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WEIGHTED AVERAGING TECHNIQUES IN OSCILLATION THEORY FOR SECOND ORDER DIFFERENCE EQUATIONS

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ABSTRACT. We consider the self-adjoint second-order scalar difference equation (1) $\Delta(r_n\Delta x_n) + p_n x_{n+1} = 0$ and the matrix system (2) $\Delta(R_n\Delta X_n) + P_n X_{n+1} = 0$, where $\{r_n\}_0^\infty$, $\{p_n\}_0^\infty$ ($\{R_n\}_0^\infty$, $\{P_n\}_0^\infty$) are sequences of real numbers ($d \times d$ Hermitian matrices) with $r_n > 0$ ($R_n > 0$). The oscillation and nonoscillation criteria for solutions of (1) and (2), obtained in [3, 4, 10], are extended to a much wider class of equations by Riccati and averaging techniques.

1. **Introduction.** In a number of recent papers, [1, 3–10], the oscillation properties of solutions of the following self-adjoint second order difference equations have been extensively studied:

$$\Delta(r_n\Delta x_n) + p_n x_{n+1} = 0, \quad n \geqq 0 \tag{1.1}$$

$$\Delta(R_n\Delta X_n) + P_n X_{n+1} = 0, \quad n \geqq 0. \tag{1.2}$$

Here Δ is the forward difference operator $\Delta x_n = x_{n+1} - x_n$ and, in (1.1) (resp. (1.2)), r_n, p_n (resp. R_n, P_n) denote sequences of real numbers (resp. $d \times d$ Hermitian matrices) with $r_n > 0$ ($R_n > 0$), $n = 0, 1, 2, \ldots$. We remark that Hermitian matrix inequalities $A > 0$ ($A \geqq 0$) are in the sense of positive (nonnegative) definiteness.

A real solution $x = \{x_n\}_{n=0}^\infty$ of (1.1) ($X = \{X_n\}_{n=0}^\infty$ of (1.2)) is said to be nonoscillatory if there exists $N \geqq 0$ such that $x_n x_{n+1} > 0$ ($X_n^* R_n X_{n+1} > 0$) for all $n \geqq N$ and is oscillatory otherwise. Either all solutions of (1.1) ((1.2)) are oscillatory or none are, (cf. [1], [3]).

As usual, a solution of (1.2) is said to be prepared in case

$$X_n^* R_n X_{n+1} = X_{n+1}^* R_n X_n\ . \tag{1.3}$$

In this paper, we will employ the discrete version of the Riccati equation to extend the results of [3, 4] to the more general case.

First, we introduce the Riccati equations. If x is a solution of (1.1) (X for (1.2)) with $x_n x_{n+1} > 0$ ($X_n^* R_n X_{n+1} > 0$) for $n \geqq N \geqq 0$, let

$$u_n = \frac{r_n\Delta x_n}{x_n}; \quad (U_n = (R_n\Delta X_n)X_n^{-1}).$$

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Then we have

$$\Delta u_n + \frac{u_n^2}{u_n + r_n} + p_n = 0, \quad n \geqq N \tag{1.4}$$

and

$$\Delta U_n + U_n(U_n + R_n)^{-1} U_n + P_n = 0, \quad n \geqq N. \tag{1.5}$$

It is known (cf. [1, 3]) that (1.1) ((1.2)) is nonoscillatory if and only if there exists a solution of (1.4) ((1.5)) such that $r_n + u_n > 0$ ($R_n + U_n > 0$).

2. **Scalar Case.** We will denote by F the set of all sequences of real numbers $b = \{b_n\}_{n=0}^{\infty}$ with $0 \leqq b_n \leqq 1$ and $\sum_{n=0}^{\infty} b_n = +\infty$.

Let

$$B_n = \sum_{j=0}^{n} b_j, \quad B_{n,m} = \sum_{j=m}^{n} b_j.$$

We introduce the following conditions:

(A_1^α) $$\limsup_{n\to\infty} B_n^{-(\frac{3}{2}+\alpha)} \sum_{j=0}^{n} b_j r_{j+1} < +\infty;$$

(A_2^α) $$\limsup_{n\to\infty} B_n^{-(1+\alpha)} \sum_{j=0}^{n} b_j r_{j+1} < +\infty;$$

(A_3^α) $$\limsup_{n\to\infty} B_n^{-\alpha} r_n < +\infty;$$

where $\alpha \geqq 0$. Obviously $(A_3^\alpha) \Longrightarrow (A_2^\alpha) \Longrightarrow (A_1^\alpha)$.

