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FINITE GROUPS ADMITTING FIXED POINT FREE AUTOMORPHISMS OF ORDER pq

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By the results of Rickman [7] and Ralston [6], a finite group G admitting a fixed point free automorphism α of order pq, where p and q are primes, is soluble. If $p=q$, then $|G|$ is necessarily coprime to $|\alpha|$, and it follows from Berger [1] that G has Fitting height at most 2, the composition length of $\langle\alpha\rangle$. The purpose of this paper is to prove a corresponding result in the case when $p\neq q$.

Following Thompson [9], Shamash and Shult [8] obtained an exponential bound for the Fitting heights of finite soluble groups admitting cyclic fixed point free automorphism groups. For the special case we consider in the present note, the bound (namely, 2) we obtain follows by no means from that implied in [8]. It is also easy to see that this bound is best possible (see e.g. a construction of Gross [3]).

All groups considered are finite, and the notation used is standard.

We shall need the following well-known lemmas. For the convenience of the reader, we include the proofs.

Lemma 1. *Let H be a finite group and let V be an H-module over a field k. If $\bar{k}$ is an extension field of k and $\bar{V}=\bar{k}\bigotimes_k V$, then $\mathbf{C}_{\bar{V}}(K)=\bar{k}\bigotimes_k C_V(K)$, for any $K\leqq H$.*

Proof. This can be shown easily by writing $\bar{V}=\bigoplus_{\omega\in\Omega}(\omega\otimes V)$, where Ω is a k-basis for $\bar{k}$.

Lemma 2. *Let A be an abelian normal subgroup of a finite group G. Let W be an A-homogeneous component of V_A for an irreducible G-module V over a splitting field k for A. Put $B=\mathbf{C}_A(W)$ and assume that $B\neq A$. Then*

(a) $V\cong(W_{\mathbf{C}_G(A/B)})^G$.

(b) *If A is a self-centralizing minimal normal subgroup of G possessing a cyclic complement C in G, then $\mathbf{C}_G(A/B)=A$. Consequently $V\cong(W_A)^G$ and $V_C\cong\bigoplus k[C]$. Hence C has a fixed point on V and (in view of Lemma 1) this conclusion also holds without the hypothesis on k.*

Proof. (a) In view of Clifford's Theorem it suffices to show that $\mathbf{N}_G(W)=\mathbf{C}_G(A/B)$. Clearly, $\mathbf{C}_G(A/B)\leqq\mathbf{N}_G(W)$. On the other hand the homogenous action on W of the

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cyclic group A/B forces $A/B \underset{N_G(W)}{\cong} AC_{N_G(W)}(W)/C_{N_G(W)}(W)$ to be a central factor of $N_G(W)/C_{N_G(W)}(W)$; that is to say, $N_G(W) \leqq C_G(A/B)$.

(b) Since $C_C(A/B) \trianglelefteq C, C_A(C_C(A/B)) = A$, it follows that $C_C(A/B) = 1$ and $C_G(A/B) = A$. Now $V \cong (W_A)^G$ is a consequence of (a), while $V_C \cong \cdot \oplus k[C]$ is obtained by application of Mackey's Theorem. In particular C has a fixed point on V.

Theorem. *A finite group G admitting a fixed point free automorphism α of order pq, where p and q are distinct primes, is metanilpotent.*

Proof. In view of [6], G is soluble. Let G be a minimal counter-example. Then every α-invariant proper subgroup of G is metanilpotent. Moreover, if we put $H = G\langle\alpha\rangle$ (the semidirect product), then by [4, V.8.10] G/N is metanilpotent for every non-trivial normal subgroup N of H which is contained in G. In what follows these observations will be used without further reference.

Clearly $\langle\alpha\rangle$ is a Carter subgroup of H. We set $\alpha = \alpha_p\alpha_q$, where $|\alpha_p| = p$ and $|\alpha_q| = q$.

