PART IV.

Considerations on Localized Velocity Fields in Stellar Atmospheres: Prototype — The Solar Atmosphere.

B. - Consideration of Convective Instability from the Viewpoint of Physics.

Summary-Introduction: Similarity arguments for fully developed turbulence.

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Introduction.

In the study of turbulent flows similarity arguments are used to explore the consequences of non-mechanistic assertions concerning the general behavior of the flow. For example, it is currently assumed that viscosity plays no role in the determination of the mean velocity profile of turbulent shearing flow far from a boundary. The consequences of this assumption are that the amplitude of the mean velocity will be determined by the momentum transported into such a region and that the velocity profile will be a solution to Euler's equations.

The first section of this work will attempt to critically re-assess the experimental results used to support the assertion underlying conventional similarity theories. The second section discusses alternative assertions from which the qualitative experimental results can be deduced. A final section outlines the quantitative theory which has been constructed within the framework described in section two.

1. – Two quite different turbulent flows will be explored in order to test the generality of the conventional and new assertions. The first of these is turbulent shearing flow between parallel surfaces. The second is the turbulent convection of heat between horizontal surfaces.

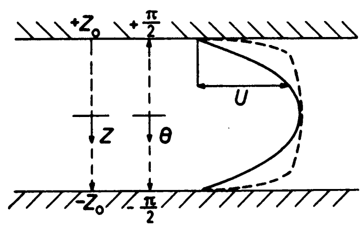

Fig. 1. – The geometry for turbulent shearing flow.

Current similarity arguments for shear flow. – In Fig. 1 the x axis is the direction of the mean velocity $U=U(Z)$. The solid velocity profile represents the parabolic solu-

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tion for laminar flow. The dashed line represents a mean velocity profile for the turbulent regime. For incompressible steady state flow the momentum transfer per unit mass is

$$\tau = \nu\beta + \overline{wu}\,, \tag{1}$$

where

$$\tau = \frac{Z}{Z_0}\tau_0\,, \qquad \tau_0 = -\left(\frac{Z_0}{\varrho}\frac{\partial \overline{P}_0}{\partial x}\right), \qquad \beta \equiv -\frac{\partial U}{\partial Z}\,,$$

ν is the kinematic viscosity, ϱ is the density, Z_0 the channel half-width, $\partial P_0/\partial x$ is the downstream pressure gradient at the boundary, w is the cross stream velocity fluctuation, u is the downstream velocity fluctuation and the horizontal superscript bar indicates an ensemble average. This flow is determined by the Reynolds number $R \equiv Z_0 U_m/\nu$ if U_m is fixed, where the subscript m indicates an average over the entire flow. The flow is determined by the alternate Reynolds number $R_\tau \equiv Z_0 U_\tau/\nu$ if the downstream pressure gradient is fixed, where $U_\tau \equiv \sqrt{\tau_0}$. The mean velocity profile is written then either as

$$U = U\left(R, \frac{Z}{Z_0}\right) \qquad \text{or} \qquad U = U\left(R_\tau, \frac{Z}{Z_0}\right). \tag{2}$$

Among the first observations on this turbulent flow was the discovery of the « velocity-defect law » shown in Fig. 2. At high Reynolds numbers all the data can be fitted to this one non-dimensional curve. The curve is logarithmic beyond a small linear « boundary layer », becoming parabolic in the mid-regions of the flow.

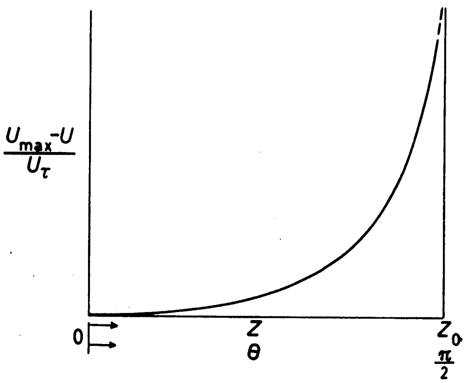

Fig. 2. – The velocity-defect « law ».

