Canad. Math. Bull. Vol. 15 (2), 1972

NOETHERIAN TENSOR PRODUCTS

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1. **Introduction**. Relatively little is known about the ideal structure of $A \otimes_R A'$ when A and A' are R-algebras. In [4, p. 460], Curtis and Reiner gave conditions that imply certain tensor products are semi-simple with minimum condition. Herstein considered when the tensor product has zero Jacobson radical in [6, p. 43]. Jacobson [7, p. 114] studied tensor products with no two-sided ideals, and Rosenberg and Zelinsky investigated semi-primary tensor products in [9].

All rings considered in this paper are assumed to be commutative with identity. Furthermore, R will always denote a field.

We are concerned with the question: When is $A \otimes_R A'$ Noetherian? Our main result gives sufficient conditions for $A \otimes_R A'$ to be Noetherian where A is a Zariski ring and A' is an algebraic extension field of R. The best known previous result on this problem gives sufficient conditions for the tensor product of two fields to be Noetherian [8, p. 168]. However, if A is a perfect field of characteristic $p \neq 0$, R is an imperfect subfield of A, and $A = A'$, then $A \otimes_R A'$ is not Noetherian. (If $x \in A$ such that $x^{1/p} \notin R$, then the ascending chain of principal ideals $(x_1) \subset (x_2) \subset \cdots \subset (x_n) \subset \cdots$, where $x_i = 1 \otimes 1 - 1/x^{1/p^i} \otimes x^{1/p^i}$, is an infinite chain of ideals.) This example is a coherent ring, but an example, due to Soublin [11], shows that the tensor product of two coherent rings need not be coherent.

In general, a ring S with an ideal M can be made into a topological ring by taking powers of M as a basis of neighborhoods of zero. This topology is called the M-adic topology and is Hausdorff if and only if $\bigcap_{n=1}^{\infty} M^n = (0)$. If the M-adic topology is Hausdorff, then a metric can be defined on S so that the M-adic topology is a metric topology and then a completion $\hat{S}$ of S can be constructed by taking equivalence classes of Cauchy sequences. Moreover, $\hat{S}$ can be regarded as the inverse limit of the sequence $S/M \leftarrow S/M^2 \leftarrow S/M^3 \leftarrow \cdots$, where each map is the natural homomorphism [14, p. 434]. Furthermore, if M is finitely generated, then $\hat{S}$ is Noetherian if and only if S/M is Noetherian [2, p. 48].

We are particularly interested in $S = A \otimes_R A'$, where A is Noetherian and A' is an algebraic extension field of the field R. We shall assume M_1 is an ideal of A and that A is Hausdorff under the M_1-adic topology. If $M = M_1 S$, then S is Hausdorff under the M-adic topology [10, p. 64]. We shall denote the completion

Received by the editors November 4, 1970 and, in revised form, February 25, 1971.

(1) The authors received support under NSF Grant GP-8424 during the writing of this paper.

$\hat{S}$ of S under the M-adic topology by $A \mathbin{\widehat{\otimes}_R} A'$ and we note that $\hat{S}$ is Noetherian if $A/M_1 \otimes_R A'$ is Noetherian.

According to Gilmer [5], a ring S is said to have property C with respect to an overring T if for each ideal B of S, $BT \cap S = B$. Gilmer observes that it is sufficient to consider only finitely generated ideals B of S. Furthermore, if T is Noetherian and S has property C with respect to T, then S is Noetherian.

Our basic approach to the problem is a simple one. We shall consider two cases when $S = A \otimes_R A'$ has property C with respect to its completion $\hat{S}$. Then it will follow that S is Noetherian whenever $\hat{S}$ is Noetherian.

Since for a finitely generated ideal B of S the closure of B is $B\hat{S} \cap S$ [13, p. 256], S will have property C with respect to $\hat{S}$ if each finitely generated ideal of S is closed. Furthermore, an overring T of S is faithfully flat as an S-module if and only if S has property C with respect to T and T is a flat S-module [1, p. 33], and in the presence of flatness, property C follows if each maximal ideal N of S is such that $NT \neq T$ [1, p. 44].

In Proposition 1, we show that if A is a Zariski ring and A' is an algebraic extension of the field R, then each finitely generated ideal of $A \otimes_R A'$ is closed. Under the weaker hypothesis that A is Noetherian with a Hausdorff M-adic topology, but under the additional assumption that the completion $A \mathbin{\widehat{\otimes}_R} A'$ of $A \otimes_R A'$ is Noetherian, we show in Proposition 2 that $A \mathbin{\widehat{\otimes}_R} A'$ is a flat module over $A \otimes_R A'$. The main result follows as a corollary to either of the propositions.

2. **Results**. Recall that A is a Zariski ring under the M-adic topology if A is Noetherian and each ideal of A is closed [13, p. 263]. In particular, A is Hausdorff under the M-adic topology, and if B is a ring which is a finite A-module, then B is also a Zariski ring.