(1) *$V = F(G) = F(H)$ is the unique minimal normal subgroup of the soluble group H; in particular $V = C_H(V)$ is an elementary abelian r-group for some prime r, and is complemented in H by a maximal subgroup X (so that V is a faithful and irreducible X-module over $GF(r)$), which can be chosen such that $\langle\alpha\rangle \leqq X$:*

By the minimality of G, there exists precisely one minimal normal subgroup V of H contained in G. Assume that V^* is a minimal normal subgroup of H not contained in G. Then $V^* \leqq Z(H)$, for V^* is H-isomorphic to a central chief factor of H in $H/G \cong \langle\alpha\rangle$. Thus V^* is contained in the Carter subgroup $\langle\alpha\rangle$ of H. Hence one of the Sylow subgroups of $\langle\alpha\rangle$ (namely V^*) centralises G, while the other necessarily acts fixed point freely on G. The nilpotency of G now follows from a celebrated theorem of Thompson. Thus V^* cannot exist, i.e. V is the unique minimal normal subgroup of H. It follows that $F(H) = V$ and by [4, III.4.2(c)]$\Phi(H) = 1$. In particular H splits over V: $H = XV$ with $X \cap V = 1$. In view of $H = XF(H)$, a Carter subgroup of X is contained in a Carter subgroup of H (using Gaschütz [2, II.13]), i.e. in a conjugate of $\langle\alpha\rangle$. From $X/X \cap G \cong XG/G = H/G \cong \langle\alpha\rangle$ we infer that a Carter subgroup of X (which covers $X/X \cap G$) has order $|\langle\alpha\rangle|$ and so is a conjugate of $\langle\alpha\rangle$. Without loss of generality we set $\langle\alpha\rangle \leqq X$.

(2) *Put $Y = X \cap G \cong G/V$. Then $Y = TS$, where $S = O_s(Y)$ is a special s-group for some prime $s \neq r$, $T \in Syl_t(Y)$ for some prime $t \neq s$ is α-invariant, and $[T, S^*] = 1$ for every $T\langle\alpha\rangle$-invariant proper subgroup S^* of S. Further, $Y/F(Y)$ is a t-chief factor of X, and $F(Y)/(O_t(Y) \times \Phi(S)) \cong S/\Phi(S)$ is a s-chief factor of X satisfying $C_X(S/\Phi(S)) = F(Y)$ (i.e. $\langle\alpha\rangle\tilde{T}$ acts faithfully and irreducibly on $\tilde{S} = S/\Phi(S)$, where $\tilde{T} = T/O_t(Y)$). We may assume that $[\tilde{T}, \alpha_p] = 1$. Then $C_{\tilde{S}}(\alpha_q) \neq 1$ and α_p acts fixed point freely on $\tilde{S}$; in particular $p \neq s$:*

Let G/G_0 be a chief factor of H. Then G/G_0 is a t-group for some prime t, and G_0 is metanilpotent by induction. In fact $G_0 = F_2(G)$. We have $G_0 \cap Y = F(Y)$ and $Y/F(Y) \cong G/G_0$. From (1) we see that $F(Y)$ is an r'-group, where r is the prime dividing $|V|$. Since $Y/F(Y) \neq 1$ is a t-group, there exists a prime $s \neq t$ and a $T \in Syl_t(Y)$ such that

$T\mathbf{O}_s(Y) \in \mathrm{Hall}_{\{s,t\}}(Y)$ is not nilpotent. As in (1) we obtain without loss of generality that $\langle\alpha\rangle \leqq \mathbf{N}_X(T) \leqq \mathbf{N}_X(T\mathbf{O}_s(Y))$. From $s \neq r$ it is easy to deduce that $T\mathbf{O}_s(Y)V$ is not metanilpotent. Minimality of G now implies that $Y = T\mathbf{O}_s(Y)$.

Let Y_0 be an α-invariant proper subgroup of Y. Assume that Y_0 is not nilpotent. In view of $Y = T\mathbf{O}_s(Y)$ with $T \in Syl_t(Y)$, this implies the existence of a non-central s-chief factors of Y_0 (below $\mathbf{O}_s(Y_0)$). Let $V = V_n \geqq \cdots \geqq V_0 = 1$ be a Y_0-composition series of V. From $V = \mathbf{C}_H(V)$ we have that $\bigcap_{i=1}^n \mathbf{C}_{Y_0}(V_i/V_{i-1}) = \mathbf{O}_r(Y_0)$. Now $s \neq r$ yields an $i \in \{1, \ldots, n\}$ such that $Y_0/\mathbf{C}_{Y_0}(V_i/V_{i-1})$ is not nilpotent. Consequently Y_0V is not metanilpotent, which is against the minimality of G. Thus every α-invariant proper subgroup of Y is nilpotent. In particular $[T, S^*] = 1$ for every $T\langle\alpha\rangle$-invariant proper subgroup S^* of S.