A recent presentation of the similarity arguments is given by TOWNSEND (1958). This argument is based on two general assertions. The first assertion is that viscosity plays no role in the determination of the mean profile far from the boundaries. From eq. (2) for R_τ fixed, the mean velocity in the midregions of the flow is then

$$U_c = U_{\max} + U_\tau F\left(\frac{Z}{Z_0}\right), \tag{3}$$

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where an arbitrary velocity of translation is chosen as the maximum velocity and F is an arbitrary function of Z/Z_0. The second assertion is that flow near the wall is only a function of distance from the wall and independent of the (large) channel half-width, Z_0. Defining a «running» Reynolds' number

$$R_{Z'} \equiv \frac{Z' U_\tau}{\nu} \,, \qquad Z' \equiv Z_0 - Z \,, \tag{4}$$

eq. (2) is re-written without loss of generality as

$$U = U_\tau G\left(R_{Z'}, \frac{Z'}{Z_0}\right), \tag{5}$$

where G is an arbitrary function of its arguments. The consequences of the second assertion is that the dependence on Z_0 in eq. (5) must vanish. Hence near the boundary,

$$U_B = U_\tau G(R_{Z'}) \,. \tag{6}$$

It is argued, that in the region of overlap, the boundary law, eq. (6), and the mid-region law, eq. (3), must be the same. Only one choice for F and G is then possible, and the overlap law becomes

$$U_0 = A + B \ln (R_{Z'}) \,, \tag{7}$$

where $A = A(R_\tau)$ only and B is a universal (Von Karman's) constant.

The experiments indicate that eq. (7) holds not just for some small overlap region of the flow but for most of the profile. It has been believed that this fact establishes the correctness of the similarity assertions. However, one might also interpret the experimental results as indicating that the first assertion is incorrect and that a region of completely inviscid flow does not exist. This possibility will be explored shortly.

Current similarity arguments for turbulent convection. – In Fig. 3 the solid line represents the temperature profile which would exist in the absence of motion between horizontal conducting plates separated by a distance d. The dashed line represents a mean temperature profile for the turbulent regime. For flows with a small total temperature drop, ΔT, the kinematical heat flux is

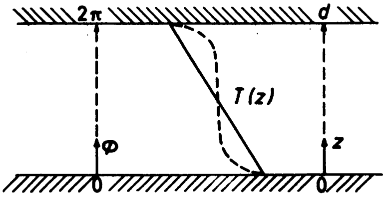

Fig. 3. – The geometry for turbulent convection.

$$H \equiv \frac{\mathscr{H}}{\varrho_m C_v} = \varkappa\beta + \overline{WT} \,, \qquad \beta \equiv -\frac{\partial \overline{T}}{\partial Z} \,. \tag{8}$$

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where $\mathscr{H}$ is the actual heat flux, ϱ is the density, $\varkappa$ is the coefficient of kinematic thermal conductivity, C_v is the specific heat of the fluid at constant volume, W is the vertical velocity and T is the temperature. For a fixed ΔT, this flow is determined by the Prandtl number $\sigma \equiv \nu/\varkappa$ and the Rayleigh number $R \equiv \alpha g \Delta T\, d^3/\varkappa\nu$, where α is the coefficient of expansion of the fluid and g is the acceleration of gravity. Alternatively, for fixed heat flux, the Prandtl number and $R_H = RH\, d/\varkappa \Delta T$ determine the flow. Hence the mean temperature profile may be written either as

$$\bar{T} = \bar{T}\left(R, \sigma, \frac{Z}{d}\right) \quad \text{or} \quad \bar{T} = \bar{T}\left(R_H, \sigma, \frac{Z}{d}\right). \tag{9}$$