PROPOSITION 1. *If A is a Zariski ring under the M-adic topology and if A' is an algebraic extension of the field R, then each finitely generated ideal of $A \otimes_R A'$ is closed in the $M(A \otimes_R A')$-adic topology.*

Proof. We let $S_\alpha = A \otimes_R B_\alpha$, where B_α is a finite extension of R. By flatness, we can consider the direct limit $S = A \otimes_R A'$ as the directed union of the subsets S_α. For $\alpha \leq \beta$, S_α is a subspace of S_β since S_α is Noetherian and S_β is a finite S_α-module [8, p. 52]. Thus if $x \in (M^n S) \cap S_\alpha$, then $x = \sum r_{\alpha_i} a_i$, where $r_{\alpha_i} \in S_{\alpha_i}$ and $a_i \in M^n$.

For $\lambda \geq \alpha$, and $\lambda \geq \alpha_i$ for each i, $x \in M^n S_\lambda$. Since $(M^n S_\lambda) \cap S_\alpha = M^n S_\alpha$, it follows that $x \in M^n S_\alpha$. Thus $(M^n S) \cap S_\alpha = M^n S_\alpha$ and S_α is a subspace of S for each α.

If B is a finitely generated ideal of S, there is an α_0 and a finitely generated ideal C of S_{α_0} such that B is the union (direct limit) of CS_β for $\beta \geq \alpha_0$. Since each S_β is a Zariski ring, CS_β is a closed ideal of S_β. Furthermore, for $\lambda \geq \delta \geq \alpha_0$, $(CS_\lambda) \cap S_\delta$ is the closure of CS_δ in S_δ. Thus $(CS_\lambda) \cap S_\delta = CS_\delta$. Since $B \cap S_\delta$ is the union of $(CS_\lambda) \cap S_\delta$ for $\lambda \geq \delta \geq \alpha_0$, it follows that $B \cap S_\delta = CS_\delta$ and $B \cap S_\delta$ is closed in S_δ for each $\delta \geq \alpha_0$. Therefore, B is closed in S.

PROPOSITION 2. *Let A be a Noetherian ring which is Hausdorff under the M-adic topology, and let A' be an algebraic extension of the field R such that $A \mathbin{\widehat{\otimes}_R} A'$ is Noetherian. Then $A \mathbin{\widehat{\otimes}_R} A'$ is a flat $A \otimes_R A'$-module.*

Proof. Let $S = A \otimes_R A'$, let $T = A \mathbin{\widehat{\otimes}_R} A'$, and let $S_\alpha = A \otimes_R B_\alpha$, where B_α is a finite extension of R. Then $T/M^nT \simeq S/M^nS$ since M^nS is an open ideal of S under the MS-adic topology [14, p. 434]. Now

$$S = A \otimes_R A' \simeq A \otimes_R (B_\alpha \otimes_{B_\alpha} A') = (A \otimes_R B_\alpha) \otimes_{B_\alpha} A' = S_\alpha \otimes_{B_\alpha} A'$$

and since A' is a free B_α-module, S is a flat S_α-module. Furthermore,

$$S \otimes_{S_\alpha} (S_\alpha/M^nS_\alpha) \simeq (S \otimes_{S_\alpha} S_\alpha)/(M^nS_\alpha)(S \otimes_{S_\alpha} S_\alpha) \simeq S/M^nS \simeq T/M^nT.$$

Therefore, T/M^nT is a flat S_α/M^nS_α-module for each integer n. If $T_\alpha = T$ for each α, then in order to show that $T = \lim_\rightarrow T_\alpha$ is a flat $S = \lim_\rightarrow S_\alpha$-module, it suffices to show that $T = T_\alpha$ is a flat S_α-module for each α[1, p. 35]. Since $(MS_\alpha)T$ is contained in the Jacobson radical of T, T is ideally separated for MS_α [2, p. 101]. Therefore, T is a flat S_α-module [2, p. 98].

If we also assume, in addition to the hypothesis of Proposition 2, that each maximal ideal N of $A \otimes_R A'$ is such that $N(A \mathbin{\widehat{\otimes}_R} A') \neq A \mathbin{\widehat{\otimes}_R} A'$, then $A \otimes_R A'$ has property C with respect to $A \mathbin{\widehat{\otimes}_R} A'$ and $A \otimes_R A'$ is Noetherian.

MAIN RESULT. *Suppose A is a Zariski ring under the M-adic topology and that A' is an algebraic extension of the field R. If the completion of $A \otimes_R A'$ under the $M(A \otimes_R A')$-topology is Noetherian, then $A \otimes_R A'$ is Noetherian.*

Proof. The result follows immediately from Proposition 1. We also show that the main result is a corollary of Proposition 2. By the above remark, it is sufficient to show that each maximal ideal N of $S = A \otimes_R A'$ is such that $N\hat{S} \neq \hat{S}$ where $\hat{S}$ is the completion of S.

If all the maximal ideals are open in the MS-adic topology (that is, MS is contained in the Jacobson radical $J(S)$ of S), then for each maximal ideal N of S, $\hat{S}/\hat{N} \simeq S/N$, where $\hat{N}$ is the closure of N in $\hat{S}$. Thus, under the assumption that N is open, $N\hat{S} \subseteq \hat{N} \neq \hat{S}$. The proof will be complete if we show that each maximal ideal N of S is open in the MS-adic topology. Each maximal ideal of A contains M since A is a Zariski ring. Also, A' algebraic over R implies S is integral over A [3, p. 14]; consequently each maximal ideal of S lies over a maximal ideal of A. Hence each maximal ideal of S contains MS, and is therefore open in the MS-adic topology.

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https://doi.org/10.4153/CMB-1972-043-x Published online by Cambridge University Press