseteq \varphi(x) \vee \varphi(y)$. Let $u \in \varphi(x \vee y)$. Then we consider two cases.

(1) If x and y are in J, then $x \vee y \in J$, so $\sigma_1(\pi_1(u)) \in J$. Thus, $\sigma_2(\pi_1(u))$ must be 0, and we have

$$u \leqq^* \langle\langle x \vee y, 0\rangle, 0\rangle = \langle\langle x, 0\rangle, 0\rangle \vee \langle\langle y, 0\rangle, 0\rangle$$

in $\mathscr{L}^*$ by Remark 4.2. But clearly

$$\langle\langle x, 0\rangle, 0\rangle \in \varphi(x) \text{ and } \langle\langle y, 0\rangle, 0\rangle \in \varphi(y),$$

so $u \in \varphi(x) \vee \varphi(y)$.

(2) If one of x and y, say x, is not in J, let $\sigma_2(\pi_1(u)) = k$. Then

$\langle\langle x, k+1\rangle, 0\rangle \in \varphi(x)$, $\langle\langle y, 0\rangle, 0\rangle \in \varphi(y)$, and

$$u \leqq^* \langle\langle x \vee y, k+1\rangle, 0\rangle = \langle\langle x, k+1\rangle \vee \langle y, 0\rangle, 0\rangle = \langle\langle x, k+1\rangle, 0\rangle \vee \langle\langle y, 0\rangle, 0\rangle,$$

hence again $u \in \varphi(x) \vee \varphi(y)$. This proves the claim.

Now suppose there is a φ-transfer, $\psi : \mathscr{L} \to \mathscr{L}^*$. We will derive a contradiction, proving the theorem. For any $x \in L$, we claim that $\sigma_1(\pi_1(\psi(x))) = x$. Indeed, $\psi(x) \in \varphi(x)$, so $\sigma_1(\pi_1(\psi(x))) \leqq x$, while if $y = \sigma_1(\pi_1(\psi(x)))$, then $\psi(x) \in \varphi(y)$, so $x \leqq y$ by definition of φ-normal.

For convenience of notation, let g denote the join-preserving function $\sigma_2 \circ \pi_1 \circ \psi : \mathscr{L} \to \omega$. Observe that $g(d) = 0$ so that $c \leqq a \wedge b \leqq c \vee d$ implies that

$$g(c) \leqq g(a \wedge b) \leqq g(c \vee d) = g(c) \vee g(d) = g(c),$$

so equality holds throughout. Thus, with $n = g(c)$, both $\pi_1(\psi(a \wedge b))$ and $\pi_1(\psi(c \vee d))$ are in $C_n = [a \wedge b, c \vee d] \times \{n\}$. Then $\psi(a \wedge b) \leqq^* \psi(c \vee d)$ implies that $\pi_2(\psi(a \wedge b)) \leqq \pi_2(\psi(c \vee d))$. However, by hypothesis $\pi_1 \circ \psi\{a, b, c, d\}$ is disjoint from C_n, so by Remark 4.2, $\pi_2(\psi(a \wedge b)) = \pi_2(\psi(a) \wedge \psi(b)) = 1$, and $\pi_2(\psi(a \vee b)) = \pi_2(\psi(a) \vee \psi(b)) = 0$, a contradiction. This completes the proof of the theorem.

Next, we show that in the presence of $(SD_\wedge)$, (W') can be strengthened to (W).

4.5 LEMMA. *Let $\mathscr{L}$ be a finite lattice satisfying $(SD_\wedge)$ and (W'). Then $\mathscr{L}$ satisfies (W).*

Proof. Let $\mathscr{L}$ fail to satisfy (W). Choose $a, b, c, d \in L$ such that $a \wedge b \leqq c \vee d$ and $\{a, b, c, d\} \cap [a \wedge b, c \vee d] = \emptyset$. From this failure of (W) we will produce a failure of (W'). Since L is finite, we can assume that $c \vee (a \wedge b)$ covers c and $d \vee (a \wedge b)$ covers d. Let $e = c \wedge a \wedge b$ and $f = d \wedge a \wedge b$. Then $a \wedge b \leqq e$ or $a \wedge b \leqq f$ would imply $a \wedge b \leqq c$ or $a \wedge b \leqq d$, contrary to hypothesis. Therefore, if $e \vee f = a \wedge b$, then $\{a, b, e, f\}$ yields a failure of (W').