A similarity argument for turbulent convection was given by PRIESTLY (1954). Paralleling the shear flow study, the assertion was made that viscosity and conductivity play no role in the determination of the mean temperature profile far from the boundaries. The consequences of this assertion were sought by establishing the possible dimensional relations in an equation such as eq. (9). Beyond the boundary region

$$\left(\bar{T}(Z) - T_m\right) = \text{const}\,(Z)^a(\mathscr{H})^b(\alpha g)^c(C_v)^d\,, \tag{10}$$

where relations between the powers a, b, c and d are to be found so that the right side of eq. (10) has the dimensions of a temperature. In contrast to the shear flow study, PRIESTLY discovered that a unique result is obtained for each of these powers. This is

$$\left(\bar{T}(Z) - T_m\right) = \text{const}\,(\mathscr{H}^2/\alpha g C_v^2)^{\frac{1}{3}}(Z)^{-\frac{1}{3}}\,. \tag{11}$$

In turbulent convection, then, an explicit form for the « inviscid » region is found without studying the overlap with a « boundary » region.

The experimental evidence in the laboratory (TOWNSEND, 1959) does not support the $Z^{-\frac{1}{3}}$ law of eq. (11) but fits a Z^{-1} law rather closely. Hence the correctness of the assertion that molecular transport coefficients are unimportant in the body of the flow is in doubt.

2. – The similarity argument to be advanced in this paper rests on two general assertions concerning steady state turbulent flows. The first of these assertions is that the mean profiles (of temperature and velocity) approach but never exceed, the local condition for marginal inviscid instability.

The problem of turbulent convection will be treated first. The condition for inviscid instability in convection is that

$$\beta > 0\,, \tag{12}$$

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that is that there be lighter fluid below and heavier fluid above. Thus the first assertion may be written

$$\frac{\varkappa\beta}{H} = I^2 \geqslant 0 \,, \tag{13}$$

where I is to be represented by its Fourier expansion

$$I = \sum_{n=-\infty}^{+\infty} I_n \exp[in\varphi]\,, \qquad 0 \leqslant \varphi \leqslant 2\pi\,, \tag{14}$$

where the co-ordinate φ is shown in Fig. 3 and where $I_n = I^*_{-n}$ in order that I be real.

The second assertion concerning steady state turbulence is that the smallest scale of motion effective in the transport of heat (or momentum) is a monotonically increasing function of the Rayleigh (or Reynolds) number of the flow. In Fig. 4 a possible spectral description is given of the mean squared velocity and convective heat transport at some fixed Rayleigh number. The wave number $n_0 = n_0(R, \sigma)$ is defined as the largest wave number which contributes to the transport of heat. That is, the wave number (n_0+1) makes no effective contribution to heat transport. It is possible, indeed necessary to the Kolmogoroff-like studies of isotropic turbulence, that the « disorganized » spectrum of the mean squared velocity extend to higher wave numbers than the « organized » correlations responsible for convection.

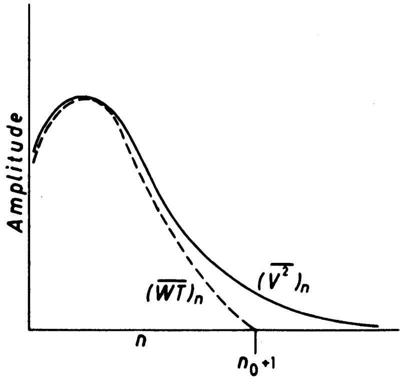

Fig. 4. – Characteristic spectra for « organized » $(\overline{WT})$ and « disorganized » $(\overline{V}^2)$, moments in turbulent convection.

From eq. (8)

$$\frac{\beta}{\beta_m} = 1 + \frac{(WT)_m - \overline{WT}}{\varkappa\beta_m}\,, \tag{15}$$

hence a consequence of the second assertion is that the spectrum of I, eq. (14), terminate at some $n_0 = n_0(R, \sigma)$. We now wish to establish the conditions under which the restricted sum

$$I(n_0, \varphi) = \sum_{n=-n_0}^{+n_0} I_n \exp[in\varphi] \tag{16}$$

leads to qualitative laws for the mean temperature profile as R (and n_0) approach infinity.