THEOREM 2.1. *Assume that* (A_1^α) *holds for some* $b \in F$ *and that (1.1) is nonoscillatory. Then the following are equivalent:*

(i) $\limsup_{n\to\infty} B_n^{-1-\alpha} \left| \sum_{k=0}^{n} b_k \sum_{j=0}^{k} p_j \right| < +\infty$;

(ii)

$$\liminf_{n\to\infty} B_n^{-1-\alpha} \sum_{k=0}^{n} b_k \sum_{j=0}^{k} p_j > -\infty; \tag{2.1}$$

(iii) For any solution x *of (1.1) with* $x_n x_{n+1} > 0$, $n \geqq N$ *for some* $N \geqq 0$, *the sequence* $u_n = \frac{r_n \Delta x_n}{x_n}$ *satisfies:*

$$\limsup_{n\to\infty} B_{n,N}^{-1-\alpha} \sum_{k=N}^{n} b_k \sum_{j=N}^{k} \frac{u_j^2}{u_j + r_j} < +\infty \tag{2.2}$$

PROOF. Obviously, (i) $\Longrightarrow$ (ii). Now we prove (ii) $\Longrightarrow$ (iii). Suppose not, let $\rho_n = \frac{u_n^2}{u_n + r_n}$. Then we have

$$\limsup_{n\to\infty} B_{n,N}^{-1-\alpha} \sum_{k=N}^{n} b_k \sum_{j=N}^{k} \rho_j = +\infty.$$

We may assume, without loss of generality, that $n > N$ is sufficiently large so that $B_{n,N} > 0$ in what follows. Then from (1.4) we have:

$$(2.3)\qquad B_{n,N}^{-1-\alpha}\sum_{k=N}^{n} b_k \sum_{j=N}^{k}\rho_j + B_{n,N}^{-1-\alpha}\sum_{k=N}^{n} b_k\sum_{j=N}^{k} p_j - B_{n,N}^{-\alpha}u_N = B_{n,N}^{-1-\alpha}\sum_{k=N}^{n} b_k(-u_{k+1}).$$

Therefore

$$(2.4)\qquad \limsup_{n\to\infty} B_{n,N}^{-\alpha-1}\sum_{k=N}^{n} b_k(-u_{k+1}) = +\infty.$$

Hence,

$$(2.5)\qquad \limsup_{n\to\infty} B_{n,N}^{-1-\alpha}\sum_{k=N}^{n} b_k|u_{k+1}| = +\infty.$$

Dividing (2.3) by $B_{n,N}^{1/2}$, then in view of (ii), $(A_1^{\alpha},)$ and the fact that $-u_{k+1} < r_{k+1}$, we have

$$(2.6)\qquad \limsup_{n\to\infty} B_{n,N}^{-\frac{3}{2}-\alpha}\sum_{k=N}^{n} b_k\sum_{j=N}^{k}\rho_j < +\infty.$$

Now for any fixed $n \geqq N+1$, we can find $m > n$ such that $B_{n,N} \leqq B_m - B_n \leqq 2B_{n,N}$ and hence $B_{m,N} \leqq 3B_{n,N}$. Therefore combining with (2.6) we obtain:

$$(2.7)\qquad \begin{aligned} B_{n,N}^{-\frac{1}{2}-\alpha}\sum_{j=N}^{n}\rho_j &= B_{n,N}^{-\frac{3}{2}-\alpha}B_{n,N}\sum_{j=N}^{n}\rho_j \\ &\leqq B_{n,N}^{-\frac{3}{2}-\alpha}\sum_{k=n+1}^{m} b_k\sum_{j=N}^{k}\rho_j \\ &\leqq 3^{\frac{3}{2}}B_{m,N}^{-\frac{3}{2}-\alpha}\sum_{k=N}^{m} b_k\sum_{j=N}^{k}\rho_j \\ &\leqq M < +\infty. \end{aligned}$$