If $e \vee f \neq a \wedge b$, we claim $e \neq f$. Indeed, if $e = f$, then $(SD_\wedge)$ implies $e = (c \vee d) \wedge (a \wedge b) = a \wedge b$, whence $a \wedge b \leqq c$, again contrary to hypothesis. Thus, either $f \nleqq c$ or $e \nleqq d$. Suppose $e \nleqq d$, the other case being handled similarly. Then $d < e \vee d \leqq (a \wedge b) \vee d$. But $(a \wedge b) \vee d$ covers d, so $e \vee d = (a \wedge b) \vee d$. Thus, $e \leqq a \wedge b \leqq e \vee d \leqq c \vee d$, and it is easily seen that $\{a, b, e, d\} \cap [a \wedge b, e \vee d] = \emptyset$, which is a failure of (W'). This completes the proof.

4.6 *Remark.* It is clear from the proof that the assumption that $\mathscr{L}$ is finite could be replaced by the requirement that for every $x, y \in L$ with $x < y$ there exists $z \geqq x$ covered by y. That this latter requirement is essential is shown by the lattice in Figure 2, which satisfies $(SD_\wedge)$ and (W') but not (W).

4.7. *Remark.* The proof of Lemma 4.5 is essentially due to I. Rival and R. Antonius (unpublished) and is part of a proof that a finite lattice satisfying $(S\ D_\vee)$ and $(S\ D_\wedge)$ satisfies (W) if and only if it contains no sublattice isomorphic to either of those in Figure 3.

Finally by 4.4, 4.5, and 3.16, we have the main result of this section.

4.8 THEOREM. *A finite sharply transferable lattice satisfies* (W).

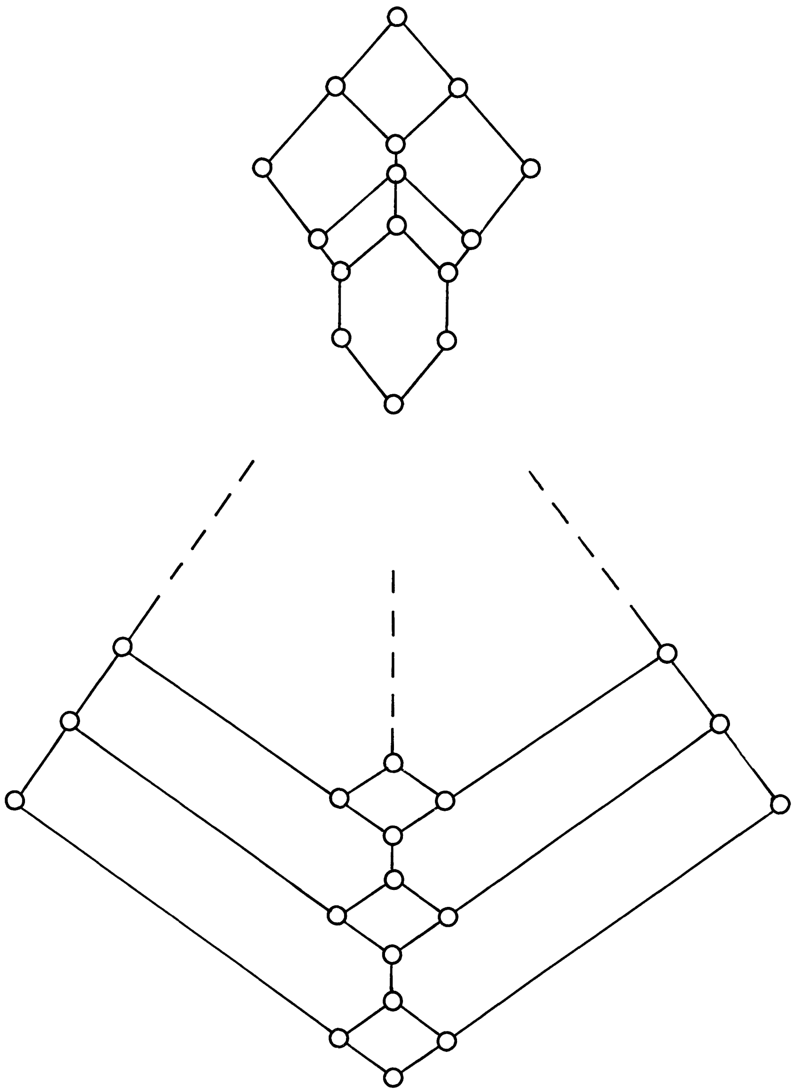

FIGURE 2

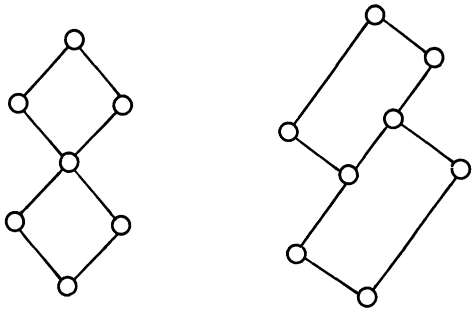

FIGURE 3

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The University of Manitoba,
Winnipeg, Manitoba