One such condition on the structure of I_n can be found by summing eq. (16) by parts:

$$\sum_{n=-n_0}^{+n_0} I_n \exp[in\varphi] = \left(I_{n_0} \sum_{n=-n_0}^{+n_0} \exp[in\varphi]\right)_r - \\ - \frac{1}{\exp[i\varphi]-1} \sum_{n=-n_0}^{+n_0} \Delta I_n \exp[in\varphi]\,. \tag{17}$$

If then I_n is smooth in the sense that

$$\Delta I_n \simeq \frac{I_n}{n_0}\,, \tag{18}$$

the first term on the right of eq. (17) is an asymptotic « law » of order n_0 larger than the second term, for $\varphi \gg \pi/n_0$.

Before investigating this consequence, an alternate statement of the conditions will be explored. If one writes

$$I_n = G\left(\frac{n}{n_0+1}, n_0\right), \tag{19}$$

and expands G in a power series

$$G = \sum_{m=0}^{\infty} G_m \left(\frac{n}{n_0+1}\right)^m, \qquad G_m = (-1)^m G_m^*\,, \tag{20}$$

it is possible to perform the partial sums and explicitly order terms in l/n_0. The sum

$$\sum_{n=-n_0}^{+n} \exp[in\varphi] = \frac{\exp[i\varphi(n_0+1)]}{\exp[i\varphi]-1} + \frac{\exp[-i\varphi(n_0+1)]}{\exp[-i\varphi]-1} \equiv \Phi\,, \tag{21}$$

permits one to determine the sums

$$\sum_{n=-n_0}^{+n_0} n^m \exp[in\varphi] = (-1)^m \frac{\partial^m \Phi}{\partial \varphi^m} = \\ = (n_0+1)^m \left\{\frac{\exp[i\varphi(n_0+1)]}{\exp[i\varphi]-1} + (-1)^m \frac{\exp[-i\varphi(n_0+1)]}{\exp[-i\varphi]-1}\right\} - \\ - m(n_0+1)^{m-1} \left\{\frac{\exp[i\varphi(n_0+2)]}{(\exp[i\varphi]-1)^2} + (-1)^m \frac{\exp[-i\varphi(n_0+2)]}{(\exp[-i\varphi]-1)^2}\right\} + \dots\,. \tag{22}$$

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Hence

$$I(n_0, \varphi) = \sum_{m=0}^{\infty} \left[G_m \left\{ \frac{\exp\,[i\varphi(n_0+1)]}{\exp\,[i\varphi]-1} + (-1)^m \frac{\exp\,[-i\varphi(n_0+1)]}{\exp\,[-i\varphi]-1} \right\} \cdot \right. \tag{23}$$

$$\cdot \frac{-mG_m}{n_0+1} \left\{ \frac{\exp\,[i\varphi(n_0+2)]}{(\exp\,[i\varphi]-1)^2} + (-1)^m \frac{\exp\,[-i\varphi(n_0+2)]}{(\exp\,[-i\varphi]-1)^2} \right\} +$$

$$\left. + \ldots + \frac{m^a G_m}{(n_0+1)^a} \{\} + \ldots \right].$$

One obvious, but strong, condition that, away from the boundaries, the leading term be of order n_0 larger than all other terms in eq. (23) is that G be a finite polynomial. The conditions that I_n be smooth, eq. (18), or that it be properly represented by a finite polynomial therefore lead to the same law at large R. One may write

$$\sum_{m=0}^{\infty} G_m \equiv g_r + i g_i$$

then from Eq. (20) and (23)

$$I_{n_0 \to \infty} = g_r \frac{\sin\,(2n_0+1)(\varphi/2)}{\sin\,(\varphi/2)} - g_i \frac{\cos\,(2n_0+1)(\varphi/2)}{\sin\,(\varphi/2)}. \tag{24}$$

Our physical problem requires that β be symmetric around the mid-point of the region. Therefore either g_i or g_r must be zero. If one chooses $g_i = 0$ and defines $\theta = \varphi - \pi$ then

$$\left(\frac{\varkappa\beta}{H}\right)_{n_0 \to \infty} = I^2 = g_r^2 \frac{\cos^2\,(2n_0+1)\theta}{\cos^2\theta}. \tag{25}$$

Integrating eq. (25) to obtain the temperature field one finds

$$\frac{\varkappa(T_m - \overline{T}(\theta))}{H} = \frac{g_r^2}{2}\left(\operatorname{tg}\theta + 0\left(\frac{1}{n_0}\right)\right), \tag{26}$$

outside a region within π/n_0 of the boundaries. An identical law results for $g_r = 0$ and $g_i \neq 0$.