Now one can modify the proof of the Hall-Higman reduction (see [**4**, III.13.5]) to obtain that $S = \mathbf{O}_s(Y)$ is a special s-group, $T\langle\alpha\rangle$ acts irreducibly on $S/\mathbf{\Phi}(S)$, and T acts nontrivially on this group. Set $\tilde{S} = S/\mathbf{\Phi}(S)$ and $\tilde{T} = T/\mathbf{O}_t(Y)$. If $\mathbf{C}_{\tilde{T}\langle\alpha\rangle}(\tilde{S}) \neq 1$, then as in the proof of (1), $\mathbf{C}_{\tilde{T}\langle\alpha\rangle}(\tilde{S})$ is a central subgroup of $\tilde{T}\langle\alpha\rangle$ and so is contained in the Carter subgroup $\langle\alpha\rangle$. We get a contradiction against Thompson's theorem. Thus $\mathbf{C}_{\tilde{T}\langle\alpha\rangle}(\tilde{S}) = 1$. By Lemma 2(b), $\mathbf{C}_{\langle\alpha\rangle}(\tilde{T}) \neq 1$: observe that $\tilde{T} = Y/\mathbf{F}(Y)$, a chief factor of $X = Y\langle\alpha\rangle$. We set without loss of generality $\mathbf{C}_{\langle\alpha\rangle}(\tilde{T}) = \langle\alpha_p\rangle$. Application of Lemma 2(b) to a (faithful) $\langle\alpha_q\rangle\tilde{T}$-composition factor of $\tilde{S}$ now yields $\mathbf{C}_{\tilde{S}}(\alpha_q) \neq 1$. Finally, $\mathbf{C}_{\tilde{S}}(\alpha_p) = 1$ follows from $\langle\alpha_p\rangle \trianglelefteq \langle\alpha\rangle\tilde{T}$.

(3) *Set* $k = \mathbf{GF}(r)$ *and let* $n \in \mathbb{N}$ *be such that* $\bar{k} = \mathbf{GF}(r^n)$ *is a splitting field for all subgroups of* X. *Consider an irreducible* $\bar{k}[X]$*-submodule* W *of* $V \bigotimes_k \bar{k}$. *Then* $\mathbf{C}_X(W) = 1$ *and* $\mathbf{C}_W(\alpha) = 0$:

First of all, note that the existence of n is guaranteed by a theorem of Brauer (cf. [**5**, VII.2.6.b]). Further, Lemma 1 yields $\mathbf{C}_{V \otimes_k \bar{k}}(X_0) = \mathbf{C}_V(X_0) \bigotimes_k \bar{k}$ for all $X_0 \leqq X$. Thus $\mathbf{C}_X(W) \trianglelefteq X$ centralises a non-trivial k-subspace of the irreducible $k[X]$-module V and hence acts trivially on V. That is, $\mathbf{C}_X(W) \leqq \mathbf{C}_X(V) = 1$ (see (1)). Finally $\mathbf{C}_W(\alpha) \leqq \mathbf{C}_{V \otimes_k \bar{k}}(\alpha) = \mathbf{C}_V(\alpha) \bigotimes_k \bar{k} = 0$.

(4) $\mathbf{\Phi}(S) \neq 1$:

Assume false. Then $S = \tilde{S}$ is elementary abelian, and is faithful and irreducible module over $\mathbf{GF}(s)$ for $\tilde{T}\langle\alpha\rangle$. Let W be as in (3). Consider the decomposition $W_S = W_1 \oplus \cdots \oplus W_h$ of the $\bar{k}[S]$-module W_S into homogeneous components W_i and put $S_i = \mathbf{C}_S(W_i)$ (so that S_i is the kernel of S on any irreducible component V_i of W_i). Clearly, $\bigcap_{i=1}^h S_i = \mathbf{C}_S(W) = 1$, $\bigcap_{i=1}^h \mathbf{C}_X(S/S_i) = \mathbf{C}_X(S) = \mathbf{O}_t(Y)S$. From this we get an $i \in \{1, \ldots, h\}$ such that $\alpha_q \notin \mathbf{C}_X(S/S_i)$ while $\alpha_p \notin \mathbf{C}_X(S/S_i)$ is a consequence of (2) and $S = \tilde{S}$. Thus $\mathbf{N}_X(W_i) \cap \langle\alpha\rangle = 1$, and by taking the elements of $\langle\alpha\rangle$ as part of a transversal to $\mathbf{N}_X(W_i)$, we see that the induced module $W_i^{ST\langle\alpha\rangle}|_{\langle\alpha\rangle}$ contains a free $k[\langle\alpha\rangle]$-module. This gives a fixed point of α in W, a contradiction.