Let

$$a_n = \begin{cases} u_n + r_n & \text{if } \ u_n \neq 0 \\ 0 & \text{if } \ u_n = 0. \end{cases}$$

Then we have $r_n \geqq a_n - u_n$, so

$$(2.8)\qquad B_{n,N}^{-\frac{3}{2}-\alpha}\sum_{k=N}^{n} b_k r_{k+1} \geqq B_{n,N}^{-\frac{3}{2}-\alpha}\sum_{k=N}^{n} b_k a_{k+1} + B_{n,N}^{-\frac{3}{2}-\alpha}\sum_{k=N}^{n} b_k(-u_{k+1}).$$

Since $B_{n+1,N}^{\frac{1}{2}+\alpha} = (B_{n,N} + b_{n+1})^{\frac{1}{2}+\alpha} \leqq 2^{\frac{1}{2}+\alpha}B_{n,N}^{\frac{1}{2}+\alpha}$ and

$$\begin{aligned} \Big(\sum_{k=N}^{n} b_k|u_{k+1}|\Big)^2 &= \Big(\sum_{k=N}^{n} b_k\sqrt{a_{k+1}\rho_{k+1}}\,\Big)^2 \\ &\leqq \Big(\sum_{k=N}^{n} b_k a_{k+1}\Big)\Big(\sum_{k=N}^{n} b_k\rho_{k+1}\Big) \\ &\leqq \sum_{k=N}^{n}\rho_{k+1}\sum_{k=N}^{n} b_k a_{k+1} \\ &\leqq MB_{n,N}^{\frac{1}{2}+\alpha}\sum_{k=N}^{n} b_k a_{k+1}, \end{aligned}$$

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it follows that

$$B_{n,N}^{-\frac{3}{2}-\alpha}\sum_{k=N}^{n} b_k a_{k+1} \geqq \frac{1}{M}\Big(B_{n,N}^{-1-\alpha}\sum_{k=N}^{n} b_k|u_{k+1}|\Big)^2. \tag{2.9}$$

By (2.8–2.9), we get:

$$\begin{aligned} B_{n,N}^{-\frac{3}{2}-\alpha}\sum_{k=N}^{n} b_k r_{k+1} &\geqq B_{n,N}^{-\frac{3}{2}-\alpha}\sum_{k=N}^{n} b_k a_{k+1} + B_{n,N}^{-\frac{3}{2}-\alpha}\sum_{k=N}^{n}(-u_{k+1})b_k \\ &\geqq \frac{1}{M}\Big(B_{n,N}^{-1-\alpha}\sum_{k=N}^{n} b_k|u_{k+1}|\Big)^2 + B_{n,N}^{-\frac{3}{2}-\alpha}\sum_{k=N}^{n} b_k(-u_{k+1}). \end{aligned}$$

Then (2.4) and (2.5) imply that

$$\limsup_{n\to\infty} B_{n,N}^{-\frac{3}{2}-\alpha}\sum_{k=N}^{n} b_k r_{k+1} = +\infty$$

which contradicts (A_1^α). Therefore, (ii) $\Longrightarrow$ (iii). Next we prove (iii) $\Longrightarrow$ (i). As in (2.7) and using (2.2), we first observe that $B_{n,N}^{-\alpha}\sum_{j=N}^{n}\rho_j \leqq M$, which means $\sum_{j=N}^{n}\rho_j \leqq B_{n,N}^{\alpha}M$. Next let

$$v_n = \sum_{j=N}^{n} b_j|u_{j+1}|.$$

Then from (iii), we have

$$\begin{aligned} v_n^2 &= \Big(\sum_{j=N}^{n}\sqrt{\rho_{j+1}}\,\sqrt{r_{j+1}+u_{j+1}}\,b_j\Big)^2 \\ &\leqq \sum_{j=N}^{n} b_j\rho_{j+1}\sum_{j=N}^{n} b_j(u_{j+1}+r_{j+1}) \\ &\leqq \sum_{j=N}^{n}\rho_{j+1}\sum_{j=N}^{n} b_j(u_{j+1}+r_{j+1}) \\ &\leqq MB_{n,N}^{\alpha}\Big(v_n+\sum_{j=N}^{n} b_j r_{j+1}\Big) \\ &\leqq 2MB_{n,N}^{\alpha}\max\Big\{v_n, \sum_{j=N}^{n} b_j r_{j+1}\Big\}. \end{aligned} \tag{2.11}$$