Near the boundary eq. (26) leads to

$$T_m - \overline{T}(Z) \simeq Z^{-1} \tag{27}$$

in keeping with the experimental results.

The preceding arguments for thermal turbulence are easily adapted to the shear flow problem. The sufficient condition for inviscid stability in parallel

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flow is that the curvature of the flow do not change sign. Paralleling eq. (13) one may write

$$\frac{\partial^2 (U/U_\tau)}{\partial (Z/Z_0)^2} = I^2 \geqslant 0. \tag{28}$$

Since the conditions on I are just those of the thermal problem,

$$I^2_{n_0 \to \infty} = g_r^2 \frac{\cos^2 (2n_0 + 1)\theta}{\cos^2 \theta}, \tag{29}$$

where n_0 is now a monotonically increasing function of the Reynolds' number. Integrating eq. (29) twice one obtains a velocity-defect law

$$\frac{U_{\max} - U}{U_\tau} = \frac{1}{2} g_r^2 \left\{ \ln \left(\frac{1}{\cos \theta} \right) + 0 \left(\frac{1}{n_0} \right) \right\}. \tag{30}$$

This profiles adheres closely to the experimental results (LAUFER, 1950) not only in its logarithmic behavior near the boundary but in its parabolic character in the mid-regions of the flow.

The possibility remains that the heat and momentum transport spectra could have « tails » extending well beyond n_0+1. It has been found that a weak exponential « tail » beyond n_0, responsible for only one percent of the total transport, can significantly modify eq. (24) for I. Hence one must conclude that the available experimental evidence supports the second assertion as well as the first.

3. – The qualitative conclusions, eq. (26) and (30), have been obtained in two previous studies (MALKUS, 1954, 1956). However they were immersed in the complexity of a quantitative analysis and it was not clear at that time whether these « laws » were immediate consequences of the basic assertions or whether they resulted from the several mathematical approximations. The formulation of the problem in Section **2** was made to isolate these asymptotic consequences of the two assertions from two more explicit assertions on which the quantitative theory rests.

The first of these more explicit assertions is that the smallest scale of motion contributing to the transport of heat (or momentum) is that smallest motion which is unstable on the mean profile. This statement replaces the second assertion of Section **1**. It is based on the belief that there is a negligible transfer of organization down the spectrum by non-linear processes. Hence n_0+1 is to be found by a conventional stability analysis of the mean profile.

Still, within the constraints so imposed on the fields of motion, many pos-

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sibilities remain. The second of the more explicit assertions was that the constrained flow would approach that extreme state which maximized the total dissipation rate. The determination of this extreme state proves to be a tractable, if difficult, variational problem for the optimum I_n. With I_n quantitatively determined, a mean field of flow from boundary to boundary and the dependence of the transport on the boundary conditions is given. Comparison with experiment can then establish the range of validity of the assertions with little opportunity for self-deception.

First attempts at the determination of the extreme states for shear turbulence and convective turbulence have been made (MALKUS, 1954, 1956). The quantitative results for Von Karman's constant ($\frac{1}{2}g_r^2$ in eq. (30)), agree with the data of LAUFER (1950). The quantitative results for the convective constant ($\frac{1}{2}g_r^2$ in eq. (26)) is twenty percent less than the value found by TOWNSEND (1959). However, in this latter case, more mathematical care must be taken to satisfy boundary conditions and more data must be gathered. A report on steps in both these directions and a new simplification of the mathematical problem will be presented in a forthcoming study.

* * *

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