(5) $\mathbf{C}_W(\alpha_q) = 0$, *where* W *is as in* (3):

Recall that $\mathbf{\Phi}(S) \neq 1$ by (4). Let $W_{\mathbf{\Phi}(S)} = W_1 \oplus \cdots \oplus W_h$ be the decomposition of the restriction of W to $\mathbf{\Phi}(S)$ into $\mathbf{\Phi}(S)$-homogeneous components W_i. By Clifford's Theorem, $W \cong ((W_1)_A)^X$, where $A = \mathbf{N}_X(W_1) \geqq Y$, as (by (2)) $\mathbf{\Phi}(S) \leqq \mathbf{Z}(Y)$. Moreover W_1 is an irreducible A-module. By Mackey's Theorem, as $X=\langle\alpha\rangle Y=\langle\alpha\rangle A$,

$$W_{\langle\alpha\rangle\Phi(S)}\cong((W_1)_A^X)_{\langle\alpha\rangle\Phi(S)}\cong((W_1)_{A\cap\langle\alpha\rangle\Phi(S)})^{\langle\alpha\rangle\Phi(S)}=((W_1)_{(A\cap\langle\alpha\rangle)\Phi(S)})^{\langle\alpha\rangle\Phi(S)},$$

where $A\cap\langle\alpha\rangle\in\{1,\langle\alpha_p\rangle,\langle\alpha_q\rangle,\langle\alpha\rangle\}$.

Suppose $A\cap\langle\alpha\rangle=1$. Then Mackey's Theorem yields the existence of a regular $\langle\alpha\rangle$-submodule of $W_{\langle\alpha\rangle}$ and thus a fixed point of $\langle\alpha\rangle$ on W, contradicting $\mathbf{C}_W(\alpha)=0$ (see (3)). $A\cap\langle\alpha\rangle=\langle\alpha\rangle$ would imply that $\Phi(S)$ acts homogeneously and thus (by (2) and the choice of W) as a scalar on $W=W_1$, where $A=\langle\alpha\rangle Y=X$. Consequently we would have $1\neq\Phi(S)\leqq\mathbf{Z}(X)$, as $\mathbf{C}_X(W)=1$, contradicting the fixed point free action of α on Y.

$A\cap\langle\alpha\rangle=\langle\alpha_q\rangle$ gives the desired conclusion $\mathbf{C}_W(\alpha_q)=0$ as follows. If $\mathbf{C}_W(\alpha_q)\neq 0$, then we may assume that $\mathbf{C}_{W_1}(\alpha_q)\neq 0$; indeed from $W=((W_1)_{\langle\alpha_q\rangle Y})^{\langle\alpha\rangle Y}$, we see that each W_i is some $(W_1)^{\alpha_p^j}=W_1\otimes\alpha_p^j$, where $j\in\{0,1,\dots,p-1\}$. Further, $[\alpha_p,\alpha_q]=1$ shows that each W_i is α_q-invariant. This gives the decomposition of $\mathbf{C}_W(\alpha_q)$ into α_q-invariant subspaces $\mathbf{C}_{W_1}(\alpha_q)\oplus\cdots\oplus\mathbf{C}_{W_h}(\alpha_q)$. On the other hand α_p induces a transitive permutation on $\{W_1,\dots,W_h\}$ and thus on $\{\mathbf{C}_{W_1}(\alpha_q),\dots,\mathbf{C}_{W_h}(\alpha_q)\}$ too. Now it is clear that there exists a trivial $\langle\alpha\rangle$-submodule of $(((\mathbf{C}_{W_1}(\alpha_q))_{\langle\alpha_q\rangle})^{\langle\alpha\rangle}$, and thus a fixed point of $\langle\alpha\rangle$ on $W=((W_1)_{\langle\alpha_q\rangle Y})^X$, a contradiction.