Thus

$$v_n \leqq \max\Big\{2MB_{n,N}^{\alpha}, \Big(2MB_{n,N}^{\alpha}\sum_{j=N}^{n} b_j r_{j+1}\Big)^{1/2}\Big\}. \tag{2.12}$$

From (A_1^α) and (2.12), we have

$$\lim_{n\to\infty} B_{n,N}^{-1-\alpha} v_n = 0. \tag{2.13}$$

Thus, from (2.3) and (2.13), the proof is complete.

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REMARK. If $\alpha = 0$, one can show $B_{n,N}^{-1} \sum_{k=N}^{n} b_k \sum_{j=N}^{k} \rho_j$ is monotone with respect to n. This means

$$\limsup_{n\to\infty} B_{n,N}^{-1} \sum_{k=N}^{n} b_k \sum_{j=N}^{k} \rho_j = \lim_{n\to\infty} B_{n,N}^{-1} \sum_{k=N}^{n} b_k \sum_{j=N}^{k} \rho_j.$$

Therefore (i) in Theorem 2.1 can be replaced by the condition $\lim_{n\to\infty} B_{n,N}^{-1} \sum_{k=N}^{n} b_k \sum_{j=N}^{k} p_j$ exists (constant). This is the same as Theorem 2.11 of [3].

COROLLARY 2.2. *If (A_1^α) and (2.1) hold for some $b \in F$, then equation (1.1) is oscillatory provided:*

$$\limsup_{n\to\infty} B_n^{-1-\alpha} \sum_{k=0}^{n} b_k \sum_{j=0}^{k} p_j = +\infty. \tag{2.14}$$

EXAMPLE 2.1. Consider $\Delta(n\Delta x_n) + n^{-\beta} x_{n+1} = 0$. Here $r_n = n$, $p_n = n^{-\beta}$, $\beta \geqq 0$. Let $b_j = 1$. Then $B_n = n$. Choosing $\alpha = \frac{1}{2}$, we see that (A_1^α) holds. Evidently, if $\beta < \frac{1}{2}$, we have $\frac{1}{n^{3/2}} \sum_{k=1}^{n} \sum_{j=1}^{k} p_j \longrightarrow +\infty$ as $n \longrightarrow \infty$. By Corollary 2.2, we know that this equation is oscillatory. Through a computation, we note that with increasing β, the "distance between zeros" increases.

THEOREM 2.3. *Assume that (A_2^α) holds for some $b \in F$. If*

$$\limsup_{n\to\infty} B_n^{-1-\alpha} \sum_{k=1}^{n} b_k \sum_{j=1}^{k} p_j = +\infty,$$

then (1.1) is oscillatory.

PROOF. Suppose (1.1) is nonoscillatory. From (A_2^α) we have

$$M_2 := \limsup_{n\to\infty} B_{n,N}^{-1-\alpha} \sum_{k=N}^{n} b_k r_{k+1} \geqq \limsup_{n\to\infty} B_{n,N}^{-1-\alpha} \sum_{k=N}^{n} b_k(-u_{k+1}).$$

Therefore, from (2.3), we have

$$\begin{aligned} M_2 &\geqq \limsup_{n\to\infty} B_{n,N}^{-1-\alpha} \sum_{k=N}^{n} b_k(-u_{k+1}) \\ &\geqq \liminf_{n\to\infty} B_{n,N}^{-1-\alpha} \sum_{k=N}^{n} b_k \sum_{j=N}^{k} \rho_j + M_3 \\ &\quad + \limsup_{n\to\infty} B_{n,N}^{-1-\alpha} \sum_{k=N}^{n} b_k \sum_{j=N}^{k} p_j \\ &\geqq \limsup_{n\to\infty} B_{n,N}^{-1-\alpha} \sum_{k=N}^{n} b_k \sum_{j=N}^{k} p_j + M_3. \end{aligned}$$

for some constant M_3 ($= 0$ if $\alpha > 0$). This contradiction shows that (1.1) is oscillatory.

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EXAMPLE 2.2. In (1.1), let $\alpha > 0$ and let

$$r_n = \begin{cases} n^\alpha & \text{if} \quad n = \text{odd} \\ n^{\alpha+2} & \text{if} \quad n = \text{even}, \end{cases}$$

$$p_n = \begin{cases} 1 & \text{if} \quad n = 1 \\ (-1)^{[\frac{n}{2}]} n^{\beta_0} - \sum_{k=1}^{n-1} p_k, & n = 2, 3, \dots , \end{cases}$$

$$b_n = \begin{cases} 1 & n \quad \text{even} \\ 0 & n \quad \text{odd}. \end{cases}$$

Then (A_2^α) holds. Furthermore, we have $\sum_{j=1}^{n} p_j = (-1)^{[\frac{k}{2}]} n^{\beta_0}$ and

$$\limsup_{n\to\infty} B_n^{-\beta_0} \sum_{k=1}^{n} b_k \sum_{j=1}^{k} p_j = \frac{1}{2} > 0.$$

Hence, by Theorem 2.3, if $\beta_0 > 1 + \alpha$, (1.1) is oscillatory. This result is not obtainable by Theorem 1 or by any of the results of the references.

THEOREM 2.4. *If (A_3^α) holds for some $b \in F$, then (1.1) is oscillatory provided*

$$\limsup_{n\to\infty} B_n^{-\alpha} \sum_{k=1}^{n} p_j = +\infty.$$

PROOF. Suppose not. From (1.4), we have

$$u_{n+1} - u_N + \sum_{k=N}^{n} \rho_k + \sum_{k=N}^{n} p_k = 0.$$

Now in view of (A_3^α) and the fact that $-u_{n+1} < r_{n+1}$, we have

$$B_{n,N}^{-\alpha} \sum_{k=N}^{n} p_k \leqq -B_{n,N}^{-\alpha} \sum_{k=N}^{n} \rho_k + m_1 \leqq m.$$

This contradiction shows that (1.1) is oscillatory.

3. **Matrix Case.** Set $a_n = \lambda_d(R_n)$, $A_n = \lambda_1(R_n)$, where we suppose that the eigenvalues of R_n are ordered with $\lambda_1(R_n) \geqq \dots \geqq \lambda_d(R_n)$.

We introduce the following conditions which will be needed in the results to follow: Suppose there exists $b \in F$ such that:

(H)
$$\limsup_{n\to\infty} B_n^{-\frac{1}{2}} \left(\frac{A_n}{a_n}\right) < +\infty;$$

$(\bar{A}_1^\alpha)$
$$\limsup_{n\to\infty} B_n^{-\frac{3}{2}-\alpha} \sum_{j=0}^{n} b_j A_{j+1} < +\infty;$$

$(\bar{A}_2^\alpha)$
$$\limsup_{n\to\infty} B_n^{-1-\alpha} \sum_{j=0}^{n} b_j A_{j+1} < +\infty;$$

$(\bar{A}_3^\alpha)$
$$\limsup_{n\to\infty} B_n^{-\alpha} A_n < +\infty;$$

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Here $\alpha \geqq 0$ and clearly, $(\bar{A}_3^\alpha) \Longrightarrow (\bar{A}_3^\alpha) \Longrightarrow (\bar{A}_1^\alpha)$.

In a similar way as in [4] and in the scalar case above, we can prove the following results:

THEOREM 3.1. *If R satisfies (H) and $(\bar{A}_1^\alpha)$ for some $b \in F$, and equation (1.2) is nonoscillatory, then the following are equivalent:*

(i)

(3.1) *If $C_n := B_n^{-1-\alpha} \sum_{k=0}^{n} b_k \sum_{j=0}^{k} P_j$, then $|(C_n)_{ij}| \leqq m$, for some $m > 0$ and for $n \geqq N$, $i,j = 1,\ldots,d$.*

(ii)

$$\limsup_{n\to\infty} B_n^{-1-\alpha} \sum_{k=0}^{n} b_k \sum_{j=0}^{k} tr\, P_j > -\infty; \tag{3.2}$$

(iii) For any prepared solution X of (1.2) with $X_{n+1}^* R_n X_n > 0$ for $n \geqq N$ for some $N \geqq 0$, the sequence $U_n = (R_n \Delta X_n)^* X_n^{-1}$ satisfies:

$$B_{n,N}^{-1-\alpha} \sum_{k=N}^{n} b_k \sum_{j=N}^{k} \bar{\rho}_j \leqq M_0; \quad \text{(constant Hermitian matrix)}.$$

(Here $\bar{\rho}_j = U_j^*(U_j + R_j)^{-1} U_j$, for any $n \geqq N$).