Consider lastly $A\cap\langle\alpha\rangle=\langle\alpha_p\rangle$. Then as in the previous case $\mathbf{C}_W(\alpha_p)=0$. Set $S_0=\mathbf{C}_S(\alpha_q)$. Then $S_0\neq 1$. This is obvious when $q=s$. When $q\neq s$, we have $S=\mathbf{C}_S(\alpha_q)[S,\alpha_q]$, where $[\alpha_q,S]\neq S$, as $\mathbf{C}_{\bar{S}}(\alpha_q)\neq 1$ (see (2)). Since α_p acts fixed point freely on both S_0 and W, the (semidirect) product S_0W admits a fixed point free automorphism of prime order, thus S_0W is nilpotent. From (1) and (2) we have $r\neq s$, so $S_0W=S_0\times W$, with $S_0\in Syl_s(S_0W)$ and $W\in Syl_r(S_0W)$. In particular $1\neq S_0\leqq\mathbf{C}_X(W)=1$. With this we complete the proof of (5).

In what follows we shall consider a $\bar{k}[X]$-module W as defined in (3).

(6) *The case* $s=q$.

Set $S_1=\langle\alpha_q\rangle S$. Let A be a maximal abelian normal subgroup of S_1. Then A is self-centralising, as S_1 is an s-group.

Now assume that $\alpha_q\notin A$. Then $[\alpha_q,A]\neq 1$, and there exists an $\langle\alpha_q\rangle A$-composition factor U of W such that $[U,[\alpha_q,A]]\neq 0$—note that $\mathbf{C}_X(W)=1$ and $r\neq s=q$. Let $U_A=U_1\oplus\cdots\oplus U_h$ with A-homogeneous components U_i. Clifford's Theorem gives $U\cong((U_1)_B)^{\langle\alpha_q\rangle A}$, where $B=\mathbf{N}_{\langle\alpha_q\rangle A}(U_1)$ acts irreducibly on U_1. Clearly $B\geqq A$. Suppose $B=A$. Then $U=((U_1)_A)^{\langle\alpha_q\rangle A}$, and by Mackey's Theorem, $U_{\langle\alpha_q\rangle}\cong((U_1)_{A\cup\langle\alpha_q\rangle})^{\langle\alpha_q\rangle}=((U_1)_1)^{\langle\alpha_q\rangle}\cong\oplus\bar{k}[\langle\alpha_q\rangle]$ contains a fixed point of α_q, contradicting (5). Thus $B=\langle\alpha_q\rangle A$. Then $U=U_1$ is a homogeneous A-module, $A/\mathbf{C}_A(U)$ is abelian and faithful on U, and thus is cyclic. Moreover α_q normalises $\mathbf{C}_A(U)$ and therefore induces a non-trivial automorphism on $A/\mathbf{C}_A(U)$: observe that $[U,[\alpha_q,A]]\neq 0$. This, however, contradicts the fact that U is a homogeneous A-module. We have therefore shown that $\alpha_q\in A$.

In particular $[S_1,\alpha_q,\alpha_q]\leqq[A,\alpha_q]=1$; i.e. $\tilde{x}^{(\alpha_q-1)^2}=0$ for every $\tilde{x}\in\tilde{S}=S/\Phi(S)$. Since $[\alpha_q,T]\neq 1$, a result of G. Higman (see e.g. [**5**; IX.1.10]) states that the minimal polynomial of α_q on $\tilde{S}$ is x^q-1. Consequently $q=2$. Now consider the action of α on the semidirect product TW. From $[\tilde{T},\alpha_p]=1$ (see (2)) we get $\mathbf{C}_T(\alpha_p)\neq 1$. Thus α_q induces a fixed point free automorphism of order 2 on $\mathbf{C}_T(\alpha_p)W$, whence this group is abelian. In particular $\mathbf{C}_T(\alpha_p)\leqq\mathbf{C}_X(W)=1$, a contradiction.

(7) *$t=r$ and $\mathbf{O}_t(Y)=1$ (i.e. $T=\tilde{T}$):*

From $[\tilde{T},\alpha_p]=1$ we deduce that $C=\mathbf{C}_T(\alpha_p)\neq 1$. In view of (5), α_q acts fixed point freely on CW, which therefore is nilpotent. Now $\mathbf{C}_X(W)=1$ requires that $t=r$. In particular, $\mathbf{O}_t(Y)=1$.