COROLLARY 3.2. *Suppose that R satisfies $(\bar{A}_1^\alpha)$ for some $b \in F$ and (H), and P satisfies (3.2), then (1.2) is oscillatory provided either*

$$\limsup_{n\to\infty} B_n^{-1-\alpha} \lambda_1\Big(\sum_{k=0}^{n} b_k \sum_{j=0}^{k} P_j\Big) = +\infty.$$

or

$$\limsup_{n\to\infty} |(C_n)_{i_0 j_0}| = +\infty$$

for some $1 \leqq i_0, j_0 \leqq d$.

REMARK. In [4], condition (H) is simply $\limsup_{n\to\infty} \frac{A_n}{a_n} < +\infty$. The above results also improve the results of [10].

EXAMPLE 3.1. In (1.2), let $R_n = \begin{pmatrix} \sqrt{n} & \frac{1}{2} \\ \frac{1}{2} & n \end{pmatrix}$, $P_n = \begin{pmatrix} 1 & (-1)^n \\ (-1)^n & n^{-\frac{1}{3}} \end{pmatrix}$ or $P_n = \begin{pmatrix} 0 & n^{-\frac{1}{3}} \\ n^{-\frac{1}{3}} & 0 \end{pmatrix}$. By Corollary 3.2, it is easy to show that (1.2) is oscillatory. This may not be concluded from any of the known oscillation criteria, as far as the authors are aware.

THEOREM 3.3. *Let $(\bar{A}_2^\alpha)$ hold for some $b \in F$. Then equation (1.2) is oscillatory provided*

$$\limsup_{n\to\infty} B_n^{-1-\alpha} \lambda_1\Big(\sum_{k=0}^{n} b_k \sum_{j=0}^{k} P_j\Big) = +\infty.$$

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PROOF. From (1.5), we have

$$B_{n,N}^{-1-\alpha}\sum_{k=N}^{n} b_k \sum_{j=N}^{k} P_j = B_{n,N}^{-\alpha} U_N + B_{n,N}^{-1-\alpha}\sum_{k=N}^{n} b_k(-U_{k+1}) - B_{n,N}^{-1-\alpha}\sum_{k=N}^{n} b_k \sum_{j=N}^{k} \bar{\rho}_j.$$

By Weyl's inequality [11], and $U_n + R_n > 0$ and $(\bar{A}_2^\alpha)$ we have:

$$\begin{aligned}\lambda_1(B_{n,N}^{-1-\alpha}\sum_{k=N}^{n} b_k \sum_{j=N}^{k} P_j) &\leqq \lambda_1(B_{n,N}^{-\alpha} U_N) + \lambda_1\Big(B_{n,N}^{-1-\alpha}\sum_{k=N}^{n} b_k(-U_{k+1})\Big) \\ &\quad + \lambda_1(-B_{n,N}^{-1-\alpha}\sum_{k=N}^{n} b_k \sum_{j=N}^{k} \bar{\rho}_j) \\ &\leqq \lambda_1(B_{n,N}^{-\alpha} U_N) + \lambda_1(B_{n,N}^{-1-\alpha}\sum_{k=N}^{n} b_k R_{k+1}) \\ &\leqq \bar{M}_2 + B_{n,N}^{-1-\alpha}\sum_{k=N}^{n} b_k A_{k+1} \\ &\leqq \bar{M}_2 < +\infty.\end{aligned}$$

for some constant $\bar{M}_2$. The contradiction shows that (1.2) is oscillatory.

In the same way, we can prove:

THEOREM 3.4. *Let $(\bar{A}_3^\alpha)$ hold for some $b \in F$. Then (1.2) is oscillatory provided*

$$\limsup_{n\to\infty} B_n^{-\alpha}\lambda_1\Big(\sum_{k=0}^{n} P_k\Big) = +\infty.$$

REMARK. This result may also be concluded from the results of [10].

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