(8) *The case $q\neq s$:*

Let U be a homogeneous component of $W_{\Phi(S)}$. We have $\Phi(S)\leqq \mathbf{Z}(K)$, where $K=\langle\alpha_q\rangle TS$ (note that $[\alpha_q,\Phi(S)]=1$ follows from (5)). Thus $K\leqq \mathbf{N}_X(U)$. Since $\Phi(S)$ does not lie in $\mathbf{Z}(X)=\mathbf{Z}(\langle\alpha_q\rangle K)$, we cannot possibly have the faithful X-module W coinciding with U. This shows that $\mathbf{N}_X(U)=K$. Put $D=\mathbf{C}_K(U)$. Observing that $W=\bigoplus_{i=1}^{p} U^{\alpha_p^i}$, we see that $\bigcap_{i=1}^{p} D^{\alpha_p^i}=\mathbf{C}_K(W)=1$. Therefore D cannot contain a non-trivial subgroup of K normalised by α_p and so is contained in S: note that K/TS and TS/S are chief factors of K isomorphic to Sylow subgroups of K. Consider $R=S/D$. Since S is a special s-*subgroup* and $\Phi(S)$ is not contained in $D, R'=\Phi(R)=S'D/D=\Phi(S)D/D$ is of order s. Next note that due to (5) and Lemma 2 together with $q\neq s$, α_q centralises every abelian α_q-invariant subgroup of R. In particular $\mathbf{Z}(R)\leqq \mathbf{C}_R(\alpha_q)$. Let R_0/R' be an $\langle\alpha_q\rangle T$-invariant complement in R/R' of $\mathbf{Z}(R)/R'$. Then $\mathbf{Z}(R_0)\leqq \mathbf{Z}(R_0\mathbf{Z}(R))=\mathbf{Z}(R)$, which gives $\mathbf{Z}(R_0)=R_0\cap \mathbf{Z}(R)=R'=\Phi(R)$. Knowing already that the latter group is of order s, it is easy to see that R_0 must be an extraspecial s-group. Put $R_1=[R_0,\alpha_q]$ and $R_2=\mathbf{C}_{R_0}(\alpha_q)$. Then $[R_2,\alpha_q,R_0]=1$, $[R_0,R_2,\alpha_q]\leqq[\mathbf{Z}(R_0),\alpha_q]=1$, so that $[R_1,R_2]=[\alpha_q,R_0,R_2]=1$. Thus R_1 and R_2 are extraspecial, as also $R_0=R_1R_2$, so $\mathbf{Z}(R_1)\leqq\mathbf{Z}(R_0)$. Now from [**4**, V.17.13] we have $|R_1|=2^{2m+1}$, $q=2^{m-1}$, and in particular α_q acts irreducibly on $R_1/\Phi(R_1)\cong[\tilde{R}_0,\alpha_q]$, where $\tilde{R}_0=R_0/\Phi(R_0)$.

However considering the action of $\langle\alpha_q\rangle T$ on $\tilde{R}_0$, we find from Lemma 2(b) that $\tilde{R}_0$ becomes a free α_q-module on extending the field to the algebraic closure of $GF(s)$, and hence must be a free α_q-module itself (for two modules which become isomorphic after field extension, are already isomorphic, by the Noether–Deuring Theorem). Since α_q is irreducible on $[\tilde{R}_0,\alpha_q]$, this means $\tilde{R}_0$ is a free α_q-module of rank 1, so $|\mathbf{C}_{\tilde{R}_0}(\alpha_q)|=2$. But R_2 is also extraspecial, and so $R_2/\Phi(R_2)$ cannot have order 2. This is a final contradiction and the proof of our theorem is complete.

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REFERENCES

1. T. R. Berger, Nilpotent fixed point free automorphism groups of solvable groups, *Math. Z.* **131** (1973), 305–312.

2. W. Gaschütz, *Lectures on Subgroups of Sylow Type in Finite Soluble Groups* (Notes in Pure Mathematics **11**, The Australian National University, Canberra, 1979).

3. F. Gross, A note on fixed-point-free solvable operator groups, *Proc. Amer. Math Soc.* **19** (1968), 1363–1365.

4. B. Huppert, *Endliche Gruppen I* (Springer-Verlag, Berlin, New York, 1967).

5. B. Huppert and N. Blackburn, *Finite Groups II* (Springer-Verlag, Berlin, Heidelberg, New York, 1982).

6. E. W. Ralston, Solvability of finite groups admitting fixed-point-free automorphisms of order rs, *J. Algebra* **23** (1972), 164–180.

7. B. Rickman, Groups which admit a fixed-point-free automorphism of order P^2, *J. Algebra* **59** (1979), 77–171.

8. J. Shamash and E. Shult, On groups with cyclic Carter subgroups, *J. Algebra* **11** (1969), 564–597.

9. J. G. Thompson, Automorphisms of solvable groups, *J. Algebra* **1** (1964), 259–267.

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